Nonlocal dynamics of domains and domain walls in dissipative systems

F. J. Elmer
Institut für Physik, Universität Basel, Klingelbergstrasse 82, CH-4056 Basel, Switzerland
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The dynamics of domains and domain walls in spatially one-dimensional systems is investigated for the case that the evolution equation contains nonweighted spatial averages of the order parameter or a function of it (strongly nonlocal dynamics). Two ordinary differential equations for reduced order parameters are introduced. The first one governs the dynamics of domain states and domain-wall positions. For large systems, there occurs a separation of time scales that leads to a second reduced equation of motion governing the dynamics of the domain sizes. The time scale of the domain-size dynamics is proportional to the length of the system. Validity conditions of both reduced equations of motion are discussed. The ballast resistor and another current instability system serve as illustrations.

I. INTRODUCTION

Many dissipative systems, especially nonlinear optical, magnetic, and electrical systems, show bistability or even multistability. In spatially extended versions of such systems different locally stable states may be simultaneously realized in different parts of the system. The part of the system that is occupied by one state is usually called a domain. The domains are separated by domain walls (also called fronts or kinks).

The global macroscopic state of such dissipative systems is in general determined by an n-component field called the order parameter $\mathbf{u}$. The dynamics of the order parameter is usually modeled by a partial differential equation. The investigation of such nonlinear evolution equations will be considerably simplified if a Lyapuonov functional $\Phi[\mathbf{u}]$ [i.e., $d\Phi/dt=(\delta\Phi/\delta\mathbf{u})\partial_t\mathbf{u}<0$] exists. In this case one readily gets the following result. The domain wall between two states (which are characterized by local minima of Φ) propagates into the less stable state, i.e., the state with the higher local minimum. For a plane wall in an infinitely extended system the propagation velocity v^{wall} is constant because of the translation symmetry in homogeneous systems. For equally stable states the domain walls are at rest.[1] This picture is often also valid for systems having no Lyapunov functional. However, it is usually more difficult to obtain the sign of the wall velocity.

In this paper we investigate the dynamics of domains and domain walls under the influence of a nonlocality, or more precisely, a strong nonlocality. Strongly nonlocal dynamics means that the evolution equation contains nonweighted spatial averages of the order parameter or a function of it. We restrict our investigations to spatially one-dimensional systems

$$\partial_t\mathbf{u}=\mathbf{f}\left[\mu;\frac{1}{L}\int_0^L\mathbf{g}(\mathbf{u})dx,\mathbf{u},\partial_x\mathbf{u},\ldots,\partial_x^N\mathbf{u}\right]\quad\text{with } \mathbf{u}\in\mathbb{R}^n\ , \tag{1.1}$$

where μ is the control parameter, L is the length of the system, and $\mathbf{g}$ is an arbitrary m-dimensional vector function yielding m different types of strong nonlocalities. Strongly nonlocal evolution equations occur in several fields of physics [e.g., the Taylor-Couette system,[2] ferromagnetic resonance,[3] and current instability systems (see examples below)].

The aim of this paper is to understand the influence of nonlocalities on the dynamics of domains and domain walls. In Sec. II we derive two equations of motion for a reduced order parameter with a finite number of degrees of freedom. The first equation of motion (called "reduction 1") describes the relaxation of domain states towards equilibrium and the variation of domain sizes by domain-wall motion. For large L, the domain-size motion becomes very slow. Therefore the domain states can be adiabatically eliminated. This leads to a second equation of motion (called "reduction 2") for the size of the domains only. In Sec. III we then take a new look at the well-known ballast resistor[4,5] from our point of view. In Sec. IV a two-layer model for semiconductors and gas discharges introduced by Radehaus *et al.*[6] serves as an example where the order parameter of a domain can be either spatially uniform or spatially periodic.

II. EQUATIONS OF MOTION FOR REDUCED ORDER PARAMETERS

Before we investigate the nonlocal dynamics of domains and domain walls, we consider their local aspects.

A. Local aspects of the dynamics

In order to discuss the local aspects we introduce the local version of the nonlocal evolution equation (1.1):

$$\partial_t\mathbf{u}=\mathbf{f}(\mu,\bar{\mathbf{g}};\mathbf{u},\partial_x\mathbf{u},\ldots,\partial_x^N\mathbf{u})\ , \tag{2.1}$$

where the nonlocality $1/L\int_0^L\mathbf{g}(\mathbf{u})dx$ is replaced by $\bar{\mathbf{g}}$ which should be interpreted as an additional control parameter and where $L \to \infty$ is always considered.

Since we are interested in states with different domains, we assume at least bistability behavior of the local version (2.1). More precisely, in a certain subspace of the control parameters μ and $\bar{g}$ Eq. (2.1) has s (≥ 2) attracting (i.e., linearly stable) states which are either stationary uniform ($k_i=0$) or spatially periodic ($k_i \neq 0$) with a standing ($c_i=0$) or traveling ($c_i \neq 0$) shape:

$$\mathbf{u}_{i0}(\bar{\mathbf{g}};k_i(x-c_it)) \text{ with } \mathbf{u}_{i0}(\bar{\mathbf{g}};\phi)=\mathbf{u}_i(\bar{\mathbf{g}};\phi+2\pi)\ , \quad i=1,\ldots,s\ . \tag{2.2}$$

We exclude attractors where the space average of $\mathbf{g}(\mathbf{u})$ is time dependent.

The domain wall between two domains should be a stable solution of (2.1) having a stationary shape or envelope moving with a constant velocity v_{ij}^{wall} from a domain of type i into a domain of type j. For simplicity we assume that for all combinations i,j there exists a unique type of stable domain wall with a velocity $v_{ij}^{\text{wall}}(\bar{\mathbf{g}})$ being a unique function of $\bar{\mathbf{g}}$.

All local aspects of the nonlocal dynamics will be determined by the local version (2.1). If $1/L\int_0^L \mathbf{g}(\mathbf{u})dx$ is time independent for a process $\mathbf{u}(t)$ of (1.1), then this process has local character [i.e., will be described by (2.1)]. For example, a phase shift of a spatially periodic state is a local process.

B. Reduction 1

Now we investigate the nonlocal dynamics of a state with K domains. The order parameter $\mathbf{u}^{(i)}$ of the domain i is assumed to be a slightly disturbed equilibrium state $\mathbf{u}_{n^{(i)}0}$ of the local version (2.1). The size $q^{(i)}L$ of a domain should be much larger than the width of a domain wall (and also larger than $2\pi/k_i$ if the domain state is spatially periodic). Thus, the nonlocality is approximated by

$$\bar{\mathbf{g}}=\frac{1}{L}\int_0^L \mathbf{g}(\mathbf{u}(x))dx=\sum_{i=1}^{K} q^{(i)}\frac{1}{L}\int_0^L g(\mathbf{u}^{(i)}(x))dx\ , \tag{2.3}$$

with $0<q^{(i)}<1$ and $\sum_{i=1}^{K} q^{(i)}=1$.

First of all we neglect domain-wall motions (i.e., $q^{(i)}=\text{const}$) and investigate only the dynamics of the domain states. This is a mindful approach because, as we will see below, the time scales of domain-state motion and domain-wall motion separate for large L. The domain states may be described by the general equation of motion (1.1). If the coupling caused by the nonlocality (2.3) is zero every domain state $\mathbf{u}^{(i)}$ will relax into the equilibrium $\mathbf{u}_{n^{(i)}0}$. If the coupling is not zero every small disturbance (or mode) $\delta\mathbf{u}^{(i)}$ of the equilibrium which does not change $\bar{\mathbf{g}}$, i.e.,

$$\frac{1}{L}\int_0^L \left.\frac{d\mathbf{g}}{d\mathbf{u}}\right|_{\mathbf{u}_{n^{(i)}0}} \delta\mathbf{u}^{(i)}dx=0\ ,$$

will die out. Therefore the coupling between different domains is caused only by the modes which change $\bar{\mathbf{g}}$ (nonorthogonal modes). Thus it should be possible to reduce the full equation of motion (1.1) to an equation of motion for only these modes. For a spatially uniform domain state $\mathbf{u}^{(i)}$ the main nonorthogonal modes are uniform. Thus, we get the reduced equation of motion

$$\frac{d\mathbf{u}^{(i)}}{dt}=\mathbf{f}(\mu;\bar{\mathbf{g}};\mathbf{u}^{(i)},0,\ldots,0)\ . \tag{2.4}$$

For a spatially periodic domain state $\mathbf{u}^{(i)}$ the main nonorthogonal modes are uniform amplitude changing modes.[7] In this case the full equation of motion may be often reduced to a third- or fifth-order Landau equation for the amplitude of the oscillation.[8]

Now we introduce the notions *local stability* and *nonlocal stability.* Local stability means that any infinitesimally small disturbance which does not change the nonlocality (2.3) dies out. Nonlocal stability means that *all* infinitesimally small disturbances die out. Clearly, local stability of every domain state (which we have been assumed in Sec. II A) does not imply nonlocal stability of the total state. It should be noted that the nonlocal stability depends on the domain sizes given by $q^{(i)}$.

Up to now we have assumed that the domain walls are at rest (i.e., $q^{(i)}=\text{const}$). But the movement of the domain walls changes the domain size. Thus, we get

$$\frac{dq^{(i)}}{dt}=\frac{1}{L}[v_{n^{(i)}n^{(i+1)}}^{\text{wall}}(\bar{\mathbf{g}})-v_{n^{(i-1)}n^{(i)}}^{\text{wall}}(\bar{\mathbf{g}})]\ , \quad i=1,\ldots,K-1\ , \tag{2.5}$$

with $v_{n^{(0)}n^{(1)}}^{\text{wall}}\equiv 0$. In the case of spatially uniform domains Eqs. (2.5) and (2.4) for $i=1,\ldots,K$ coupled through the nonlocality (2.3) form a closed system of ordinary differential equations for the reduced order parameter $\{\mathbf{u}^{(1)},q^{(1)},\ldots,\mathbf{u}^{(K-1)},q^{(K-1)},\mathbf{u}^{(K)}\}$. This reduced equation of motion will be called "reduction 1."

It should be noted that Eq. (2.5) is only qualitatively correct because the domain-wall motion is generally not governed by the velocity function $v^{\text{wall}}(\bar{\mathbf{g}})$which is only defined for *stationary* domain states. In nonlocal systems domain states and domain sizes change simultaneously. This leads to an open question: What is the response of a domain wall if at least one domain state is time dependent?

C. Reduction 2

In order to get a further reduction there should exist a subspace of domain configurations $\{q^{(1)},\ldots,q^{(K-1)}\}$ that are all *nonlocally stable.* For large L the domain sizes $q^{(i)}$ become very slow variables compared with the L-independent relaxation times of the domain states. Thus, we eliminate the faster variables $\mathbf{u}^{(i)}$ adiabatically (in lowest order $d\mathbf{u}^{(i)}/dt\equiv 0$) which leads to a self-consistency equation for the nonlocality

$$\bar{\mathbf{g}}=\sum_{i=1}^{s} q_i\frac{1}{L}\int_0^L \mathbf{g}(\mathbf{u}_{i0}(\bar{\mathbf{g}};x))dx\ , \quad \text{with } q_i=\sum_{j=1}^{K}\delta_{i,n^{(j)}}q^{(j)}\ , \tag{2.6}$$

where $\delta_{i,j}$ is the Kronecker symbol, and where q_iL is the sum over sizes of those domains which are in the state

$\mathbf{u}_{i0}$. Clearly, the q's fulfill the conditions $0<q_i<1$, $i=1,\ldots,s$ and $\sum_{i=1}^{s} q_i=1$. The self-consistency equation (2.6) is an implicit definition of the function $\bar{\mathbf{g}}(q_1,\ldots,q_{s-1})$. Thus, Eq. (2.5) leads to coupled equations of motion for the q_i (reduction 2):

$$\frac{dq_i}{dt}=\frac{1}{L}\sum_{j\in\sigma_i} n_{ij}v_{ij}^{\text{wall}}(\bar{\mathbf{g}}),\quad i=1,\ldots,s-1\ , \tag{2.7}$$

where σ_i is the set of all types of domains which are neighbors of domains of type i and where n_{ij} is the number of domain walls connecting domains of types i and j.

Equation (2.7) describes correctly the dynamics up to the order L^{-1}. By taking into account the interaction between domain walls, we get the correction terms of order $e^{-qL/D}$ (D is the domain-wall width) of the Kawasaki-Ohta theory.[9] These terms will be dominant after reaching a stable fixed point of (2.7).

A further reduction will be possible if the number of nonlocalities m is smaller than $s-1$ (s is the number of domain types), and if the spatial average of $\mathbf{g}(\mathbf{u}_{i0}(\bar{\mathbf{g}};x))$ does not depend on $\bar{\mathbf{g}}$. Differentiating (2.6) and using (2.7) leads to an equation of motion for the nonlocalities:

$$\frac{d\bar{\mathbf{g}}}{dt}=\frac{1}{L}\sum_{i=1}^{s}\bar{\mathbf{g}}_i(\bar{\mathbf{g}})\sum_{j\in\sigma_i} n_{ij}v_{ij}^{\text{wall}}(\bar{\mathbf{g}})$$

$$\text{with } \bar{\mathbf{g}}_i=\frac{1}{L}\int_0^L \mathbf{g}(\mathbf{u}_{i0}(\bar{\mathbf{g}};x))dx\ . \tag{2.8}$$

In Secs. III and IV we concentrate our discussion on physical systems where reduction 2 gives a correct description. But it should be kept in mind that (2.7) is only true as long as the domain states are nonlocally stable (i.e., nonorthogonal disturbances which are connected by the nonlocality die out). Since nonlocal stability depends on q_i, we can apply bifurcation theory, where the q's act as control parameters. For example, if $\bar{\mathbf{g}}(q_1,\ldots,q_{s-1})$ defined by the self-consistency equation (2.6) is multivalued, a saddle-node bifurcation will occur at the submanifold of the space $\{q_1,\ldots,q_{s-1}\}$ where two branches of $\bar{\mathbf{g}}(q_1,\ldots,q_{s-1})$ are connected. The domain states are nonlocally stable on one branch and unstable on the other. Now let us choose a state on the stable branch. Consider the case that the equation of motion (2.7) for the q's moves the state toward this saddle-node bifurcation. At least at the saddle-node bifurcation, reduction 2 [i.e., Eq. (2.7)] fails. Usually it fails before reaching the saddle-node bifurcation because

$$\left|\frac{d\bar{\mathbf{g}}}{dt}\right|=\left|\sum_{i=1}^{s}\partial_{q_i}\bar{\mathbf{g}}\frac{dq_i}{dt}\right|\to\infty$$

and $\bar{\mathbf{g}}$ varies on a faster timescale than the nonlocal relaxation of the domain states. One might go back to reduction 1, but it is unclear whether this approach is useful because of the above-mentioned unknown status of the domain-wall velocities for time-dependent domain states.

III. THE BALLAST RESISTOR

In this section we apply our theory to one of the simplest nonlocal systems: The ballast resistor—an old electrical device. The advantages of this example are the following. (i) The order parameter has only one component (the temperature); (ii) there is only one nonlocality (the total resistance); (iii) the equilibrium states of the domains are always uniform and stationary; (iv) the local version which is a one-dimensional reaction diffusion system has a Lyapunov functional[10,11] (but not the nonlocal one); (v) existence and stability of domain walls of the local version have been rigorously proved.[12]

The ballast resistor consists of an iron wire in a cooled hydrogen atmosphere under low pressure. Connecting the ballast resistor to a voltage source leads to a linear ohmic response for low voltage. For higher voltage, the slope of the current-voltage characteristic decreases, and a plateau of constant current is reached either smoothly or by a jump to a lower level of the current. At the end of the plateau, the current increases again, usually linearly, but with a higher resistivity than for low voltages. In the plateau regime the resistor shows a nonuniform spatial structure of a hot glowing region surrounded by cooler regions of the wire. The hot region spreads by increasing voltage.

Although since the beginning of the century the ballast resistor was used to stabilize the current in technical applications,[4] an explanation of this behavior was given for the first time by Busch in 1921.[4] Since the mid 1970s, the ballast resistor was taken as an example of a nonlinear system.[10,11,13–15] In the past 30 years several similar electrothermal instabilities in metals, semiconductors, and superconductors have been investigated (for review and further references see Ref. 5).

A. The equation of motion

The ballast resistor is an essentially one-dimensional system. The equation of motion for the temperature profile $T(x,t)$ along the wire is

$$c_v\partial_t T=\lambda\partial_x^2 T-A(T)+I^2R(T)\ , \tag{3.1}$$

where c_v is the specific heat per unit length, λ the heat conductivity, $A(T)$ the heat loss into the gas, and $R(T)$ the temperature-dependent resistance of the wire per unit of length. For simplicity we have neglected the Thomson effect (which leads to a term $\sigma I\partial_x T$) and the temperature dependence of c_v and λ. The nonlocality results from the fact that the current I is determined by the total resistance which is an integral of $R(T)$ over the sample,

$$I=\frac{E}{\frac{1}{L}\int_0^L R(T(x))dx}\quad\text{with } E=\frac{U_0}{L}\ , \tag{3.2}$$

where U_0 is the external voltage and L the length of the wire. In order to suppress the effects of boundary layers we use the following boundary conditions:

$$\partial_x T(0,t)=\partial_x T(L,t)=0\ . \tag{3.3}$$

Instead of the averaged resistivity $\bar{R}$ we will use the current I as the nonlocality. Therefore, the local version (3.1) of the ballast resistor has a natural meaning: It is the resistor in a current-controlled circuit. In any other circuit where the current through the ballast resistor is not externally given, we have to take into account the nonlocal character of the system. In the literature the ballast resistor is often treated as a current-controlled system, and voltage U_0 is computed afterwards by using (3.2). Such an approach fails, however, in the case of nonstationary solutions. In a paper by Bedeaux and Mazur[16] on the stability of domainlike stationary states, they distinguish clearly between fixed current and fixed voltage. However, they made an error in the computation of the spectrum of the linear operator which occurs in the linear stability analysis. For no obvious reason they *symmetrize* the original non-Hermitian linear operator. But it is evident that the spectrum of the symmetrized operator differs generally from the spectrum of the original one. In the case of Neumann boundary conditions (3.3) this leads to the wrong result that the largest eigenvalue for large L is proportional to L^{-2}. From the second reduced equation of motion (2.7) (reduction 2) it may be seen that the largest eigenvalue is proportional to L^{-1} [see Eq. (3.17) below]. Computing the spectrum of the nonsymmetrized operator of Bedeaux and Mazur leads to the correct result.

The local version. The local (i.e., current-controlled) version (3.1) of the ballast resistor has the Lyapunov functional

$$\Phi[T]=\int_0^L \left[\frac{\lambda}{2}(\partial_x T)^2-\phi(T)\right]dx \ , \tag{3.4}$$

where $\phi(T)$ is defined by

$$\phi(T)=\int^T [I^2R(T')-A(T')]dT' \ . \tag{3.5}$$

Clearly, the stationary linearly stable uniform states $T_{i0}(I)$ of the local version (3.1) are given by the relative minima of $\Phi[T]$ [i.e., relative maxima of $\phi(T)$]:

$$A(T_{i0})=I^2R(T_{i0}) \quad \text{with } d_T A(T_{i0})>I^2 d_T R(T_{i0}) \quad \text{for } i=1,\ldots,s \ , \tag{3.6}$$

where $d_T \equiv d/dT$. Figure 1 shows the graphical solution of (3.6): Locally *stable* uniform states are given by intersection of A with I^2R, where the slope of A is larger than the slope of I^2R. In order to get even bistability R should have at least one inflection point if heat conduction dominates the heat loss (i.e., $A \propto T-T_G$). Such behavior of R is typical near a phase transition point T_c of the conductor.[11] Bistability is possible only if $T_G<T_C$ and if $I_{\min}<I<I_{\max}$ (see Fig. 1). Spatially periodic states of the local version are all unstable.[16]

Consider a domain wall between two uniform stable states T_{i0} and $T_{j0}>T_{i0}$. As mentioned in the introduction, the sign of the domain wall velocity $v_{ij}^{\text{wall}}(I)$ is given by the sign of

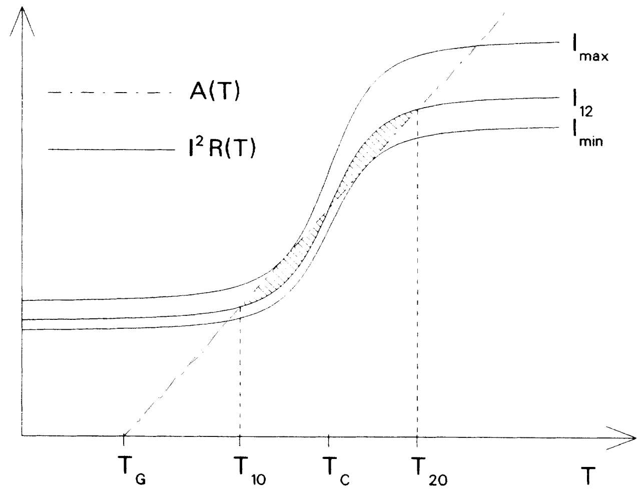

FIG. 1. The uniform states T_{i0} of the ballast resistor defined by Eq. (3.6) (for more details see text).

$$\phi(T_{j0}(I))-\phi(T_{i0}(I))=\int_{T_{i0}(I)}^{T_{j0}(I)} A(T)dT - \int_{T_{i0}(I)}^{T_{j0}(I)} I^2R(T)dT \ .$$

This leads to the well-known equal-area rule[10] which says that the domain wall rests if the two hatched areas in Fig. 1 are equal. Since the local version is a one-component reaction diffusion system the domain wall is always stable in the comoving frame of reference if there does not exist a stable state T_0 with $T_{i0}<T_0<T_{j0}$.[12]

The existence of a Lyapunov functional implies that the local version has no limit cycle solutions. But the nonlocal (i.e., voltage-controlled) ballast resistor does not have a Lyapunov functional. A forthcoming paper will give examples for such limit cycles in the framework of a detailed discussion on the failure of reductions 1 and 2.

B. Reduced equations of motion

We assume that there exists only two locally stable domain states T_{10} and T_{20} with $T_{20}>T_{10}$, as in Fig. 1. The simplest nonuniform state has two domains which are separated by a domain wall. Applying our theory from section 2 [i.e., Eqs. (2.4) and (2.5)], we get reduction 1:

$$c_v \frac{dT^{(i)}}{dt}=-A(T^{(i)})+I^2R(T^{(i)}) \quad i=1,2 \ , \tag{3.7}$$

$$\frac{dq}{dt}=\frac{1}{L}v_{21}^{\text{wall}}(I) \ , \tag{3.8}$$

with

$$I=\frac{E}{(1-q)R(T^{(1)})+qR(T^{(2)})} \ , \tag{3.9}$$

where qL is the size of domain 2.

In order to test the domain states on nonlocal stability we keep q fixed. Now the stationary solution of (3.7) is $(T^{(1)},T^{(2)})=(T_{10}(I(q)),T_{20}(I(q)))$, $i=1,2$, where $I(q)$ is a solution of

$$I(q)=\frac{E}{(1-q)R(T_{10}(I(q)))+qR(T_{20}(I(q)))} . \tag{3.10}$$

Solving the linearized equation of motion (3.7) with the ansatz $\delta T^{(i)}=c_i\exp(\lambda t/c_v)$ leads to the characteristic polynomial for the eigenvalue λ:

$$\lambda^2-\lambda N_{tr}+N_{det}=0 , \tag{3.11}$$

where

$$N_{tr}=\lambda_1^{loc}-\lambda_2^{loc}-2\frac{I^3}{E}(q_1R_1d_TR_1+q_2R_2d_TR_2) , \tag{3.12}$$

$$N_{det}=\lambda_1^{loc}\lambda_2^{loc}-2\frac{I^3}{E}(q_1R_1d_TR_1\lambda_2^{loc}+q_2R_2d_TR_2\lambda_1^{loc}) , \tag{3.13}$$

with

$$q_1\equiv 1-q,\quad q_2\equiv q,\quad R_i\equiv R(T_{i0}) ,$$
$$d_TR_i\equiv d_TR(T_{i0}),\quad \lambda_i^{loc}\equiv -d_TA(T_{i0})+I^2d_TR(T_{i0}) . \tag{3.14}$$

Now the condition for nonlocal stability (i.e. $\mathrm{Re}\lambda<0$) is simply $N_{tr}<0$ and $N_{det}>0$. Because E, I, q_i, and R_i are positive and λ_i^{loc} is negative in accordance with (3.6), a sufficient condition for nonlocal stability is $d_TR(T_{10})>0$ and $d_TR(T_{20})>0$.

Now assuming nonlocally stable domain states and applying Eq. (2.7), we get for the second reduced equation of motion (reduction 2)

$$\frac{dq}{dt}=\frac{1}{L}v_{21}^{wall}(I(q)) , \tag{3.15}$$

where the current $I(q)$ is defined by (3.10). The stationary state is given for equal-area-rule current I_{12} [i.e., $v_{21}^{wall}(I_{12})=0$]:

$$q_0=\frac{E/I_{12}-R(T_{10}(I_{12}))}{R(T_{20}(I_{12}))-R(T_{10}(I_{12}))} . \tag{3.16}$$

In order to test the stability we make the usual ansatz $q=q_0+\delta q\, e^{\lambda_q t}$ and linearize the equation of motion around q_0. Clearly q_0 is stable if

$$\lambda_q=n_{21}\frac{1}{L}\frac{dv_{21}^{wall}}{dI}\bigg|_{I_{12}}\frac{dI}{dq}\bigg|_{q_0} \tag{3.17}$$

is negative. The existence of a Lyapunov functional for the local version implies immediately

$$\frac{dv_{21}^{wall}}{dI}\bigg|_{I_{12}}\propto \partial_I\phi(T_{20})|_{I_{12}}-\partial_I\phi(T_{10})|_{I_{12}}$$
$$=2I_{12}\int_{T_{10}(I_{12})}^{T_{20}(I_{12})}R(T)dT>0 .$$

Differentiating (3.10) with respect to q and using $A_i\equiv A(T_{i0}(I_{12}))$ and the definitions (3.14) at $I=I_{12}$ we find

$$\frac{dI}{dq}\bigg|_{q_0}=\frac{-(A_2-A_1)}{E+I_{12}^2((1-q_0)d_TR_1d_IT_1+q_0d_TR_2d_IT_2)} . \tag{3.18}$$

Differentiating (3.6) with respect to I leads to

$$d_IT_i=\frac{2I_{12}R_i}{d_TA_i-I_{12}^2d_TR_i}>0 . \tag{3.19}$$

Using (3.13) and (3.19) we find

$$\frac{dI}{dq}\bigg|_{q_0}=\frac{-(A_2-A_1)\lambda_1^{loc}\lambda_2^{loc}}{EN_{det}} . \tag{3.20}$$

Since N_{det} is assumed to be positive (i.e., nonlocal stability of domain states) the stationary solution (3.16) is stable if $A(T_{20})>A(T_{10})$.

C. Comparison between numerical simulations, reductions 1 and 2

We choose the following (physically reasonable) functions:

$$A(T)=T , \tag{3.21}$$

$$R(T)=1+r(T-T_0)/[(\Delta T)^2+(T-T_0)^2]^{1/2} \quad \text{with } 0<r<1 , \tag{3.22}$$

where T measures the temperature difference from the gas temperature T_G. Furthermore, we set in Eq. (3.1) $c_v=1$ and $\lambda=1$. Since $R(T)$ increases with T we see immediately by applying the results of Sec. III B that a two-domain state is always stable. For the simulation a Crank-Nicolson scheme was used. All computations were done with a grid point distance of 0.1 in space as well as in time.

Figure 2 illustrates the typical dynamical behavior of

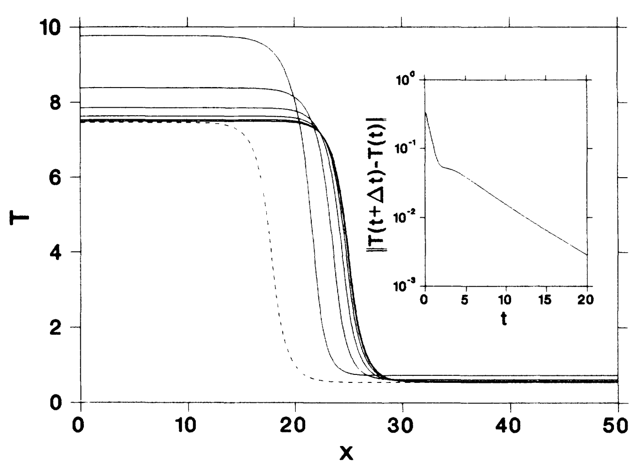

FIG. 2. Numerical simulation of the ballast resistor with $r=0.9$, $T_0=4$, $\Delta T=1$, $E=2$, and $L=50$. Mesh point distance $\Delta x=0.1$. The dotted line indicates the temperature profile at $t=0$ which is the stationary state for $E=1.5$. The solid lines are snapshots of the system at $t=5,10,15,20,25,30$. The inset depicts the Euclidean distance in phase space between successive simulation steps ($\Delta t=0.1$).

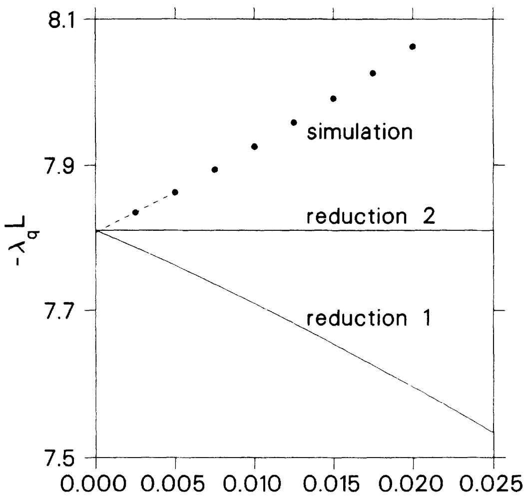

FIG. 3. Relaxation of the domain wall. Comparison between simulations, reduction 1 and reduction 2. The dashed line is a linear extrapolation of the last two data points. The parameters are the same as in Fig. 2.

the ballast resistor. The simulation starts with a two-domain state which is stationary for a lower value of E. The solid lines are snapshots of the temperature profile for equidistant time steps. First of all the temperatures of both domains increase rapidly. Afterwards the temperatures of both domains slowly decrease down to the values they had started with. On the same time scale the domain wall relaxes into a new equilibrium position. In 1921 in a footnote of his paper,[4] Busch gave a nice description of an observation of this dynamical behavior. The inset of Fig. 2 depicts clearly the two regimes of the fast and the slow relaxation of the domain states and the domain wall, respectively.

In Fig. 3 the time scale of a domain-wall relaxation given by the largest eigenvalue λ_q of a linear stability analysis is shown as a function of $1/L$. The data of the simulation are given by the slope of $\ln\|T(t+\Delta t)-T(t)\|$ for large t. The curves for reductions 1 and 2 are calculated analytically (see appendix). Reduction 2 gives the leading term of a $1/L$ expansion which agrees very well with the simulation. The higher-order corrections of reduction 1 are clearly wrong. The reason for that is the fact mentioned in Sec. II that the domain-wall motion is not known quantitatively for nonstationary domain states.

IV. NONLOCAL DYNAMICS IN A TWO-LAYER MODEL OF A CURRENT INSTABILITY

This section exemplifies the typical nonlocal behavior of a bistable system where one state is spatially periodic. The system is a two-layer model for semiconductors and gas discharges introduced by Radehaus *et al.*[6]

The system consists of distinct planes layers between two plane metal electrodes which are connected to a constant voltage source via a load resistor. One layer is assumed to be a medium with an S-shaped negative differential conductivity and the other layer is assumed to be a purely ohmic medium with constant resistivity. In the model of Radehaus *et al.*,[6] the potential $v(x,y,t)$ across the ohmic layer and the normal component of the current density $w(x,y,t)$ from the ohmic layer into the nonlinear layer are governed by the following equations of motion:

$$\begin{aligned} \tau\partial_t v &= \Delta v + w - v - \mu_{\text{eff}} \\ \partial_t w &= \sigma\Delta w - v - f(w) \end{aligned} \tag{4.1}$$

with the nonlocality

$$\mu_{\text{eff}} = \mu - r\frac{1}{F}\int_F w\,dx\,dy\ , \tag{4.2}$$

where all variables and parameters are measured in dimensionless units, F is the area of the boundary between the two layers, μ and μ_{eff} are proportional to the external voltage and to the voltage difference between the metal electrodes, respectively, r is proportional to the load resistance, τ is given by the dielectric constants, and σ is proportional to the ratio between the charge carrier diffusion constant and an effective "diffusion" constant of v (for more details see Ref. 6). The nonlinearity $f(w)$ is the current-voltage characteristic of the nonlinear medium.

Now we restrict our investigation to the one-dimensional version of (4.1) (i.e., $\Delta \to \partial_x^2$) and choose as boundary conditions

$$\partial_x v(0,t) = \partial_x v(L,t) = \partial_x w(0,t) = \partial_x w(L,t)\ . \tag{4.3}$$

For f we choose a third-order polynomial which is the simplest analytic function leading to negative differential resistivity (i.e., $f' \equiv d_w f < 0$):

$$f(w) = w^3 - \gamma w\ , \tag{4.4}$$

where the inflection point of f has been shifted into the origin which does not change the form of (4.1) because f is the only nonlinearity.

A. Stationary states of the local version and their stability

Solutions of

$$\mu_{\text{eff}} = w + f(w) = w^3 + (1-\gamma)w \tag{4.5}$$

lead to spatially uniform states of the local version (4.1) with $v = -f(w)$. If $\gamma > 1$ we find bistability for

$$|\mu_{\text{eff}}| \le 2\left[\frac{\gamma-1}{3}\right]^{3/2}\ .$$

Testing the stability of a uniform solution against a perturbation of the form $(\delta v, \delta w) = (\delta v_k, \delta w_k)\exp(ikx+\lambda t)$ leads to the eigenvalue (3.11) with

$$N_{\text{tr}} = -f'(w) - (\sigma + \tau^{-1})k^2 - \tau^{-1}\ , \tag{4.6}$$

$$\tau N_{\text{det}} = 1 + (1+k^2)[f'(w) + \sigma k^2]\ . \tag{4.7}$$

The uniform state becomes unstable either by $N_{\text{tr}} > 0$ which implies a Hopf bifurcation or by $N_{\text{det}} < 0$ which

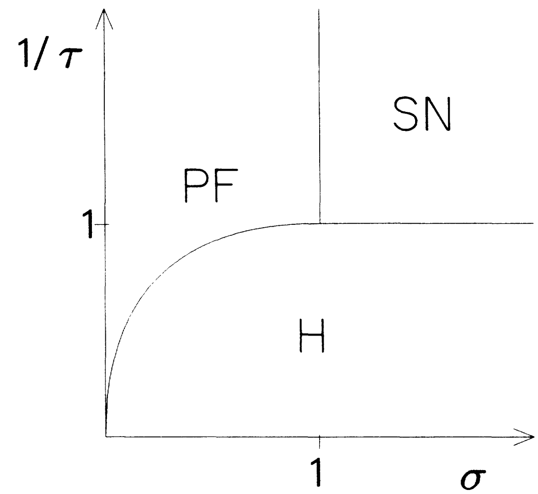

FIG. 4. Instability and bifurcation types of the spatially uniform states of the local version of the two-layer model (4.1), where H indicates Hopf bifurcations, SN saddle-node bifurcations ($k=0$), and PF pitchfork bifurcations ($k=k_c\neq 0$).

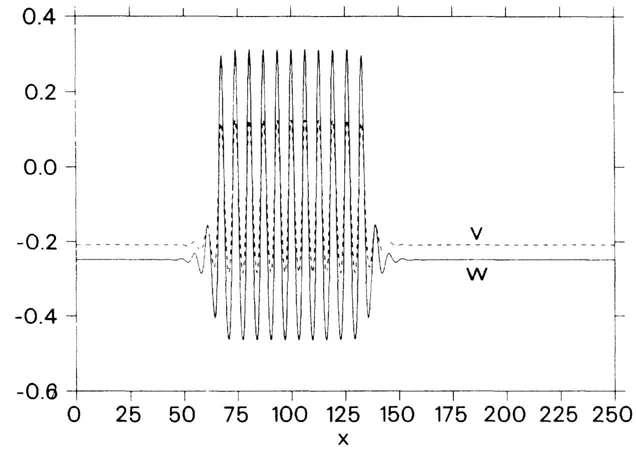

FIG. 5. A stationary state of the two-layer model (4.1) for $\mu=-0.47$, $r=2$, $\sigma=\frac{1}{4}$, $\tau=1$, $\gamma=0.9$, and $L=250$. It results from a numerical simulation ($\Delta x=0.25$, $\Delta t=0.1$) started with the state (4.11) (a sharp Gaussian peak at $x=100$ on the unstable uniform state).

implies a saddle-node bifurcation ($k=0$) or a pitchfork bifurcation ($k\neq 0$). The Hopf bifurcation occurs if

$$f'\leq f'_{\rm H}\equiv -\tau^{-1}\ , \tag{4.8}$$

the saddle-node bifurcation occurs if

$$f'\leq f'_{\rm SN}\equiv -1\ , \tag{4.9}$$

and the pitchfork bifurcation occurs if

$$f'\leq f'_{\rm PF}\equiv \sigma-2\sqrt{\sigma}\quad \text{with } k=k_c\equiv\left[\frac{1}{\sqrt{\sigma}}-1\right]^{1/2}. \tag{4.10}$$

Which kind of instability actually occurs depends on σ and τ (see Fig. 4).

B. The nonlocal pinning behavior

We choose $\tau=1$, $\sigma=\frac{1}{4}$, and $\gamma=0.9$ to avoid bistability between uniform states. Nevertheless, the local version shows numerically bistability between a uniform state and spatially periodic states.

Figures 5–7 present results of a simulation for $\mu=-0.47$, $r=2$, and $L=250$. Again a Crank-Nicolson scheme was used. The simulation was started at the unstable uniform state with a sharp Gaussian peak:

$$\begin{aligned} v(x,0)&=-0.186\,461+e^{-\left[\frac{x-100}{2}\right]^2}\ ,\\ w(x,0)&=-0.218\,821+e^{-\left[\frac{x-100}{2}\right]^2}\ . \end{aligned} \tag{4.11}$$

This large perturbation of the uniform state guarantees that only one domain with a spatially periodic order parameter will be created (see Fig. 5). For $0<t<20$ a nonlocal relaxation occurs into a small spatially periodic domain with one spike.[17] For $20<t<500$ the nonlocal domain-wall motion moves the system into a state with 11 spikes as seen in Fig. 5. Oscillations of $\|\mathbf{u}(t+\Delta t)-\mathbf{u}(t)\|$ [i.e., the L_2 norm of the difference of the order parameter $\mathbf{u}\equiv(v(x),w(x))$ between successive simulation steps] indicate acceleration and deceleration

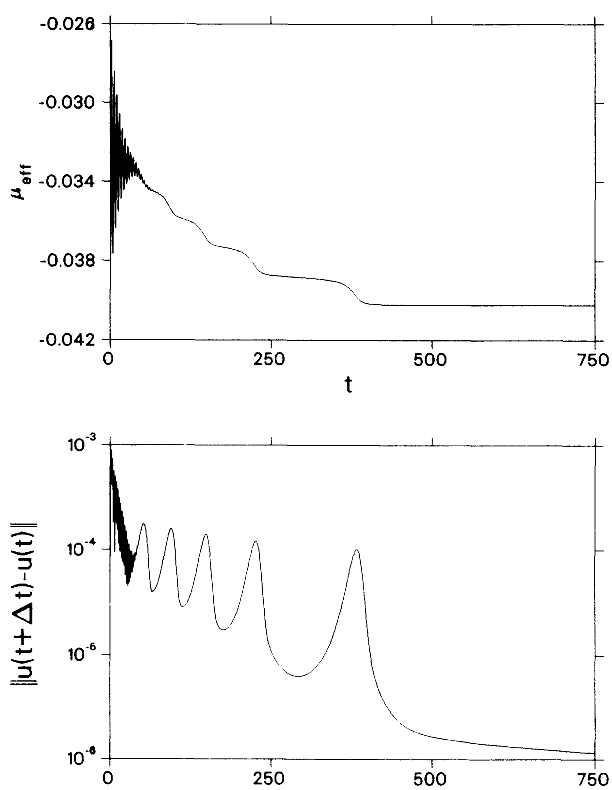

FIG. 6. The evolution of the nonlocality $\mu_{\rm eff}$ and the distance in phase space between successive steps of the simulation of the two-layer model mentioned in Fig. 5.

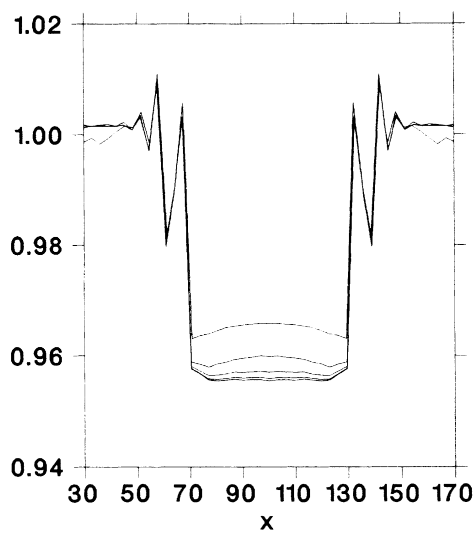

FIG. 7. The wave number $k(x)$ from the simulation of the two-layer model mentioned in Fig. 5 for $t=500$, 1500, 2500, 3500, 4500. Phase diffusion flattens $k(x)$ in the domain with the spatially periodic substate. $k(x)$ was computed by measuring the distance between successive maxima or minima.

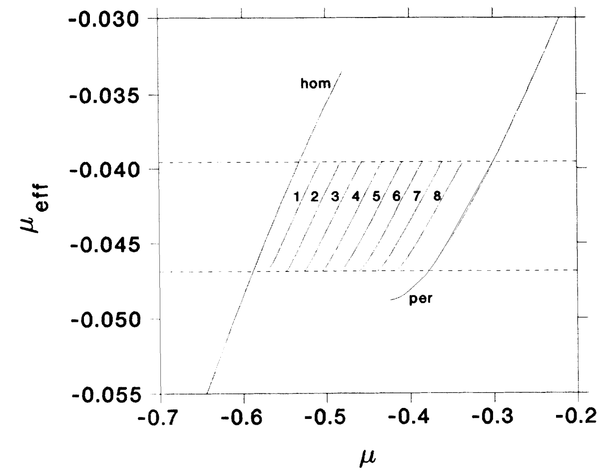

FIG. 8. Stationary stable states of the two-layer model for $r=2$, $\sigma=\frac{1}{4}$, $\tau=1$, $\gamma=0.9$, and $L=20\pi$ found by numerical simulation ($\Delta x=\pi/10$). The uniform state is labeled with *hom*, the spatially periodic state ($k=0.95$) with *per*, and the two-domain state with the number of spikes.

of the domain-wall motion. At every maximum of $\|\mathbf{u}(t+\Delta t)-\mathbf{u}(t)\|$ a new spike occurs. A pair of spikes will always be created since there are two domain walls. At the same time the nonlocality μ_{eff} decreases rapidly. The time intervals between two successive spike creations become larger and larger because the averaged domain wall velocity decreases. After the creation of the last pair of spikes the domain wall comes to rest. For $t>500$ the nonlocality μ_{eff} remains constant. But a local process remains: Phase diffusion in the spatially periodic domain (Fig. 7).

The acceleration and deceleration of the domain-wall motion indicate that the domain wall "feels" the periodicity of the nonuniform domain state. This is a generic behavior of a domain wall between a uniform state and a spatially periodic state.[18] As consequence of this behavior the averaged domain-wall velocity $v^{\text{wall}}(\mu_{\text{eff}})$ of the local system is zero in a *finite* interval of μ_{eff}, i.e., the domain wall is pinned. In a numerical experiment we are able to measure this pinning interval by varying μ_{eff} until an instability occurs, and either a spike appears of disappears. Figure 8 shows the result for a smaller system than the above-mentioned $[L=10(2\pi/k_c)]$. Thus, $v^{\text{wall}}=0$ for $-0.0468<\mu_{\text{eff}}<-0.0396$. In both the local and the nonlocal systems, the domain-wall pinning leads to multistability between states with different numbers of spikes, where the number of states is proportional to L and to the width of the interval of μ_{eff} where $v^{\text{wall}}=0$. Furthermore, multistability is also caused by the fact that the number and the positions of spatially periodic domains may be varied. In the extreme case single-spike domains may be randomly distributed over the system if the Kawasaki-Ohta forces between the domain walls are smaller than the pinning force.[19]

Multistability is a very common feature of current instabilities. Many experiments[6,20–22] show that multistability is caused by the occurrence of filaments (i.e., narrow current trails where the current density is much higher than elsewhere) which occur randomly everywhere in the sample. By identifying filaments with spikes the analogy is evident. But the matter is not as simple as one would expect because one has to keep in mind that the two-layer model has two different uniform states if $\gamma>1$. In this case (which is the more typical one) it is difficult to distinguish between spikes which are small uniform domains and spikes which are spatially periodic domains with one oscillation.

V. CONCLUSION

In this paper the influence of nonlocalities on the dynamics of domains and domain walls has been studied. The existence of domains implies that the local version of the equation of motion (i.e., the nonlocalities are treated as control parameters) shows bistability or even multistability. Two systems of first-order differential equations, called reductions 1 and 2, were introduced. Reduction 1 [Eqs. (2.4) and (2.5) with (2.3)] describes the dynamics of nonorthogonal modes (orthogonal modes do not contribute to nonlocalities) of the domain states coupled by the nonlocalities and the dynamics of the domain sizes. The time scales of the domain states do not depend on L, whereas the time scales of domain sizes are proportional to L. Thus, for large L the nonorthogonal modes can be adiabatically eliminated. This yields reduction 2 [Eq. (2.7) with (2.6)] which is a closed set of first-order differential equations for the domain sizes. Reduction 2 may be interpreted as the first term of a $1/L$ expansion.

The adiabatical elimination yielding reduction 2 is correct as long as all nonorthogonal modes of the domain states are damped (nonlocal stability of the domain states). Nonlocal instabilities are possible even though every single domain state is locally stable. For the ballast resistor, for example, this may be possible if $d_T R(T_{i0})<0$. Nonlocal stability depends clearly on the actual domain sizes. Therefore, it is conceivable that the dynamics governed by reduction 2 moves the system into a state where the validity condition of reduction 2 fails. Using reduction 1 in this case the system may be described qualitatively very well. But, as we have seen in Sec. III C, quantitative results may be incorrect since the domain-wall velocities used in (2.5) are generally wrong because they are computed for *stationary* domain states.

It should be noted that the idea which leads to reduction 2 may be found implicitly and some times explicitly in the literature, e.g., for electrothermal instabilities see Refs. 5 and 13, and for spiral turbulence in Taylor-Couvette systems see Ref. 23. But it is the first time that reduction 2 is generally discussed, and almost more important, its limitations are shown.

ACKNOWLEDGMENTS

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APPENDIX: LINEAR STABILITY ANALYSIS OF THE TWO-DOMAIN STATE OF THE BALLAST RESISTOR

In this appendix we compute the stationary two-domain state and its stability for heat loss $A(T)$ and resistivity $R(T)$ defined by (3.21) and (3.22), respectively. We do this only for reduction 1 [i.e., (3.7) and (3.8) with (3.9)]. The results, especially for λ_q, for reduction 2 can be obtained by taking the limit $1/L \to 0$.

Because of the symmetry of $R(T)$ the equal-area rule leads immediately to

$$I_{12}=\sqrt{T_0}\ . \tag{A1}$$

Thus, the stationary state of reduction 1 is given by

$$T_{10/20}=T_0 \mp \Delta T\sigma,\quad q_0=\frac{1}{2}\left[1+\frac{E\sqrt{T_0}-T_0}{\Delta T\sigma}\right], \tag{A2}$$

where $\rho=rT_0/\Delta T$ and $\sigma=(\rho^2-1)^{1/2}$. Next, we linearize reduction 1:

$$\frac{d}{dt}\begin{bmatrix}\delta T_1\\ \delta T_2\\ \delta q\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12} & b_1\\ a_{21} & a_{22} & b_2\\ c_1 & c_2 & d\end{bmatrix}\begin{bmatrix}\delta T_1\\ \delta T_2\\ \delta q\end{bmatrix}, \tag{A3}$$

with

$$a_{ij}=\lambda_i^{\mathrm{loc}}\delta_{i,j}-\frac{2I_{12}^3}{E}q_j R_i d_T R_j\ , \tag{A4}$$

$$b_i=-\frac{2I_{12}^3}{E}(R_2-R_1)R_i\ , \tag{A5}$$

$$c_i=-\frac{1}{L}\frac{dv_{21}^{\mathrm{wall}}}{dI}\bigg|_{I_{12}}\frac{I_{12}^2}{E}q_i d_T R_i\ , \tag{A6}$$

$$d=-\frac{1}{L}\frac{dv_{21}^{\mathrm{wall}}}{dI}\bigg|_{I_{12}}\frac{I_{12}^2}{E}(R_2-R_1)\ , \tag{A7}$$

where the definitions (3.14) with $q=q_0$ and $I=I_{12}$ are used. The slope of $v_{21}^{\mathrm{wall}}(I)$ at $I=I_{12}$ can be given without knowing the domain-wall solution:

$$\frac{dv_{21}^{\mathrm{wall}}}{dI}\bigg|_{I=I_{12}}=\frac{\partial_I\phi(T_{20})|_{I=I_{12}}-\partial_I\phi(T_{10})|_{I=I_{12}}}{\int_{T_{10}}^{T_{20}}\{2[\phi(T_{10/20})-\phi(T)]\}^{1/2}dT}=\frac{4\sqrt{T_0}\sigma}{\Delta T[\rho\sigma-\ln(\rho+\sigma)]}\ . \tag{A8}$$

This result is easily found by interpreting the equation of motion (3.1) in the comoving frame of the domain wall as an equation of motion of a damped (damping constant $=v^{\mathrm{wall}}$) particle in the potential ϕ. Solving the characteristic polynomial of (A3) leads to the eigenvalues

$$\lambda_{1/2}=\frac{1+\rho^2}{2\rho^2}\left\{-1+\frac{\lambda_q^{(1)}\rho^2}{\rho^2-1}L^{-1}\pm\left[1+\frac{2\lambda_q^{(1)}\rho^2(\rho^2-3)}{\rho^4-1}L^{-1}+\left[\frac{\lambda_q^{(1)}\rho^2}{\rho^2-1}L^{-1}\right]^2\right]^{1/2}\right\}, \tag{A9}$$

$$\lambda_3=-1+\frac{1}{\rho^2} \tag{A10}$$

where

$$\lambda_q^{(1)}=\frac{-8\sqrt{T_0}\sigma^4}{E(\rho^2+1)[\rho\sigma-\ln(\rho+\sigma)]}\ . \tag{A11}$$

Expanding the largest eigenvalue $\lambda_q \equiv \lambda_1$ (which characterizes the relaxation time of a domain wall) in powers of L^{-1} we get

$$\lambda_q=\lambda_q^{(1)}L^{-1}+O(L^{-2})\ . \tag{A12}$$

For reduction 2 λ_q is given by the leading term of (A12).

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