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Radiations by "solitons" at the zero group-dispersion wavelength of single-mode optical fibers

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Nonlinear pulse propagation at the zero group-dispersion wavelength is studied analytically. It is discovered that, after the characteristic initial frequency splitting, evolution of the pulse envelope that is shifted down to the anomalous regime is described by the nonlinear Schrödinger equation with higher-order dispersion as a perturbation. The effect of the perturbation on the pulse is to excite radiation at a frequency inversely proportional to the small parameter β. The amplitude of the radiation is exponentially small [$\propto \exp(-1/\beta)$] and can be calculated only by the perturbation method that goes beyond all orders.

I. INTRODUCTION

In recent years, many activities have been centered on the propagation of nonlinear pulses (solitons) in single-mode optical fibers. The goal is to determine the feasibility of utilizing solitons in long-distance optical communication systems. The information-carrying capacity of such a nonlinear system is estimated to be many gigabits per second, while that of the present system operating in the linear regime is about hundreds of megabits per second. The idea was suggested by Hasegawa and Tappert in 1973.[1] Taking into account the intrinsic Kerr nonlinearity of silica, they showed that pulse propagation in the anomalous dispersion regime of optical fiber is governed by the nonlinear Schrödinger equation. It is well known[2] that the nonlinear Schrödinger equation can be solved completely by the inverse scattering method. Among other characteristics, the equation possesses solutions whose pulse shape either propagates without distortion (fundamental soliton) or changes periodically (breathers). Such solutions, the former in particular, are ideal information carriers of a long-distance optical communication system. In the linear regime, the maximum bit rate of a single-mode fiber is limited by dispersion. Optical pulses traveling along the fiber would broaden and overlap with their neighbors, and eventually become indistinguishable at the receiver end. With solitons, this pulse broadening dispersive effect is countered by the Kerr nonlinearity. Much shorter pulses could be used, and the bit rate could then be increased significantly.

Experimental verification came in 1980 when propagation of both fundamental solitons and breathers were observed by Mollenauer, Stolen, and Gordon,[3] There has been much progress since then. For example, the contracting phase of breathers is used in pulse compression experiments, the invention of the soliton laser which emitted 2-breathers readily, and the use of stimulated Raman scattering to periodically amplify the solitons so as to eliminate the need for slow and expensive electronic repeaters.[4–8] There are also a theoretical report[9] and subsequent experimental observation[10] of soliton propagation at the zero dispersion point. In a single-mode fiber, dispersion is adequately described by the second-order dispersion coefficient, $k''(\omega)$, in most frequencies. The quantity $k(\omega)$ is the propagation constant, and ω is the angular frequency. For pure silica, k'' equals zero at around 1.27 μm. The wavelength λ_0 at which $k''=0$ is called the zero dispersion point. It divides the spectrum into the normal dispersion regime ($k''>0$), and the anomalous dispersion regime ($k''<0$). Dispersion at λ_0 is of course nonzero because higher-order dispersion effects in general do not vanish there. Pulses launched at the zero dispersion point are reported to shift up part of its energy into the normal dispersion regime, and shift down part of its energy into the anomalous dispersion regime (Fig. 1). The latter portion then develops into a solitonlike pulse. Questions arise on whether the observed wave packet is indeed a solitary wave, or even a soliton. Contrary to the famed nonlinear Schrödinger equation, the evolution equation at the zero dispersion point is not integrable by the inverse scattering method, and one therefore would not expect it to have soliton solutions. On the other hand, numerical simulations[9] indicate that solitary solutions exist if the ratio between the pulse width and frequency shift satisfies certain criteria.

In this paper we show that the wave packet shifted down to the anomalous dispersion region is not a "true" solitary wave, hence not a soliton. In fact, the high-frequency component at a value determined by the second- and third-order dispersion coefficient is radiated out continuously, although its amplitude is exponentially small. We will show that the effect of the frequency downshift is to transfer part of the initial pulse into the regime governed by the nonlinear Schrödinger equation with the third-order dispersion as a small perturbation. Therefore subsequent evolution of that portion of the pulse can be studied by the perturbed nonlinear Schrödinger equation. Using a perturbation method known as "asymptotic beyond all orders" developed by Kruskal and Segur,[11] we show that the effect of the small third-order dispersion is a radiation of a high-frequency component with exponentially small amplitude. This

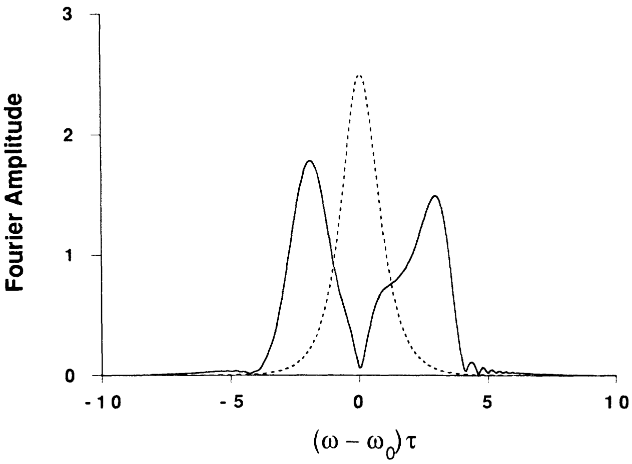

FIG. 1. Pulse launched at the zero dispersion point splits into two peaks in the frequency domain, which then evolve independently. The dashed curve is the initial frequency spectrum. The solid curve is the spectrum after $\xi=1.7$. The peak downshifted to the anomalous dispersion regime ($\omega<\omega_0$) exhibits its solitonlike properties.

small-amplitude radiation cannot be detected from the first few orders of direct perturbation expansion. In fact, we show that solitary wave solutions exist to all orders of an asymptotic series expansion in terms of the small parameter. But this is not sufficient to determine the true nature of the solution. Infinitesimal terms that violate the solitary wave condition might still exist. These corrections, if present, are certainly smaller than any polynomial order terms in the series. That is why they are described as beyond all orders. One method to recover these infinitesimal contributions is to continue the series solution to its singularities in the complex plane. There, the ordering of the original equation changes, and terms that were infinitesimal become large enough to be noticed. We pick up the beyond all order corrections, and then continue the solution back to the real axis along a contour on which the original series is convergent. The solution is further continued out to infinity in order to determine the effect of the perturbation which is the high-frequency radiation mentioned earlier.

The present paper is arranged as followed. In Sec. II we introduce the evolution equation at the zero dispersion point and show that it can be transformed to the perturbed nonlinear Schrödinger equation by a shift in the carrier frequency of the pulse. In Sec. III we formulate the perturbation problem, and give an asymptotic series solution which satisfies the solitary wave condition to all orders. We then discuss the meaning and strategy of looking for corrections that are beyond all orders. In Sec. IV the perturbed equation is expanded at one of the essential singularities of the asymptotic series in order to recover terms that are beyond all orders in the original equation. The new equation is also solved perturbatively, and the correction term is determined by matching the inner and outer series solutions near the singularity in their common region of validity. The presence of this infinitesimal correction violates the solitary wave condition. Finally, in Sec. V, we show that this violation is manifested in the form of high-frequency radiations.

II. NONLINEAR PULSE EVOLUTION EQUATIONS

We first introduce the equation governing the nonlinear pulse evolution in the anomalous dispersion regime, i.e., the nonlinear Schrödinger equation,

$$i\frac{\partial u}{\partial \xi}+\frac{1}{2}\frac{\partial^2 u}{\partial s^2}+|u|^2u=0\ , \tag{2.1}$$

where u is the amplitude of the normalized wave envelope, ξ and s represent normalized distance along the fiber and normalized time variable, respectively. Equation (2.1) can be derived from Maxwell's equations using the reductive perturbation method.[12,13] The second term in Eq. (2.1) represents the effect of anomalous dispersion, which has a positive sign relative to the nonlinear term. In an optical fiber, the refractive index (n) depends on the incident pulse intensity (I) as well as the frequency, i.e.,

$$n(\omega,I)=n_0(\omega)+n_2I\ . \tag{2.2}$$

For pure silica, $n_0=1.5$, and $n_2=1.22\times10^{-22}$ (V/m).[2] Equation (2.2) is known as the optical Kerr effect, and it is represented by the last term in Eq. (2.1). The nonlinear Schrödinger equation has a very rich mathematical structure. Here, we concentrate on a special class of localized solutions that propagate without change in shape, i.e., the fundamental solitons. It is given by[14]

$$q(\xi,s)=A\ \mathrm{sech}[A(s-C\xi-s_0)]\exp[iC(s-C\xi-s_0)+i\tfrac{1}{2}(A^2+C^2)\xi+i\phi_0]\ , \tag{2.3}$$

where A is the amplitude, C is the normalized group delay, and s_0 and ϕ_0 are the initial time and phase, respectively. The pulse width ($=1/A$) is related to the amplitude A, a manifestation of the nonlinear nature of the pulse. From Eq. (2.3), we see that the pulse envelope $|q(\xi,s)|$ is independent of the distance parameter ξ, hence the pulse shape is invariant. Next, we discuss the evolution equation at the zero dispersion point which is given by

$$i\frac{\partial u}{\partial \xi'}-i\frac{1}{6}\frac{\partial^3 u}{\partial s'^3}+|u|^2u=0\ . \tag{2.4}$$

Equations (2.1) and (2.4) are similar except that the second-order dispersion term in Eq. (2.1) is replaced by the third-order dispersion term in Eq. (2.4). Equation (2.4) does not share the same mathematical properties of the nonlinear Schrödinger equation. Namely, it is not integrable by the inverse scattering method. There are no solitons, and no solitary wave solutions in analytic form. However, numerical results suggest that solitary wave solutions exist if

$$0>\frac{\sigma}{\Omega_0}>-0.24\ . \tag{2.5}$$

In Eq. (2.5), σ is the pulse width, and Ω_0 is the frequency shift of the pulse from the zero dispersion point. Outside this regime, the pulse is observed to radiate continuously. The negative sign in the inequality means that the frequency shift is towards the anomalous dispersion side.

We account for the frequency downshift of the "solitary" pulse explicitly by applying the following transformation to Eq. (2.4):

$$q(\xi,s)=u(\xi',s')\exp(i\Omega_0 s'-i\Omega_0^3\xi'/6)\ , \tag{2.6}$$

where s', and ξ' are given by

$$s=(s'-\tfrac{1}{2}\Omega_0^2\xi')/\sqrt{\Omega_0}\ , \tag{2.7}$$

$$\xi=\xi'\ . \tag{2.8}$$

We obtain

$$i\frac{\partial q}{\partial \xi}+\frac{1}{2}\frac{\partial^2 q}{\partial s^2}+|q|^2 q=i\beta\frac{\partial^2 q}{\partial s^3}\ , \tag{2.9}$$

where the coefficient $\beta=1/6(\Omega_0)^{3/2}$. Equation (2.6) represents a shift of the carrier frequency from the zero dispersion value into the anomalous dispersion region by an amount Ω_0. Equations (2.7) and (2.8) are scaling and Galilean transformations. The relation between Eqs. (2.4) and (2.9) holds as long as one is sufficiently close to the zero dispersion point that the second-order dispersion coefficient at the operating wavelength can be approximated by the linear extrapolation from the third-order dispersion coefficient at the zero dispersion point. Inspection of Eq. (2.9) shows that it is the nonlinear Schrödinger equation with an additional third-order dispersion term. Under the transformation in Eqs. (2.7) and (2.8), relation (2.5) implies that solitary wave solutions of Eq. (2.9) exist when $A\beta<0.04$, where A is the pulse amplitude. For pulses with $A\gtrsim 1$, the right-hand side of Eq. (2.9) can therefore be treated as a small perturbation. In subsequent discussions, we shall take advantage of this and use Eq. (2.9) to analyze the solitary wave solutions of Eq. (2.4). Equation (2.9) has been studied numerically.[15] It is reported that the fundamental soliton of the nonlinear Schrödinger equation radiates at a frequency $\omega\approx 1/2\beta$ (Figs. 2 and 3) due to the presence of the small third-order dispersion term.

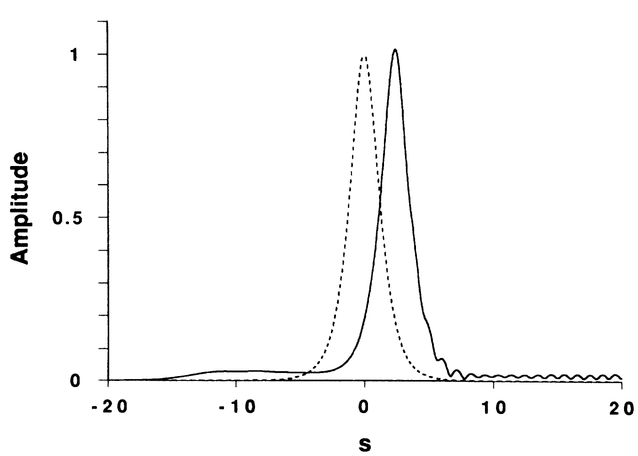

FIG. 2. The solution in time domain of the perturbed nonlinear Schrödinger equation with $\beta=0.1$ is plotted. The dashed curve is for $\xi=0.0$, and the solid curve is for $\xi=5\pi$. The soliton is observed to radiate continuously.

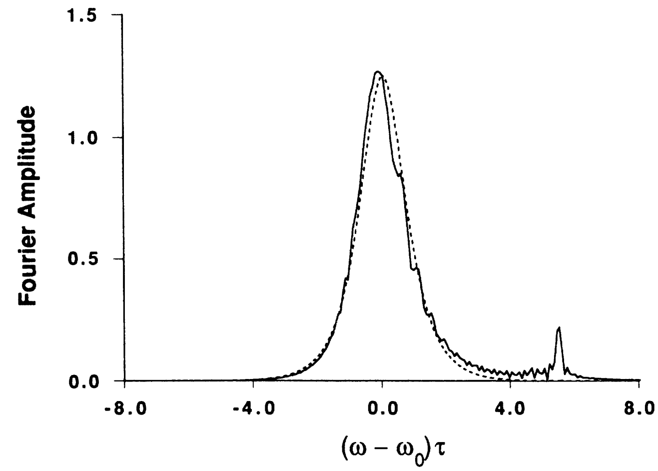

FIG. 3. The same solutions as in Fig. 2 are plotted in frequency domain for $\xi=0.0$ (dashed curve), and $\xi=5\pi$ (solid curve). A peak is observed at $(\omega-\omega_0)\approx 1/2\beta$.

III. FORMULATION OF PERTURBATION TO ALL ORDERS

In this section we show that solitary wave solutions of Eq. (2.9) exist to all orders in a perturbation expansion in the parameter β. First, the partial differential equation is reduced to an ordinary differential equation by the following assumption:

$$q(\xi,s)=\tilde{q}(\theta)\exp(ik\xi)\ , \tag{3.1}$$

where $\theta=s-v\xi$. Equation (2.9) is transformed to

$$-k\tilde{q}-iv\frac{\partial\tilde{q}}{\partial\theta}+\frac{1}{2}\frac{\partial^2\tilde{q}}{\partial\theta^2}+|\tilde{q}|^2\tilde{q}=i\beta\frac{\partial^3\tilde{q}}{\partial\theta^3}\ . \tag{3.2}$$

Our aim is to find solitary wave solutions, i.e., solutions that obey the following boundary conditions:

$$|\tilde{q}|\to 0 \quad \text{as}\ \theta\to\pm\infty\ . \tag{3.3}$$

Alternatively, we make use of the fact that if $\tilde{q}(\theta)$ satisfies Eq. (3.2), then $\tilde{q}^*(-\theta)$ is also a solution, to demand the symmetry condition on a solitary wave solution of Eq. (3.2), i.e.,

$$\tilde{q}(\theta)=\tilde{q}^*(-\theta)\ . \tag{3.4}$$

Equation (3.4) means that $\mathrm{Re}\tilde{q}(\theta)$ is symmetric, while $\mathrm{Im}\tilde{q}(\theta)$ is antisymmetric with respect to θ. Then Eq. (3.3) can be rewritten as

$$|\tilde{q}|\to 0 \quad \text{as}\ \theta\to-\infty\ , \tag{3.5}$$

$$\frac{\partial}{\partial\theta}[\mathrm{Re}\tilde{q}(0)]=0\ , \tag{3.6}$$

$$\frac{\partial^2}{\partial\theta^2}[\mathrm{Im}\tilde{q}(0)]=0\ . \tag{3.7}$$

Equation (3.2), and (3.5)–(3.7) are overdetermined as written, but a well-posed problem may be defined by dropping Eq. (3.7). We show in Appendix A that Eqs. (3.2), (3.5), and (3.6) define $q(\theta;\beta)$ uniquely for all real values of β. the solution so defined need not satisfy Eq. (3.3) as $\theta\rightarrow+\infty$. We identify the solution of Eqs. (3.2) that obeys (3.5) and (3.6) as a solitary wave solution if it also satisfies Eq. (3.7). Our approach is therefore to calculate $\partial^2[\mathrm{Im}\tilde{q}(\theta=0)]/\partial\theta^2$ for $0<\beta\ll 1$. We find that it is zero to all orders of the expansion, but will be nonzero only if we go beyond this expansion.

For small third-order dispersion, one can develop a formal asymptotic expansion,

$$\tilde{q}(\theta,\beta)=\tilde{q}^{(0)}(\theta)+\beta\tilde{q}^{(1)}(\theta)+\beta^2\tilde{q}^{(2)}(\theta)+\cdots, \tag{3.8}$$

where higher-order corrections are defined by successive truncation of Eq. (3.2). We also expand the parameters k and ν in powers of β,

$$k=k^{(0)}+\beta k^{(1)}+\beta^2 k^{(2)}+\cdots, \tag{3.9}$$

$$\nu=\nu^{(0)}+\beta\nu^{(1)}+\beta^2\nu^{(2)}+\cdots. \tag{3.10}$$

Higher-order correction terms are chosen to eliminate the secularities in lower orders $\tilde{q}^{(n)}$. The solutions are given as (Appendix B)

$$\tilde{q}^{(n)}=\sum_{m=1}^{n+1}C_m^{(n)}U_m, \tag{3.11}$$

where the U_m's are defined as

$$U_{2m+1}=|\tilde{q}^{(0)}(\theta)|^{2m}\tilde{q}^{(0)}(\theta), \tag{3.12}$$

$$U_{2m+2}=i|\tilde{q}^{(0)}(\theta)|^{2m}\frac{\partial\tilde{q}^{(0)}}{\partial\theta}, \tag{3.13}$$

$$\tilde{q}^{(0)}=A\,\mathrm{sech}(A\theta)\exp(i\nu^{(0)}\theta), \tag{3.14}$$

and the constants $C_m^{(n)}$, $m=1,\ldots,n+1$, are determined by

$$\begin{aligned}&2m(m+1)A^2C_{2m+1}^{(n)}-2(m+1)\nu C_{2m}^{(n)}-(2m-3)(m+1)C_{2m-1}^{(n)}=d_{2m-1},\\&2m(m+1)A^2C_{2m+2}^{(n)}-(m+1)(2m-1)C_{2m}^{(n)}=d_{2m}.\end{aligned} \tag{3.15}$$

The d_m, $m=1,\ldots,n+1$ are calculated from $\tilde{q}^{(m)}$, $m=1,\ldots,n-1$. Since all the U_m's are localized functions (in a coordinate frame moving with the renormalized group delay ν), the series solution (3.8) represents that of a solitary wave.

Moreover, since the U_m's obey the symmetry conditions

$$U_m(\theta)=U_m^*(-\theta) \tag{3.16}$$

for all m's, the series solution obeys the same symmetry condition. Equivalently, Eq. (3.7) is satisfied to all orders of the expansion in Eq. (3.8). It follows that, as $\beta\rightarrow 0$,

$$\mathrm{Im}\frac{\partial^2\tilde{q}(0;\beta)}{\partial\theta^2}\ll\beta^m \text{ for all } m. \tag{3.17}$$

Equation (3.17) set a bound on how well the series (3.8) approximates a solitary wave solution. Although Eq. (3.17) holds for all integer m, we are going to show that the solution of Eq. (3.2) does not satisfy Eq. (3.7), so it is not a solitary wave solution, and the deviation from a solitary wave is exponentially small, i.e., $\approx\exp(-1/\beta)$. This property of the solution cannot be inferred from asymptotic expansion to all polynomial orders. It is in this sense that Eq. (2.9) requires analysis beyond all orders.

Let us discuss in more detail what is meant by going beyond all orders in an asymptotic expansion, and the method to obtain the information. The asymptotic expansion provides an approximate description of $\tilde{q}(\theta;\beta)$ in its own region of validity. However, in the present problem, this description is not accurate enough to provide the information required. It can be shown that, at large θ, Eq. (3.2) contains a mode that grows, a mode that decays, and a mode that is oscillatory. Condition (3.3) asserts that neither of the nondecaying modes should be present in $\tilde{q}(\theta,\beta)$ as $\theta\rightarrow+\infty$. Condition (3.7) can be interpreted as stating that neither of the nondecaying modes is excited at finite θ, so neither can be present as $\theta\rightarrow+\infty$. However, any deviation of $\mathrm{Im}[\partial^2\tilde{q}(0,\beta)/\partial\theta^2]$ from zero, even something exponentially small, excites at least one of the nondecaying modes. This explains why Eq. (3.17) is less restrictive than Eq. (3.7). The latter excludes the existence of the nondecaying modes, while Eq. (3.17) only bounds their initial amplitudes. Therefore it is necessary to go beyond all orders to determine whether the solution of Eq. (3.2) satisfies Eq. (3.7).

In general, it is meaningless to ask for information beyond all orders in an asymptotic expansion, because the series diverges. However, if the series happens to converge somewhere in its region of validity, there and only there can one determine the difference between the sum of the asymptotic series and the function represented by it, i.e., the information beyond all orders. In the present problem, the infinite series in Eq. (3.8) is constructed such that $\mathrm{Re}[\partial\tilde{q}^{(n)}(0)/\partial\theta]=0$ at every order, and we show that $\mathrm{Im}[\partial^2\tilde{q}^{(n)}(0)/\partial\theta^2]=0$ for all n. As a result, both the series,

$$\sum_{n=0}^{\infty}\beta^n\mathrm{Re}\left[\frac{\partial\tilde{q}^{(n)}(0)}{\partial\theta}\right]$$

and

$$\sum_{n=0}^{\infty}\beta^n\mathrm{Im}\left[\frac{\partial^2\tilde{q}^{(n)}(0)}{\partial\theta^2}\right],$$

converge trivially. Thus we are able to calculate the possible infinitesimal corrections to these sums. For other values of θ, the series diverges, and the question is meaningless.

What remains to be done is to devise a recipe to go beyond all orders. The approach is an application of the method of matched asymptotic expansions. The asymptotic series is analytically continued into the neighborhood of one of its singularities off the real θ axis. There, the series breaks down. Terms that had been small often become large, and the effect that had been beyond all orders may become important enough to be noticed. Of course, such a singularity lies outside the region of validity of the asymptotic expansion. In the present case, the series breaks down in the complex-θ plane near $\theta=\pm i\pi/2A$, where Eq. (3.14) becomes singular. We rescale the problem with different dominant terms than those of Eq. (3.2) in the neighborhood of the singularity. After the rescaling, the effects that were beyond all orders in Eq. (3.2) appear at low-order terms in the rescaled problem. We then solve the rescaled problem to some lower order in its expansion, and recover the term that is beyond all orders. The solution with this additional piece is then analytically continued back to the real-θ axis along a path in the complex-θ plane on which the original expansion converges. The imaginary-θ axis is such a path. This solution is further continued out to $\theta=+\infty$ to determine the asymptotic behavior of the solution which is a radiation of frequency $1/2\beta$.

Following the above procedure, we first extend Eq. (3.2) to the complex-θ plane. In addition, we write the equation of the complex quantity $\tilde{q}(\theta)$ as two coupled differential equations,

$$\begin{aligned} -kR-iv\frac{\partial R}{\partial\theta}+\frac{1}{2}\frac{\partial^2 R}{\partial\theta^2}+R^2S&=i\beta\frac{\partial^3 R}{\partial\theta^3}\ , \\ -kS+iv\frac{\partial S}{\partial\theta}+\frac{1}{2}\frac{\partial^2 S}{\partial\theta^2}+S^2R&=-i\beta\frac{\partial^3 S}{\partial\theta^3}\ , \end{aligned} \tag{3.18}$$

where $R=\tilde{q}(\theta)$ and $S=\tilde{q}^*(\theta)$ when θ is real. The boundary conditions are chosen as

$$\begin{aligned} &|R\exp(-i\lambda_2\theta)|,\ |S\exp(-i\lambda_2\theta)|\to D \\ &\qquad\qquad \text{as } \mathrm{Re}(\theta)\to-\infty,\quad \mathrm{Im}(\theta)=0\ , \\ &\mathrm{Re}\left[\frac{\partial R(0)}{\partial\theta}\right]=0\ , \end{aligned} \tag{3.19}$$

where the constant D is determined *a posteriori,* and λ_2 is the solution to the polynomial

$$\beta\lambda^3-\tfrac{1}{2}\lambda^2-v\lambda-k=0\ , \tag{3.20}$$

such that $\mathrm{Im}(\lambda_2)<0$. Equation (3.18) is well-posed (Appendix A), its solution is analytic in θ wherever it is defined, and by definition, $R(\theta)=S^*(\theta)$ on the real-θ axis. Thus, by analytical continuation, we obtain that

$$R^*(\theta,\beta)=S(\theta^*,\beta)\ , \tag{3.21}$$

wherever $R(\theta,\beta)$ and $S(\theta,\beta)$ are defined on the θ plane.

IV. BEYOND ALL ORDERS

In this section we shall examine the nature of the singularity of the series Eq. (3.8). The solution of the zeroth-order equation, $\tilde{q}^{(0)}=A\,\mathrm{sech}(A\theta)$, has simple poles on the complex-θ plane at $\theta_n=\pm i(2n+1)\pi/2A$, where n is a positive integer. Therefore the series solution Eq. (3.8) has an essential singularity at θ_n. We do our analysis around $\theta=i\pi/2A$. Since we want to investigate the behavior of the solution around the singularity, we apply the following scaling transformations to Eq. (3.18):

$$\begin{aligned} A\theta&=i\pi/2+\beta Az\ , \\ R&=\bar{R}/\beta\ , \\ S&=\bar{S}/\beta\ . \end{aligned} \tag{4.1}$$

In addition, we shall take $v^{(0)}=0$ without loss of generality. After substitution of Eq. (4.1) into Eq. (3.18), we obtain for the inner expansion,

$$\begin{aligned} i\frac{\partial^3\bar{R}}{\partial z^3}&=\frac{1}{2}\frac{\partial^2\bar{R}}{\partial z^2}+\bar{R}^2\bar{S}-i\beta v\frac{\partial\bar{R}}{\partial z}-\beta^2k\bar{R}\ , \\ -i\frac{\partial^3\bar{S}}{\partial z^3}&=\frac{1}{2}\frac{\partial^2\bar{S}}{\partial z^2}+\bar{S}^2\bar{R}+i\beta v\frac{\partial\bar{S}}{\partial z}-\beta^2k\bar{S}\ . \end{aligned} \tag{4.2}$$

The boundary conditions for Eq. (4.2) are determined by matching its solution to that of Eq. (3.18), which is given approximately by Eq. (3.8), i.e.,

$$\bar{R},\bar{S}\to-\frac{i}{z}\quad\text{as } \mathrm{Re}(z)\to-\infty\ . \tag{4.3}$$

The scaling in Eq. (4.2) is different from that in Eq. (3.18); the third-order derivative term is now the same order of magnitude as the second-order derivative term. Our goal is to compute $\partial^2\mathrm{Im}\tilde{q}(\theta=0;\beta)/\partial\theta^2$. We shall integrate Eq. (4.2) and Eq. (4.3) from $\mathrm{Re}(z)=-\infty$ to $\mathrm{Re}(z)=0$, along a line on which $\mathrm{Im}(z)=$const. We then continue this solution down along $\mathrm{Re}(z)=0$ [i.e., $\mathrm{Re}(\theta)=0$] to $\theta=0$ in order to calculate $\partial^2\mathrm{Im}\tilde{q}(\theta=0;\beta)/\partial\theta^2$. The imaginary-$\theta$ axis is chosen because along $\mathrm{Re}(\theta)=0$, the series $\partial^2\mathrm{Im}R/\partial\theta^2$ and $\partial^2\mathrm{Im}S/\partial\theta^2$ are trivially convergent. This is due to the fact that Eq. (3.18) is purely real along the imaginary-θ axis. It is also understood that the path of integration is chosen to lie below any singularities of Eq. (4.2) on the complex-z plane. Since only the asymptotic behavior of the solution of Eq. (4.2) and Eq. (4.3) as $z\to\infty$ ($\beta z\to 0$) is required, it can be obtained directly from Eqs. (3.8)–(3.15). Expanding the hyperbolic secant function near its singularity at $\theta=i\pi/2A$ and expressing the results in the variable z, one has

$$A\,\mathrm{sech}(A\theta)=-i\left[\frac{1}{\beta z}-\beta\frac{A^2}{6}z+\beta^3\frac{7A^2}{360}z^3+\cdots\right]. \tag{4.4}$$

Substituting Eq. (4.4) into Eq. (3.8), the solution for Eq. (4.2), as $z\to\infty$, while βz remains small, i.e., $\beta z\ll 0$, is

$$\bar{R}(z)=\frac{-i}{z}-\frac{3}{z^2}+\frac{21i}{z^3}+\cdots+\beta^2\left[i\frac{A^2}{6}z+\cdots\right]+\cdots\,,$$
$$\bar{S}(z)=\frac{-i}{z}+\frac{3}{z^2}+\frac{21i}{z^3}+\cdots+\beta^2\left[i\frac{A^2}{6}z+\cdots\right]+\cdots\,. \tag{4.5}$$

Contrary to Eq. (3.18), Eq. (4.2) is regular in the limit $\beta\to 0$. Thus the asymptotic series in Eq. (4.5) represents a regular perturbation expansion. The infinitesimal correction term that we seek is dropped by the series expansion in Eq. (4.5) to all orders. We shall show that it can be recovered by examining the corrections to a truncated series. Moreover, the leading term (in β) is sufficient for our analysis in this region. Truncation of the series in Eq. (4.5) at order N (Nth order in z, and zeroth order in β) yield, for large $|z|$, an approximate solution of Eq. (4.2), which we denote by $\bar{R}_0^{(N)}(z)$ and $\bar{S}_0^{(N)}(z)$,

$$\bar{R}_0^{(N)}=\frac{-i}{z}-\frac{3}{z^2}+\frac{21i}{z^3}+\cdots+\frac{C_N}{z^N}\,,$$
$$\bar{S}_0^{(N)}=\frac{-i}{z}+\frac{3}{z^2}+\frac{21i}{z^3}+\cdots\ \frac{D_N}{z^N}\,, \tag{4.6}$$

where C_N and D_N are constants. The corrections for each of these truncations are obtained by solving the appropriate linearized equations, i.e., by substituting

$$\begin{bmatrix}\bar{R}_0\\ \bar{S}_0\end{bmatrix}_{\text{inner}}=\begin{bmatrix}\bar{R}_0^{(N)}\\ \bar{S}_0^{(N)}\end{bmatrix}_{\text{inner}}+\begin{bmatrix}\bar{\phi}\\ \bar{\psi}\end{bmatrix}_{\text{inner}}$$

into Eq. (4.2), where $(\bar{\phi},\bar{\psi})^t$ satisfies

$$i\frac{\partial^3\bar{\phi}}{\partial z^3}=\frac{1}{2}\frac{\partial^2\bar{\phi}}{\partial z^2}+2\bar{R}_0^{(N)}\bar{S}_0^{(N)}\bar{\phi}+\bar{R}_0^{(N)2}\bar{\psi}\,,$$
$$-i\frac{\partial^3\bar{\psi}}{\partial z^3}=\frac{1}{2}\frac{\partial^2\bar{\psi}}{\partial z^2}+2\bar{R}_0^{(N)}\bar{S}_0^{(N)}\bar{\psi}+\bar{S}_0^{(N)2}\bar{\phi}\,. \tag{4.7}$$

Equation (4.7) is then solved asymptotically for large z and the solutions, $(\bar{\phi}_j,\bar{\psi}_j)^t$, $j=1,\ldots,6$, can be shown to be

$$\begin{bmatrix}\bar{\phi}_1\\ \bar{\psi}_1\end{bmatrix}=\begin{bmatrix}z^3+\cdots\\ z^3+\cdots\end{bmatrix},$$
$$\begin{bmatrix}\bar{\phi}_2\\ \bar{\psi}_2\end{bmatrix}=\begin{bmatrix}z^2-24iz-3+\cdots\\ z^2-24iz-3+\cdots\end{bmatrix},$$
$$\begin{bmatrix}\bar{\phi}_3\\ \bar{\psi}_3\end{bmatrix}=\begin{bmatrix}1/z-3/z^3+\cdots\\ 1/z-3/z^3+\cdots\end{bmatrix},$$
$$\begin{bmatrix}\bar{\phi}_4\\ \bar{\psi}_4\end{bmatrix}=\begin{bmatrix}1/z^2-6i/z^3-63/z^4+\cdots\\ 1/z^2+6i/z^3-63/z^4+\cdots\end{bmatrix},$$
$$\begin{bmatrix}\bar{\phi}_5\\ \bar{\psi}_5\end{bmatrix}=\exp(iz/2)\begin{bmatrix}-4/z^2+54i/z^3+1302/z^4+\cdots\\ 1-8i/z-64/z^2+\cdots\end{bmatrix},$$
$$\begin{bmatrix}\bar{\phi}_6\\ \bar{\psi}_6\end{bmatrix}=\exp(-iz/2)\begin{bmatrix}1+8i/z-64/z^2+\cdots\\ -4/z^2-54i/z^3+1302/z^4+\cdots\end{bmatrix}. \tag{4.8}$$

Among the six solutions, $(\bar{\phi}_5,\bar{\psi}_5)^t$ and $(\bar{\phi}_6,\bar{\psi}_6)^t$ are oscillatory for $\mathrm{Re}(z)\neq 0$, $\mathrm{Im}(z)=\text{const}$, and the others are polynomials in z. The boundary condition for Eq. (4.2) requires that as $\mathrm{Re}(z)\to-\infty$ along $\mathrm{Im}(z)=\text{const}$, none of the above solutions is present. We recall that this boundary condition simply assures that the solution of Eq. (3.18) in the inner region matches to that in the outer region. The same matching requirement applies as we try to continue this solution along $\mathrm{Re}(z)=0$, and $\mathrm{Im}(z)\to-\infty$ to $\theta=0$. The corrections can again be obtained by linearizing Eq. (4.2) in a sufficiently small neighborhood of any point in this region. We obtain the solutions in Eq. (4.8) again, but the nature of solutions $(\bar{\phi}_5,\bar{\psi}_5)^t$ and $(\bar{\phi}_6,\bar{\psi}_6)^t$ is different. As $\mathrm{Im}(z)\to-\infty$, $(\bar{\phi}_5,\bar{\psi}_5)^t$ becomes exponentially large, but $(\bar{\phi}_6,\bar{\psi}_6)^t$ becomes exponentially small. Therefore the matching condition demands that $(\bar{\phi}_1,\bar{\psi}_1)^t$ to $(\bar{\phi}_5,\bar{\psi}_5)^t$ be absent from Eq. (4.5) but places no restriction on $(\bar{\phi}_6,\bar{\psi}_6)^t$ as $\mathrm{Im}(z)\to-\infty$ on $\mathrm{Re}(z)=0$. Usually, it is meaningless to add the correction $(\bar{\phi}_6^{(N)},\bar{\psi}_6^{(N)})^t$ to $(\bar{R}^{(N)},\bar{S}^{(N)})^t$ as $\mathrm{Im}(z)\to-\infty$, because it is smaller than terms of any order in the expansion. In this problem we shall see that the presence of the infinitesimal $(\bar{\phi}_6,\bar{\psi}_6)^t$ as $\mathrm{Im}(z)\to-\infty$ violates the boundary conditions for solitary wave solutions, i.e., $\mathrm{Im}(\partial^2\tilde{q}(0)/\partial\theta^2)\neq 0$. Adding $(\bar{\phi}_6,\bar{\psi}_6)^t$ to the series, we have to Nth order,

$$\begin{bmatrix}\bar{R}\\ \bar{S}\end{bmatrix}_{\text{inner}}\approx\begin{bmatrix}\bar{R}^{(N)}\\ \bar{S}^{(N)}\end{bmatrix}+\Gamma\exp(-iz/2)\begin{bmatrix}1+8i/z-64/z^2+\cdots\\ -4/z^2-54i/z^3+1302/z^4+\cdots\end{bmatrix}, \tag{4.9}$$

where Γ is a constant to be determined by numerical integration.

In order to calculate $\mathrm{Im}(\partial^2\tilde{q}(0)/\partial\theta^2)$, we match the inner and outer expansions of Eq. (3.18) along $\mathrm{Re}(\theta)=0$. The inner expansion is obtained above in Eq. (4.9). The outer expansion can be calculated as follows. We define a real variable y along the imaginary axis by

$$\theta=iy\,. \tag{4.10}$$

Substitution into Eq. (3.18) gives

$$-\beta\frac{\partial^3 R}{\partial y^3}=-\frac{1}{2}\frac{\partial^2 R}{\partial y^2}+R^2S-v^{(0)}\frac{\partial R}{\partial y}-k^{(0)}R\,,$$
$$\beta\frac{\partial^3 S}{\partial y^3}=-\frac{1}{2}\frac{\partial^2 S}{\partial y^2}+S^2R+v^{(0)}\frac{\partial S}{\partial y}-k^{(0)}S\,. \tag{4.11}$$

Along $\mathrm{Re}(\theta)=0$, the series solution in Eq. (3.8) is purely real, hence it does not provide any information about the imaginary part of $(R,S)^t$. In other words, the imaginary part is trivially convergent as discussed in Sec. III. Let us assume

$$R(y)=R_r(y)+i\phi(y)\ , \tag{4.12}$$

$$S(y)=S_r(y)+i\psi(y)\ , \tag{4.13}$$

where $R_r(y)$, $S_r(y)$, $\phi(y)$, and $\psi(y)$ are real functions of y. The real part, R_r and S_r are given by

$$R_r=S_r=A\sec(Ay)+\cdots\ ,$$

and the imaginary parts $\phi(y)$ and $\psi(y)$ approximately satisfies a linear equation

$$-\beta\frac{\partial^3\phi}{\partial y^3}=-\frac{1}{2}\frac{\partial^2\phi}{\partial y^2}+2R_rS_r\phi+R_r^2\psi-v^{(0)}\frac{\partial\phi}{\partial y}-k^{(0)}\phi\ , \tag{4.14}$$

$$\beta\frac{\partial^3\psi}{\partial y^3}=-\frac{1}{2}\frac{\partial^2\psi}{\partial y^2}+2R_rS_r\psi+S_r^2\phi+v^{(0)}\frac{\partial\psi}{\partial y}-k^{(0)}\psi\ .$$

Equation (4.14) has six solutions, which are given approximately by

$$\begin{bmatrix}\phi(y)\\ \psi(y)\end{bmatrix}_{\text{outer}}=\sum_{i=1}^{6}\alpha_i\mathbf{f}_i(y)\ , \tag{4.15}$$

where α_i are constants and $\mathbf{f}_i(y)$ are given by

$$\begin{aligned}
\mathbf{f}_1&=\begin{bmatrix}1\\-1\end{bmatrix}A\sec(Ay)+\cdots\ ,\\
\mathbf{f}_2&=\begin{bmatrix}1\\1\end{bmatrix}A^2\sec(Ay)\tan(Ay)+\cdots\ ,\\
\mathbf{f}_3&=\begin{bmatrix}1\\-1\end{bmatrix}[Ay\sec(Ay)+\sin(Ay)]+\cdots\ ,\\
\mathbf{f}_4&=\begin{bmatrix}1\\1\end{bmatrix}[3Ay\sec(Ay)\tan(Ay)+\sin(Ay)\tan(Ay)\\
&\qquad+2\sec(Ay)]+\cdots\ ,\\
\mathbf{f}_5&=\begin{bmatrix}1\\0\end{bmatrix}\exp(y/2\beta)+\cdots\ ,\quad \mathbf{f}_6=\begin{bmatrix}0\\1\end{bmatrix}\exp(-y/\beta)+\cdots\ ,
\end{aligned} \tag{4.16}$$

and $k^{(0)}=A^2/2$. The boundary conditions in Eqs. (3.19) and (3.21) on the imaginary-θ axis are given by

$$\text{Re}[R(y)]=\text{Re}[S(-y)]\ ,\ \text{Im}[R(y)]=-\text{Im}[S(-y)]\ ,$$
$$\text{Im}\left[\frac{\partial R}{\partial y}(0)\right]=0\ ,\ \ \text{Im}\left[\frac{\partial S}{\partial y}(0)\right]=0\ . \tag{4.17}$$

In addition, since the phase factor is arbitrary in $(R,S)^t$, we require that R be real at the origin, i.e.,

$$\text{Im}[R(0)]=0\ . \tag{4.18}$$

Substituting Eqs. (4.17) and (4.18) into the solutions in Eq. (4.15), we have, for the outer expansion,

$$\begin{bmatrix}\phi\\ \psi\end{bmatrix}_{\text{outer}}=\alpha\left[-\begin{bmatrix}1\\-1\end{bmatrix}\sec(Ay)-\frac{1}{2A\beta}\begin{bmatrix}1\\1\end{bmatrix}A^2\sec(Ay)\tan(Ay)+\begin{bmatrix}\exp(y/2\beta)\\-\exp(-y/2\beta)\end{bmatrix}\right] \tag{4.19}$$

where α is a constant to be determined by matching with the inner solutions in their regions of common validity [i.e., $\text{Re}(z)=0$, $\text{Im}(z)<0$, $|z|\gg1$, $\beta|z|\ll1$].

Now, we evaluate the inner expansion along the imaginary-z axis. Note that the series solution in Eq. (4.5) is real when

$$z=it\ , \tag{4.20}$$

therefore, from Eq. (4.9) the imaginary part in leading order of $1/z$ is given by

$$\text{Im}\begin{bmatrix}\bar{R}\\ \bar{S}\end{bmatrix}_{\text{inner}}=\text{Im}\Gamma\exp(t/2)\begin{bmatrix}1\\0\end{bmatrix}\ . \tag{4.21}$$

From Eqs. (4.1), (4.10), and (4.20), we have

$$y=\pi/2A+\beta t,\quad t<0\ . \tag{4.22}$$

Using Eq. (4.22), $R=\bar{R}/\beta$, and $S=\bar{S}/\beta$, we obtain from the outer expansion in Eq. (4.19), to leading order in β,

$$\text{Im}\begin{bmatrix}\bar{R}\\ \bar{S}\end{bmatrix}_{\text{outer}}=\alpha\beta\exp(\pi/4A\beta+t/2)\begin{bmatrix}1\\0\end{bmatrix}\ . \tag{4.23}$$

Comparing Eqs. (4.21) and (4.23),

$$\alpha=\frac{\text{Im}\Gamma}{\beta}\exp(-\pi/4A\beta)\ .$$

Substituting this into Eq. (4.19), differentiating and reverting to the original variables gives, as $\beta\to0$,

$$\text{Im}\frac{\partial^2}{\partial\theta^2}\begin{bmatrix}R\\ S\end{bmatrix}=\frac{\text{Im}\Gamma}{\beta}\exp(-\pi/4A\beta)(-A^2+1/4\beta^2)\times\begin{bmatrix}1\\-1\end{bmatrix}\ . \tag{4.24}$$

Hence, to leading order,

$$\text{Im}\frac{\partial^2\bar{q}(0;\beta)}{\partial\theta^2}=\frac{\text{Im}\Gamma}{4\beta^3}\exp(-\pi/4A\beta)\ . \tag{4.25}$$

It is clear from Eq. (4.25) that $\text{Im}\partial^2\bar{q}(0)/\partial\theta^2$ is nonzero if $\text{Im}\Gamma$ is not zero. To calculate $\text{Im}\Gamma$, we do not integrate Eq. (3.2) directly because the perturbation term is singular as $\beta\to0$. Instead, we first reduced Eq. (2.4) into an ordinary differential equation by a traveling-wave assumption,[9] and then integrated it numerically. The derivative

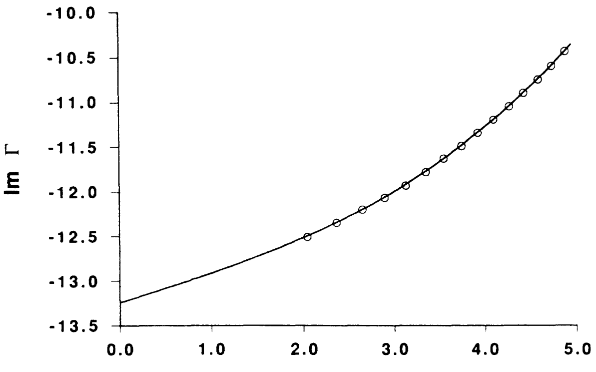

FIG. 4. The constant ImΓ is plotted against $A\beta$. The circles are numerical results and the solid line is a third-order polynomial fit to the data. The extrapolated value of ImΓ at $A\beta=0$ is 13.24.

$\mathrm{Im}\partial^2\bar{q}/\partial\theta^2$ at $\theta=0$ is calculated with the help of Eqs. (2.6)–(2.8). We use a fourth-order Runge-Kutta method and double precision in a CRAY-2 computer for different values of $A\beta$. The range of $A\beta$ is from 0.0204 to 0.05. It is found that the results approximate Eq. (4.25) very well. The constant ImΓ is plotted against $A\beta$ in Fig. 4. The circles are numerical results and the solid line is a third-order polynomial fit to the data. By extrapolating the solid curve to zero, the constant ImΓ as $A\beta\to 0$ is estimated to be 13.24. Thus the series solution Eq. (3.8) does not satisfy the solitary wave condition, although the discrepancy is exponentially small as β approaches zero. Equation (4.25) is an analytical expression for the deviation of the numerical solutions in Ref. 9 from a solitary wave in the limit $\beta\to 0$.

V. RADIATION

In this section we determine the effect of the infinitesimal violation of the solitary wave condition. We show that radiation of frequency $1/2\beta$ is excited. The magnitude of the radiation is exponentially small. This explains the robustness of the fundamental soliton under third-order dispersion perturbation.[15] The method we use is again that of matched asymptotic expansion: the solution at large s is obtained by matching it to that solved near zero. We begin with Eq. (2.9) and substitute

$$q(\xi,s)=\hat{q}(\xi,s)\exp(ik_0\xi) \tag{5.1}$$

in order to remove the explicit phase dependence of $q(\xi,s)$, so that we can assume $\hat{q}(\xi,s)$ satisfies the following boundary conditions:

$$\begin{aligned}&\mathrm{Im}[\hat{q}(s=0)]=0\ ,\\ &\mathrm{Re}\left[\frac{\partial\hat{q}(s=0)}{\partial s}\right]=0\ ,\\ &\mathrm{Im}\left[\frac{\partial^2\hat{q}(s=0)}{\partial s^2}\right]=\frac{\mathrm{Im}\Gamma}{4\beta^3}\exp(-\pi/4A\beta)\ .\end{aligned} \tag{5.2}$$

We obtain

$$i\frac{\partial\hat{q}}{\partial\xi}=k_0\hat{q}+\frac{1}{2}\frac{\partial^2\hat{q}}{\partial s^2}+|\hat{q}|^2\hat{q}+i\beta\frac{\partial^3\hat{q}}{\partial s^3}\ . \tag{5.3}$$

We have shown in the preceding section that the solutions to Eqs. (5.2) and (5.3) deviate from true solitons by an exponentially small term. Thus we can assume that

$$\hat{q}(\xi,s)=\hat{q}_s(\xi,s)+\hat{q}_r(\xi,s)\ , \tag{5.4}$$

where $\hat{q}_s(\xi,s)$ is the series solution to Eq. (5.3), and $\hat{q}_r(\xi,s)$ is a radiation component (i.e., it does not decay at infinity). The principal result of the preceding section is that $\mathrm{Im}[\partial^2\hat{q}(s=0)/\partial s^2]\neq 0$, but $\hat{q}(\xi,s)$ approaches $\hat{q}_s(\xi,s)$ to all orders as s approaches positive infinity. Hence the radiation component $\hat{q}_r(\xi,s)$ must be infinitesimally small and it satisfies approximately

$$i\frac{\partial\hat{q}_r}{\partial\xi}-k_0\hat{q}_r+\frac{1}{2}\frac{\partial^2\hat{q}_r}{\partial s^2}+2|\hat{q}_s|^2\hat{q}_r+\hat{q}_s^2\hat{q}_r^*=i\beta\frac{\partial^3\hat{q}_r}{\partial s^3}\ . \tag{5.5}$$

The boundary conditions in Eq. (5.2) become

$$\begin{aligned}&\mathrm{Im}[\hat{q}_r(s=0)]=0\ ,\\ &\mathrm{Re}\left[\frac{\partial\hat{q}_r(s=0)}{\partial s}\right]=0\ ,\\ &\mathrm{Im}\left[\frac{\partial^2\hat{q}_r(s=0)}{\partial s^2}\right]=\frac{\mathrm{Im}\Gamma}{4\beta^3}\exp(-\pi/4A\beta)\ .\end{aligned} \tag{5.6}$$

Next, we try to solve Eq. (5.5) approximately by assuming

$$\hat{q}_r(\xi,s)=\phi(k,s)\exp(ik\xi)+\psi^*(k,s)\exp(-ik\xi)\ . \tag{5.7}$$

Equation (5.5) is transformed into a set of coupled equations given by

$$\begin{aligned}&-(k+k_0)\phi+\frac{1}{2}\frac{\partial^2\phi}{\partial s^2}+2|\hat{q}_s|^2\phi+\hat{q}_s^2\psi=i\beta\frac{\partial^3\phi}{\partial s^3}\ ,\\ &(k-k_0)\psi+\frac{1}{2}\frac{\partial^2\psi}{\partial s^2}+2|\hat{q}_s|^2\psi+\hat{q}_s^{*2}\phi=-i\beta\frac{\partial^3\psi}{\partial s^3}\ .\end{aligned} \tag{5.8}$$

Our aim is to obtain an expression for $\hat{q}_r(\xi,s)$ at large s so that we can estimate the rate of loss of the main peak through the radiation channel. The strategy here is again that of matched asymptotic expansions. The s axis is divided into three regions, the inner region (I) near $s=0$, the intermediate region (II), and the outer region (III) for $s\to\infty$. Equation (5.8) is solved in each region and the solutions are matched at their respective common region of validity.

In region I ($s\approx 0$), which has a role similar to a boundary layer, we apply the following stretching transformation:

$$t=s/\beta\ ,$$

to Eq. (5.8). We obtain

$$\begin{aligned}&-\beta^2(k+k_0)\phi+\frac{1}{2}\frac{\partial^2\phi}{\partial t^2}+2\beta^2|\hat{q}_s|^2\phi+\beta^2\hat{q}_s^2\psi=i\frac{\partial^3\phi}{\partial t^3}\ ,\\ &\beta^2(k-k_0)\psi+\frac{1}{2}\frac{\partial^2\psi}{\partial t^2}+2\beta^2|\hat{q}_s|^2\psi+\beta^2\hat{q}_s^{*2}\phi=-i\frac{\partial^3\psi}{\partial t^3}\ .\end{aligned} \tag{5.9}$$

At order zero, the set of linear equations is decoupled. For $k\neq 0$, solving Eq. (5.9), and reverting to the s variable, one obtains

$$\hat{q}_r(\xi,s)=\{C_1\exp(-is/2\beta)+C_2\exp[\sqrt{2(k+k_0)s}\,]+C_3\exp[-\sqrt{2(k+k_0)s}\,]\}\exp(ik\xi)$$
$$+\{D_1\exp(is/2\beta)+D_2\exp[\sqrt{2(k_0-k)s}\,]+D_3\exp[-\sqrt{2(k_0-k)s}\,]\}\exp(-ik\xi)\ , \tag{5.10}$$

where the C_i, D_i, $i=1,3$ are constants. The boundary conditions in Eq. (5.6) should be satisfied by all k modes, but the last equations in Eq. (5.6) show that only the $k=0$ mode is driven by the main peak. Thus one only need to consider this mode in the subsequent analysis. In this case, we have $\psi=\phi^*$. Repeating the above calculations for $k=0$, and substituting into Eq. (5.6), we obtain

$$\hat{q}_r(s)\approx C_1\left[\exp(-is/2\beta)-i\frac{1}{2\beta}s-1\right]\ ,$$
$$C_1=-\frac{\mathrm{Im}\Gamma}{\beta}\exp(-\pi/4A\beta)\ . \tag{5.11}$$

In region II, we solve Eq. (5.8) with $k=0$. A similar problem was worked out in the preceding section; hence, we have to leading order in β, the following six independent solutions:

$$\begin{bmatrix}\phi\\ \psi\end{bmatrix}=\begin{bmatrix}1\\ -1\end{bmatrix}A\,\mathrm{sech}(As)+\cdots,\quad \begin{bmatrix}1\\ 1\end{bmatrix}A^2\mathrm{sech}(As)\tanh(As)+\cdots\ ,$$
$$\begin{bmatrix}1\\ -1\end{bmatrix}[As\,\mathrm{sech}(As)+\sinh(As)]+\cdots,\quad \begin{bmatrix}1\\ 1\end{bmatrix}[3As\tanh(As)+\sinh^2(As)-2]\mathrm{sech}(As)+\cdots\ , \tag{5.12}$$
$$\begin{bmatrix}1\\ 0\end{bmatrix}\exp(-is/2\beta)+\cdots,\quad \begin{bmatrix}0\\ 1\end{bmatrix}\exp(is/2\beta)+\cdots\ ,$$

where $k_0=A^2/2$. To first order, the solution of Eq. (5.5) is given by

$$\hat{q}_r(s)=E_1\exp(-is/2\beta)+E_2\mathrm{sech}(As)+E_3\mathrm{sech}(As)\tanh(As)$$
$$+E_4[As\,\mathrm{sech}(As)+\sinh(As)]+E_5[3As\,\mathrm{sech}(As)\tanh(As)+\sinh(As)\tanh(As)-2\,\mathrm{sech}(As)]\ , \tag{5.13}$$

where the E_j's, $j=1,\ldots,5$, are constants. The matching to Eq. (5.11) is done by expanding Eq. (5.13) for small s, and one has

$$E_1=C_1\ ,$$
$$E_3+2E_4=-iC_1/2A\beta\ , \tag{5.14}$$
$$E_2-2E_5=-C_1\ .$$

In anticipation of the matching with region III, we also expand Eq. (5.13) for large positive s

$$\hat{q}_r(s)\to E_1\exp(-is/2\beta)+2(E_2+E_3-2E_5)\exp(-As)+2(2E_4+3E_5)As\exp(-As)+\tfrac{1}{2}(2E_4+E_5)\exp(As)\ .$$

Finally, in region III, when s approaches positive infinity, we can neglect the potential terms in Eq. (5.8). Similar to region I, the equations becomes decoupled and can be easily solved. Hence, we find the radiative components $\hat{q}_r(s)$ to be

$$\hat{q}_r(s)=F_1\exp(-is/2\beta)+F_2\exp(As)+F_3\exp(-As)\ . \tag{5.15}$$

Again, the F_i's, $i=1,\ldots,3$, are constants. Since $\hat{q}_s(\xi,s)$ approximates $\hat{q}(\xi,s)$ to all orders as $s\to+\infty$, $F_2=0$. Matching with the solutions in region II, and using Eqs. (5.11) and (5.14), we obtain

$$F_1=E_1=C_1\ ,$$
$$F_3=-2(1+i/2A\beta)C_1\ ,$$
$$E_2=-C_1,\quad E_3=-iC_1/2A\beta\ , \tag{5.16}$$
$$E_4=E_5=0\ .$$

We thus find that at large positive s, the radiation component is

$$q_r(\xi,s)=-\frac{\mathrm{Im}\Gamma}{\beta}\exp(-\pi/4A\beta)\exp(-is/2\beta+ik_0\xi)\ . \tag{5.17}$$

This radiation has a frequency given by $1/2\beta$, which agrees with the resonant frequency observed numerically in Ref. 15. The group delay of this radiation is $1/4\beta$. The direction of propagation of the radiation is determined by the sign of β.

Equation (5.18) represents the loss of the main peak by channeling through this frequency mode. We estimate the radiation rate by calculating the rate of change in the quantity,

$$I_1=\int_{-\infty}^{\infty}|q|^2ds\ . \tag{5.18}$$

Recall that $q(\xi,s)$ approaches zero at $-\infty$ but not at

$+\infty$, hence we define

$$\frac{\partial I_1}{\partial \xi} \equiv \frac{\partial}{\partial \xi} \lim_{L\to+\infty} \int_{-\infty}^{L} |q|^2 ds \ . \tag{5.19}$$

Using Eq. (2.9), (5.4), and (5.18) we have

$$\frac{\partial I_1}{\partial \xi} = -\frac{(\mathrm{Im}\Gamma)^2}{4\beta^3} \exp(-\pi/2A\beta) \ . \tag{5.20}$$

The "solitary" wave packet of Eq. (2.4) is losing its power at a rate given by Eq. (5.21). For example, at $A\beta=0.04$, the radiation rate is

$$\sqrt{\Omega_0}\frac{\partial I_1}{\partial \xi} \approx 10^{-11} \ .$$

Since Ω_0 is of order unity, the radiation rate is very small.

VI. SUMMARY AND DISCUSSIONS

In this paper nonlinear pulse propagation at the zero group-dispersion wavelength is studied analytically. Pulses launched at the zero dispersion point are known to exhibit spectral splitting. Part of the initial pulse is shifted up into the normal dispersion regime, while the other is shifted down to the anomalous dispersion regime. It is found that the downshifted portion can be described by the nonlinear Schrödinger equation with third-order dispersion as small perturbation. We obtain a series solution to the perturbed equation in terms of the small parameter. The series satisfies the solitary wave condition to all orders. However, using a perturbation method that allows us to recover correction terms that are beyond all orders, we show that an infinitesimally small radiation is excited by the third-order dispersion. As a result, the true solution is not stationary although its decay rate due to this radiation is negligible. The calculated deviations from solitary wave behavior agree with the tolerance used in the numerical solutions reported in Ref. 9. Moreover, the frequency of the radiation $(1/2\beta)$ is consistent with that observed from previous numerical studies.[15]

The above analysis can also be considered as an analytical study of the effect of perturbation on the solitons of the nonlinear Schrödinger equation. The results indicate that the solitons are destroyed (albeit very slowly) by the small third-order dispersion term even though the perturbed equation itself is Hamiltonian. Moreover, it is obvious that order-by-order perturbative studies are not sufficient to determine the true nature of the solution. This is because the perturbation effect is singular, i.e., it does not remain small throughout the domain of interest. In fact, its strength approximates that of the second-order dispersion when $\omega \approx 1/2\beta$, and dominates at higher frequency. This reversal in ordering at $\omega \approx 1/2\beta$ manifests itself as radiations at that frequency. The effect of this radiation on fundamental solitons in optical fibers is negligible when compared with other non-Hamiltonian perturbations such as attenuations, and Raman self-frequency shift. However, the perturbation readily breaks up the breather solutions (bound state of solitons) into their constituent solitons as observed in Ref. 15.

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APPENDIX A: WELL-POSEDNESS

In this appendix we show that the following equation define (R,S) uniquely for real, nonzero β:

$$\begin{aligned} &\beta R''' + iR''/2 + \nu R' - ikR + iR^2 S = 0 \ , \\ &\beta S''' - iS''/2 + \nu S' + ikS - iS^2 R = 0 \ , \end{aligned} \tag{A1}$$

where the prime denotes derivatives with respect to the variable θ. The boundary conditions are chosen as

$$|R \exp(-i\lambda_2\theta)|, \ |S \exp(-i\lambda_2\theta)| \to D \quad \text{as } \mathrm{Re}(\theta)\to-\infty, \ \mathrm{Im}(\theta)=0 \ , \tag{A2}$$

$$\mathrm{Re}\left[\frac{\partial R}{\partial \theta}\right] = 0 \ ,$$

where the constant D is determined *a posteriori*, and λ_2 is a solution to the polynomial

$$\beta\lambda^3 - \tfrac{1}{2}\lambda^2 - \nu\lambda - k = 0 \ , \tag{A3}$$

and $\mathrm{Im}\lambda_2 < 0$. In the following, we solve Eqs. (A1) and (A2) on the real axis. We then analytically continue the solution to the complex-θ plane. To solve the equations, we construct a Green's function for the linear operator in Eq. (A1). Using this Green's function, Eq. (A1) is converted to a nonlinear integral equation. We then use the method of successive approximations to show that the integral equation has a unique solution.

On the real-θ axis, we consider the following equations:

$$\begin{aligned} &\beta G''' + iG''/2 + \nu G' - ikG = \delta(\theta-\theta') \ , \\ &\beta H''' - iH''/2 + \nu H' + ikH = \delta(\theta-\theta') \ , \end{aligned} \tag{A4}$$

where $G(\theta,\theta')$ and $H(\theta,\theta')$ are the Green's functions. The boundary conditions are

$$G(\theta,\theta') = H(\theta,\theta') = 0 \quad \text{if } \mathrm{Re}(\theta-\theta') \le 0 \ . \tag{A5}$$

Notice that Eqs. (A4) are decoupled and linear. To solve Eqs. (A4), we assume that λ_1, λ_2, and λ_3 are the roots of the polynomial in Eq. (A3), with λ_1 real, and $\lambda_2 = \lambda_3^* = a - ib$ $(b>0)$. Equation (A3) has complex roots if

$$\left[\tfrac{1}{18} + \beta(\nu + 6\beta k)\right]^2 > \tfrac{16}{3}\left(\tfrac{1}{12} + \beta\nu\right)^3 \ . \tag{A6}$$

For small β, Eq. (A6) reduces to

$$k > \tfrac{1}{2}\nu^2 + O(\beta) \quad \text{or } k < -O(\beta^{-2}) \ . \tag{A7}$$

It can then be shown that

$$G(\theta,\theta')=\left[\frac{\exp[-i\lambda_1(\theta-\theta')]}{\beta(\lambda_2-\lambda_1)(\lambda_1-\lambda_3)}+\frac{\exp[-i\lambda_2(\theta-\theta')]}{\beta(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)}+\frac{\exp[-i\lambda_3(\theta-\theta')]}{\beta(\lambda_1-\lambda_3)(\lambda_3-\lambda_2)}\right]\Theta(\theta-\theta')\ ,$$

$$H(\theta,\theta')=\left[\frac{\exp[-i\bar\lambda_1(\theta-\theta')]}{\beta(\bar\lambda_2-\bar\lambda_1)(\bar\lambda_1-\bar\lambda_3)}+\frac{\exp[-i\bar\lambda_2(\theta-\theta')]}{\beta(\bar\lambda_2-\bar\lambda_1)(\bar\lambda_3-\bar\lambda_2)}+\frac{\exp[-i\bar\lambda_3(\theta-\theta')]}{\beta(\bar\lambda_1-\bar\lambda_3)(\bar\lambda_3-\bar\lambda_2)}\right]\Theta(\theta-\theta')\ , \tag{A8}$$

where the $\bar\lambda_j$'s, $j=1,\ldots,3$ are solutions to the polynomial

$$\beta\bar\lambda^3+\tfrac{1}{2}\bar\lambda^2-\nu\bar\lambda+k=0\ , \tag{A9}$$

and they are defined similar to the λ_i's. The function $\Theta(x)$ is the Heaviside function defined as

$$\Theta(x)=\begin{cases}1, & x>0\\ 0, & x\le 0.\end{cases} \tag{A10}$$

Using Eqs. (A8), one can turn Eqs. (A1) into a set of integral equations

$$\begin{aligned}R(\theta)&=D\exp(i\lambda_2\theta)-i\int_{-\infty}^{\theta}d\theta' G(\theta,\theta')R^2(\theta')S(\theta')\ ,\\ S(\theta)&=D\exp(i\bar\lambda_2\theta)+i\int_{-\infty}^{\theta}d\theta' H(\theta,\theta')S^2(\theta')R(\theta')\ .\end{aligned} \tag{A11}$$

We postulate the following sequence of successive approximations:

$$\begin{aligned}&R_0(\theta)=D\exp(i\lambda_2\theta),\quad S_0(\theta)=D\exp(i\bar\lambda_2\theta)\ ,\\ &R_{n+1}(\theta)=R_0-i\int_{-\infty}^{\theta}d\theta' G(\theta,\theta')R_n^2(\theta')S_n(\theta')\ ,\\ &S_{n+1}(\theta)=S_0+i\int_{-\infty}^{\theta}d\theta' H(\theta,\theta')S_n^2(\theta')R_n(\theta')\ .\end{aligned} \tag{A12}$$

If the mapping in Eqs. (A12) has a fixed point, then it satisfies Eqs. (A11). One can verify by direct differentiation that it satisfies Eqs. (A1).

The next step is to show that the sequence of functions defined in Eqs. (A12) is uniformly bounded, and analytic if $\theta<X$, for some X. We prove that by using a sequence $\{\zeta_n(\theta)\}$ defined as

$$\begin{aligned}&\zeta_0=D\exp(b\theta)>0\ ,\\ &\zeta_{n+1}=\zeta_0+\zeta_n^3/\sigma\ ,\end{aligned} \tag{A13}$$

where $\sigma=\beta b^2|\lambda_1-\lambda_2|$. We want to show that

$$R_n(\theta),S_n(\theta)\le\zeta_n(\theta)\quad\text{for all }\theta \tag{A14}$$

and then deduce the analyticity of $\{R_n\},\{S_n\}$ from that of $\{\zeta_n\}$. The relevant properties of $\{\zeta_n\}$ we will use are as follows.

(1) $\{\zeta_n\}$ is a bounded nondecreasing function of n.
(2) $\zeta_\infty(\theta)\le 3\zeta_0(\theta)/2$ if $\theta<X(D;\beta)$.
(3) $\partial\zeta_n/\partial\theta\ge b\zeta_n$.

By construction, we have

$$|R_0|,|S_0|<\zeta_0\ . \tag{A15}$$

Furthermore, we note the following:

$$\begin{aligned}&|\lambda_1-\lambda_2|=|\lambda_1-\lambda_3|\ge b\ ,\\ &|\lambda_2-\lambda_3|=2b\ ,\\ &|\exp[-i\lambda_2(\theta-\theta')]|=|\zeta_0(\theta)/\zeta_0(\theta')|\ ,\\ &\qquad\le\frac{3\zeta_n(\theta)}{2\zeta_n(\theta')}\ .\end{aligned} \tag{A16}$$

Using Eqs. (A16), one can establish that

$$\begin{aligned}&\left|i\frac{\exp[-i\lambda_1(\theta-\theta')]}{\beta(\lambda_2-\lambda_1)(\lambda_1-\lambda_3)}\right|\le\frac{1}{\beta|\lambda_1-\lambda_2|b}\ ,\\ &\left|i\frac{\exp[-i\lambda_2(\theta-\theta')]}{\beta(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)}\right|\le\frac{1}{2\beta|\lambda_1-\lambda_2|b}\frac{3\zeta_n(\theta)}{2\zeta_n(\theta')}\ .\\ &\left|i\frac{\exp[-i\lambda_3(\theta-\theta')]}{\beta(\lambda_1-\lambda_3)(\lambda_3-\lambda_2)}\right|\le\frac{1}{2\beta|\lambda_1-\lambda_2|b}\frac{3\zeta_n(\theta')}{2\zeta_n(\theta)}\ .\end{aligned} \tag{A17}$$

Notice that in the limit $\beta\to0$, $\lambda_1\to1/2\beta$, $a\to-\nu$, $b\to(\nu^2-2k)^{1/2}$, and $\beta|\lambda_1-\lambda_2|\to\frac{1}{2}$. The coefficients in Eqs. (A17) remain finite when $\beta\to0$. We then assume that $|R_n|,|S_n|\le\zeta_n$, hence

$$\begin{aligned}|R_{n+1}|&\le\zeta_0+\int_{-\infty}^{\theta}|d\theta'|\,|G(\theta,\theta')|\,|R_n|^2|S_n|\\ &\le\zeta_0+\frac{1}{\beta b^2|\lambda_1-\lambda_2|}\int_0^{\zeta_n}|d\zeta(\theta')|\left[1+\frac{3\zeta_n(\theta)}{4\zeta_n(\theta')}+\frac{3\zeta_n(\theta')}{4\zeta_n(\theta)}\right]\zeta_n^2(\theta')\\ &=\zeta_0+\frac{43}{48\beta b^2|\lambda_1-\lambda_2|}\zeta_n^3(\theta)\\ &\le\zeta_0+\frac{1}{\beta b^2|\lambda_1-\lambda_2|}\zeta_n^3(\theta)=\zeta_{n+1}(\theta)\ .\end{aligned} \tag{A18}$$

Similarly, $S_{n+1}(\theta)\le\zeta_{n+1}(\theta)$. By induction, we have shown that Eqs. (A12) define a set of uniformly bounded sequence of functions. Analyticity can also be established.

We now prove that the sequence of functions defined by Eqs. (A12) converges to a fixed point by showing that Eqs. (A12) define a contraction map. We define a norm for the functions by

$$\|R_n(\theta)-R_0(\theta)\| \equiv \int_{-\infty}^{\theta} d\sigma |R_n(\sigma)-R_0(\sigma)| \exp(-\delta b \,\mathrm{Re}\sigma) , \tag{A19}$$

where δ is a parameter to be determined. Since

$$|R_n(\sigma)-R_0(\sigma)| \le C_1 \zeta_{n-1}^3 \le C_2 \zeta_0^3 \le C_3 \exp[3b\,\mathrm{Re}(\sigma)] , \tag{A20}$$

where the C_i's, $i=1,\ldots,3$ are constants, Eq. (A19) is well defined if $\delta<3$, as $\sigma\to-\infty$. From Eq. (A19), we have

$$\begin{aligned}\|R_{n+1}(\theta)-R_n(\theta)\| &\le \int_{-\infty}^{\theta} d\sigma \exp(-\delta b\,\mathrm{Re}\sigma) \int_{-\infty}^{\sigma} d\theta' |G(\sigma,\theta')| |R_n^2 S_n(\theta')-R_{n-1}^2 S_{n-1}(\theta')| \\ &= \int_{-\infty}^{\theta} d\theta' \int_{\theta'}^{\theta} d\sigma \exp(-\delta b\,\mathrm{Re}\sigma) |G(\sigma,\theta')| |R_n^2 S_n(\theta')-R_{n-1}^2 S_{n-1}(\theta')| .\end{aligned} \tag{A21}$$

The last relation is obtained by interchanging the order of integration. We note that

$$\begin{aligned}|R_n^2 S_n(\theta')-R_{n-1}^2 S_{n-1}(\theta')| &= |(R_n-R_{n-1})R_n S_n+(R_n-R_{n-1})R_{n-1}S_n+(S_n-S_{n-1})R_n^2| \\ &\le 2\zeta_n^2|R_n-R_{n-1}|+\zeta_n^2|S_n-S_{n-1}| \\ &\le (\tfrac{12}{11})^2\zeta_0^2(2|R_n-R_{n-1}|+|S_n-S_{n-1}|) \\ &\le \frac{12^2\beta b^2|\lambda_1-\lambda_2|}{11^2\times 27}(2|R_n-R_{n-1}|+|S_n-S_{n-1}|) .\end{aligned} \tag{A22}$$

Moreover, one can show that

$$|G(\sigma,\theta')| \le \frac{1}{\beta b^2|\lambda_1-\lambda_2|}\{1+\exp[b(\sigma-\theta')]/2+\exp[-b(\sigma-\theta')]/2\} . \tag{A23}$$

Substituting Eqs. (A22) and (A23) into Eq. (A21), one obtains

$$\begin{aligned}&\|R_{n+1}-R_n\| \le \int_{-\infty}^{\theta} d\theta' J(\theta')(2|R_n-R_{n-1}|+|S_n-S_{n-1}|) , \\ &J(\theta') = \frac{12^2 b}{11^2\times 27}\int_{\theta'}^{\theta} d\sigma \{1+\exp[b(\sigma-\theta')]/2+\exp[-b(\sigma-\theta')]/2\}\exp(-\delta b\sigma) .\end{aligned} \tag{A24}$$

It can be shown that, for $\delta>1$,

$$\begin{aligned}J(\theta') &\le \frac{12^2}{11^2\times 27}\left[\frac{1}{\delta}+\frac{1}{2(\delta-1)}+\frac{1}{2(\delta+1)}\right]\exp(-\delta b\theta') \\ &\equiv r(\delta)\exp(-\delta b\theta') .\end{aligned} \tag{A25}$$

Hence we have

$$\|R_{n+1}-R_n\| \le r(\delta)(2\|R_n-R_{n-1}\|+\|S_n-S_{n-1}\|) , \tag{A26}$$

and similarly, for the sequences $S_n(\theta)$,

$$\|S_{n+1}-S_n\| \le r(\delta)(2\|S_n-S_{n-1}\|+\|R_n-R_{n-1}\|) . \tag{A27}$$

The criterion for a contraction map is that both the eigenvalues of the matrix in Eqs. (A26) and (A27) are less than 1. In terms of $r(\delta)$, they are given by $r(\delta)<1$, and $r(\delta)<\frac{1}{3}$. Solving for δ numerically, we found that if

$$1.13<\delta<3 , \tag{A28}$$

then Eq. (A12) is indeed a contraction map. Hence they converge to $R(\theta)$ and $S(\theta)$. The uniqueness property follows from the contraction property.

APPENDIX B: ASYMPTOTIC SERIES

In this section we establish an asymptotic series solution to Eq. (2.9) and show that it satisfies the solitary wave conditions to all order. We restate the problem here for convenience in subsequent discussions. The equation of motion is

$$i\frac{\partial q}{\partial \xi}+\frac{1}{2}\frac{\partial^2 q}{\partial s^2}+|q|^2 q = i\beta\frac{\partial^3 q}{\partial s^3} , \tag{B1}$$

where $\beta \ll 1$. One can develop a formal asymptotic expansion,

$$q(\xi,s;\beta)=q^{(0)}(\xi,s)+\beta q^{(1)}(\xi,s)+\beta^2 q^{(2)}(\xi,s)+\cdots , \tag{B2}$$

where the higher-order corrections are determined by successive truncation of Eq. (B1). For example, at order zero, we have the nonlinear Schrödinger equation,

$$i\frac{\partial q^{(0)}}{\partial \xi}+\frac{1}{2}\frac{\partial^2 q^{(0)}}{\partial s^2}+|q^{(0)}|^2 q^{(0)}=0 . \tag{B3}$$

The fundamental soliton solution is given by

$$q^{(0)}=A\,\mathrm{sech}(A\theta)\exp(i\nu\theta+ik\xi) , \tag{B4}$$

where $\theta=s-\nu\xi$, and $k=(A^2+\nu^2)/2$. At order β, we have

$$L(q^{(1)},(q^{(1)})^*)\equiv i\frac{\partial q^{(1)}}{\partial\xi}+\frac{1}{2}\frac{\partial^2 q^{(1)}}{\partial s^2}+2|(q^{(0)})^2|q^{(1)}+(q^{(0)})^2(q^{(1)})^* = i\frac{\partial^2 q^{(0)}}{\partial s^{(3)}} . \tag{B5}$$

The operator $L(q^{(1)},(q^{(1)})^*)$ is the linearized nonlinear Schrödinger equation. In Eq. (B5), if the inhomogeneous term $i\partial^3 q^{(0)}/\partial s^3$ contains a term $i\phi^{(1)}$ such that $\phi^{(1)}$ is a solution of the homogeneous equation, $L(\phi^{(1)},(\phi^{(1)})^*)=0$, then $q^{(1)}$ has a secular term. That is, $q^{(1)}=\xi\phi^{(1)}+u^{(1)}$, where $u^{(1)}$ is given by $L(u^{(1)},(u^{(1)})^*)=i\partial^3 q^{(0)}/\partial s^3-i\phi^{(1)}$. The term $i\phi^{(1)}$ is usually called the resonant term. For the fundamental soliton, the resonant terms are given by $q^{(0)}$ and $i\partial q^{(0)}/\partial s$, which are due to the shifts of the frequency and the group delay of the soliton, respectively. In order to avoid the secular solution, one should eliminate the resonant terms in the inhomogeneous part of Eq. (B5) by renormalizing the frequency and the group delay of the soliton. We use the following procedure to eliminate the resonant terms of Eq. (B1).

We first reduce the partial differential equation to an ordinary differential equation by the following assumption:

$$q(\xi,s)=\tilde{q}(\theta)\exp(ik\xi) . \tag{B6}$$

Equation (B1) is then transformed to

$$-k\tilde{q}-i\nu\frac{\partial\tilde{q}}{\partial\theta}+\frac{1}{2}\frac{\partial^2\tilde{q}}{\partial\theta^2}+|\tilde{q}|^2\tilde{q}=i\beta\frac{\partial^3\tilde{q}}{\partial\theta^3} . \tag{B7}$$

Besides expanding $\tilde{q}(\theta)$ as in Eq. (B2), we also expand the frequency k and group delay ν in terms of β,

$$k=k^{(0)}+\beta k^{(1)}+\beta^2 k^{(2)}+\cdots ,\quad \nu=\nu^{(0)}+\beta\nu^{(1)}+\beta^2\nu^{(2)}+\cdots , \tag{B8}$$

where the higher-order correction terms are chosen to eliminate the secularities in $\tilde{q}^{(n)}$. At order zero, we have

$$-k^{(0)}\tilde{q}^{(0)}-i\nu^{(0)}\frac{\partial\tilde{q}^{(0)}}{\partial\theta}+\frac{1}{2}\frac{\partial^2\tilde{q}^{(0)}}{\partial\theta^2}+|\tilde{q}^{(0)}|^2\tilde{q}^{(0)}=0 , \tag{B9}$$

the reduced nonlinear Schrödinger equation. The fundamental soliton is now written as

$$\tilde{q}^{(0)}(\theta)=A\,\mathrm{sech}(A\theta)\exp(i\nu^{(0)}\theta) , \tag{B10}$$

where $k^{(0)}=[A^2+(\nu^{(0)})^2]/2$. So far we have only restated previous results. At order β, we obtain

$$-k^{(0)}\tilde{q}^{(1)}-i\nu^{(0)}\frac{\partial\tilde{q}^{(1)}}{\partial\theta}+\frac{1}{2}\frac{\partial^2\tilde{q}^{(1)}}{\partial\theta^2}+2|\tilde{q}^{(0)}|^2\tilde{q}^{(1)}+(\tilde{q}^{(0)})^2(\tilde{q}^{(1)})^*=i\frac{\partial^3\tilde{q}^{(0)}}{\partial\theta^3}+k^{(1)}\tilde{q}^{(0)}+i\nu^{(1)}\frac{\partial\tilde{q}^{(0)}}{\partial\theta} . \tag{B11}$$

The term $i\partial^3\tilde{q}^{(0)}/\partial\theta^3$ can be shown to be

$$i\frac{\partial^3\tilde{q}^{(0)}}{\partial\theta^3}=-2\nu^{(0)}[A^2+(\nu^{(0)})^2]\tilde{q}^{(0)}+i[A^2-3(\nu^{(0)})^2]\frac{\partial\tilde{q}^{(0)}}{\partial\theta}-6i|\tilde{q}^{(0)}|^2\frac{\partial\tilde{q}^{(0)}}{\partial\theta} . \tag{B12}$$

To eliminate the resonant terms in Eq. (B12), $k^{(1)}$ and $\nu^{(1)}$ on the right-hand side of Eq. (B11) are chosen to be

$$k^{(1)}=2\nu^{(0)}[A^2+(\nu^{(0)})^2] ,\quad \nu^{(1)}=-[A^2-3(\nu^{(0)})^2] . \tag{B13}$$

Thus the frequency and group delay of the soliton are renormalized to first order in β. The deformation in shape is obtained by solving Eq. (B11), which gives

$$\tilde{q}^{(1)}(\theta)=6\nu^{(0)}\tilde{q}^{(0)}+3i\frac{\partial\tilde{q}^{(0)}}{\partial\theta} . \tag{B14}$$

If, for simplicity, one assumes $\nu^{(0)}=0$, then one obtains $k^{(1)}=0$, $\nu^{(1)}=-A^2$, and $\tilde{q}^{(1)}(\theta)=3i\partial\tilde{q}^{(0)}/\partial\theta$. The procedure can be repeated to any order. For example at order β^2, we have

$$k^{(2)}=0 ,\quad \nu^{(2)}=0 ,\quad \tilde{q}^{(2)}(\theta)=21\tilde{q}^{(0)}-39A^2|\tilde{q}^{(0)}|^2\tilde{q}^{(0)}/2 . \tag{B15}$$

In general, at order β^n, we have

$$\mathcal{L}(\tilde{q}^{(n)},(\tilde{q}^{(n)})^*)=\mathcal{R}^{(n)}(\tilde{q}^{(n-1)},\ldots,(\tilde{q}^{(n-1)})^*,\ldots) , \tag{B16}$$

where the operators $\mathcal{L}$ and $\mathcal{R}^{(n)}$ are defined as

$$\mathcal{L}(\tilde{q}^{(n)},(\tilde{q}^{(n)})^*)\equiv-k^{(0)}\tilde{q}^{(n)}-i\nu^{(0)}\frac{\partial\tilde{q}^{(n)}}{\partial\theta}+\frac{1}{2}\frac{\partial^2\tilde{q}^{(n)}}{\partial\theta^2}+2|\tilde{q}^{(0)}|^2\tilde{q}^{(n)}+(\tilde{q}^{(0)})^2(\tilde{q}^{(n)})^*$$

$$\mathcal{R}^{(n)}\equiv i\frac{\partial^3\tilde{q}^{(n-1)}}{\partial\theta^3}+\sum_{i=1}^{n}k^{i}\tilde{q}^{(n-i)}+i\sum_{i=1}^{n}\nu^{(i)}\frac{\partial\tilde{q}^{(n-i)}}{\partial\theta}+\sum_{i+j+k=n,i,j,k\neq n}\tilde{q}^{(i)}\tilde{q}^{(j)}\tilde{q}^{(k)} .$$

To obtain the solution of Eq. (B16), we try the ansatz

$$\tilde{q}^{(n)}=\sum_{j=1}^{n+1} C_j^{(n)} U_j \ , \tag{B17}$$

where the U_j's are defined as

$$U_{2j+1}=|\tilde{q}^{(0)}(\theta)|^{2j}\tilde{q}^{(0)}(\theta) \ ,$$
$$U_{2j+2}=i|\tilde{q}^{(0)}(\theta)|^{2j}\frac{\partial\tilde{q}^{(0)}}{\partial\theta} \ , \tag{B18}$$

and the $C_j^{(n)}$, $j=1,\ldots,n+1$, are constants to be determined later.

We observe that the U_j's are closed under differentiations, i.e., any derivatives of the U_j can be represented as a finite series of the U_j's alone. More important, they are closed under the operator $\mathcal{L}$:

$$\mathcal{L}U_{2j+1}=2j(j+1)A^2U_{2j+1}-(2j-1)(j+2)U_{2j+3} \ ,$$
$$\mathcal{L}U_{2j+2}=2j(j+1)A^2U_{2j+2}-2(j+2)\nu^{(0)}U_{2j+3}-(2j+1)(j+2)U_{2j+4} \ . \tag{B19}$$

Therefore the U_j's form an independent base for the solutions of Eq. (B1). Equation (B17) is a solution of Eq. (B16) if we can determine the coefficients $C_j^{(n)}$'s, $j=1,\ldots,n+1$. The argument goes as follows. First, we introduce the idea of rank of the U_j's. Let us assume that the quantities $\tilde{q}^{(0)}(\theta)$, $(\tilde{q}^{(0)})^*(\theta)$, and $\partial/\partial\theta$ are all of rank 1, and the ranks of the U_j's are the sum of that of its constituents. Hence the U_j's are of rank j. For reducible expressions such as $\partial^2\tilde{q}^{(0)}/\partial\theta^2$, it has an apparent rank of 3 (1 from $\tilde{q}^{(0)}(\theta)$, and 2 from $\partial/\partial\theta$). However, it can be written in term of the U_j's in the form

$$\frac{\partial^2\tilde{q}^{(0)}}{\partial\theta^2}=[A^2+(\nu^{(0)})^2]U_1+2\nu^{(0)}U_2-2U_3 \ . \tag{B20}$$

The right-hand side of Eq. (B20) contains terms of rank 1, 2, and 3, respectively. Therefore, for reducible terms, it can be represented by U_j's of rank equal to and lower than its apparent value. The quantity $\tilde{q}^{(n)}(\theta)$ has an apparent rank of $n+1$ [hence Eq. (B17)], and the operator $\mathcal{L}$ has an apparent rank of 2. Let us return to Eq. (B16). Its inhomogeneous term $\mathcal{R}^{(n)}$ has an apparent rank of $n+3$. It can be expressed in terms of the U_j's as

$$\mathcal{R}^{(n)}=\sum_{j=1}^{n+3} d_{j-2}^{(n)}U_j \ . \tag{B21}$$

The d_j, $j=-1,\ldots,n+1$ are known constants because they can be determined from the $\tilde{q}^{(j)}$, $j=1,\ldots,n-1$. As discussed earlier, the parameters $k^{(n)}$ and $\nu^{(n)}$ are chosen to eliminate the secular terms $\tilde{q}^{(0)}$ and $i\partial\tilde{q}^{(0)}/\partial\theta$ (i.e., U_1 and U_2) in $\mathcal{R}^{(n)}$. One thus has d_{-1} and d_0 equal to zero. The term $\mathcal{R}^{(n)}$ contains only U_j's of rank from 3 to $n+3$. For the left-hand side of Eq. (B16), using Eqs. (B19), it also only contains expressions of ranks from 3 to $n+3$. Consequently, Eq. (B17), when substituted into Eq. (B16), forms a closed set of equations. Equating the coefficients of each rank on both sides of Eq. (B16), we obtain $n+1$ linear equations in $n+1$ unknowns, $C_j^{(n)}$, $j=1,\ldots,n+1$. They are

$$2j(j+1)A^2C_{2j+1}^{(n)}-2(j+1)\nu^{(0)}C_{2j}^{(n)}-(2j-3)(j+1)C_{2j-1}^{(n)}=d_{2j-1} \ ,$$
$$2j(j+1)A^2C_{2j+2}^{(n)}-(2j-1)(j+1)C_{2j}^{(n)}=d_{2j} \ . \tag{B22}$$

The $C_j^{(n)}$'s can then be calculated recursively once the tedious task of determining the d_j's is performed. This shows that the expansion (B2) can be carried out to all orders in β. Since all the U_j's are localized functions (in a frame moving with the renormalized group delay ν), the series solution (B2) represents that of a solitary wave. Moreover, since the U_j's obey the symmetry conditions,

$$U_j(\theta)=U_j^*(-\theta) \ , \tag{B23}$$

for all j's, the series solution also obeys the same symmetry condition.

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