Solutions of a generalized Emden equation and their physical significance

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A generalized Emden equation is shown to appear in the description of a wide variety of critical systems as a reduced equation for multidimensional kinetic equations for the order parameter. We have surveyed some of the known solutions and found a number of types of physical behavior such as singular, decaying, and oscillatory damped. A new method of finding appropriate solutions has been given together with, in some cases, severe restrictions on its use. An important link between a generalized Emden equation and an autonomous equation for a damped anharmonic oscillator has been found and exploited.

I. INTRODUCTION

Historically, the Emden equation

$$\frac{d^2\phi}{dr^2}+\frac{2}{r}\frac{d\phi}{dr}+\mu\phi^n=0 \tag{1}$$

was first studied by Emden in 1907 (Ref. 1) in the context of the thermodynamics of interacting gas molecules in a spherical cloud, where ϕ represents the gravitational potential of the gas, μ is an empirical constant, n is an integer, and r is the distance from the center of the cloud. It appears also as an equation describing white dwarf stars.[2] An anharmonic oscillator with a damping force proportional to velocity may also be modeled using an Emden-like equation. In these earlier studies relatively few solutions have been found analytically, although series expansions and numerical integration techniques have been used extensively.[3]

In several recent publications a generalized Emden equation has found new applications to the kinetics of critical phenomena.[4–8] There appear to be two main approaches to these types of problem, both based on Landau-Ginzburg (LG) phenomenology.[9] The first postulates a Hamiltonian or Lagrangian density as a series expansion of the potential energy $V(\phi)$ with a kinetic energy and a term describing inhomogeneities. Here ϕ is an order-parameter field that may either be real or complex. This then leads to Euler-Lagrange equations (ELE) for the formulation of kinetics of the critical system, which will either be a nonlinear Klein-Gordon equation (NLKGE) or a nonlinear Schrödinger equation (NLSE). In the case of free-energy densities, a similar expansion is postulated as the starting point, and the relaxation dynamics of the order parameter is then derived in the form of a time-dependent Landau-Ginzburg equation (TDLGE). The above equations have been recently analyzed using the method of symmetry reduction (MSR).[4–9]

A number of critical systems can be described using the real order parameter ϕ and the following Hamiltonian density:[10]

$$H_1=\tfrac{1}{2}m(\phi_t)^2+\tfrac{1}{2}D(\nabla\phi)^2+A_2\phi^2+A_4\phi^4+A_6\phi^6 , \tag{2}$$

where m is the mass, $A_2=\alpha(T-T_c)$, and the transition is of second order if $A_4>0$ and of first order when $A_4<0$. The constant D is related to the nearest-neighbor interactions. The kinetics is described by ELE in the form of the NLKGE below,

$$\mathcal{D}_\epsilon\phi=-2(A_2\phi+2A_4\phi^3+3A_6\phi^5)\equiv F(\phi) , \tag{3}$$

where $\mathcal{D}_\epsilon\equiv\partial^2/\partial x_0^2+\epsilon\sum_{i=1}^{3}\partial^2/\partial x_i^2$ and $\epsilon=-\mathrm{sgn}(D)$ is a signature. The independent variables are defined as

$$x_0=m^{-1/2}t,\quad (x_1,x_2,x_3)=|D|^{-1/2}(x,y,z) .$$

Analysis of Eq. (3) using the MSR (Ref. 5) resulted in a large number of geometries for the solution space. Here, we are particularly interested in solutions leading to a generalized Emden equation of the form

$$\frac{d^2\phi}{d\xi^2}+\left[\frac{k}{\xi}\right]\frac{d\phi}{d\xi}=\lambda F(\phi)=-g(\phi) , \tag{4}$$

where ξ is a symmetry variable and the coefficients k and λ depend on the particular reduction. In general,[5] for arbitrary values of the parameters A_2, A_4, and A_6 the following cases are of particular physical interest.

(a) For $\epsilon=+1$,

$$\xi=(x_0^2+x_1^2+\cdots+x_k^2)^{1/2}\quad\text{for } 0\le k\le 3,\quad \lambda=1 .$$

This represents contracting unidirectional, circular, or spherical solutions (order-parameter structures) for $k=1$, 2, and 3, respectively.

(b) For $\epsilon=-1$,

$$\xi=(x_0^2-x_1^2-\cdots-x_k^2)^{1/2}\quad\text{for } 0\le k\le 3,\quad \lambda=1$$

which represents expanding analogs of (a).

(c) Also, for $\epsilon=-1$,

$$\xi=(x_1^2+x_2^2+\cdots+x_{k+1}^2)^{1/2}\quad\text{for } 0\le k\le 2,\quad \lambda=-1 .$$

This is a stationary hyperspherical type of symmetry variable.

(d) Also, for $\epsilon=-1$,

$$\xi=\left[\sum_{i=1}^{k+1}(\mathbf{A}_i,\mathbf{x}-\mathbf{B})^2\right]^{1/2} \quad 0\le k\le 1,\quad \lambda=-1\ ,$$

$(\mathbf{A}_i,\mathbf{A}_j)=-\delta_{ij};1\le i,\ j\le k+1$, where $\mathbf{A}_i$ and $\mathbf{B}$ are arbitrary vector functions of x_0+x_3. This is a so-called degenerate symmetry variable. We wish to emphasize that in all these cases, (a)–(d), one should reduce Eq. (3) to the ordinary differential equation (ODE), Eq. (4).

In addition, we find specific reductions that arise precisely at the tricritical point, i.e., for $A_2=A_4=0$. One of them is $\xi=(x_0^2-x_1^2-\cdots-x_k^2)(x_0+x_1)^q$, $k=1,2,3$ for $\epsilon=-1$. The solution is then expressed as $\phi=\rho(\mathbf{x})f(\xi)$, where

$$\rho=[(2q+1)/6A_6]^{1/4}(x_0+x_1)^{q/2},\quad q\neq-\tfrac{1}{2}$$

and $f(\xi)$ satisfies a special form of Eq. (4),

$$\frac{d^2f}{d\xi^2}+\frac{(3q+k)}{(2q+1)}\left[\frac{1}{\xi}\right]\frac{df}{d\xi}+f^5=0\ . \tag{5}$$

Due to the nature of q, each solution of (5) represents an infinite family of solutions leading to distinct (curved) geometries of the order-parameter field.

When ϕ is complex[6] the Hamiltonian density is

$$H_2=\tfrac{1}{2}m|\phi_t|^2+\tfrac{1}{2}D|\nabla\phi|^2+A_2|\phi|^2+A_4|\phi|^4+A_6|\phi|^6\ . \tag{6}$$

The equation of motion then takes the form of a complex NLKGE

$$\mathcal{D}_\epsilon\phi=-2[A_2+2A_4\phi\phi^*+3A_6(\phi\phi^*)^2]\phi\ . \tag{7}$$

Representing the order parameter as $\phi=\eta e^{i\psi}$, where η is the envelope and ψ the carrier wave, one can transform Eq. (7) into an equivalent system of coupled partial differential equations (PDE's), which can then be solved by imposing a special ansatz. For example,[6] by setting $(\nabla\psi)^2=\text{const}$, $\mathcal{D}_\epsilon\psi=0$ and $(\nabla\eta)\cdot(\nabla\psi)=0$ yields the now familiar NLKGE for the envelope η. The three conditions listed above Eq. (8) can be easily satisfied when the envelope and carrier waves propagate in orthogonal directions. Thus, again, we see several possibilities which lead to an Emden-like equation.

When the conjugate momentum of the complex order parameter ϕ is

$$\pi=\frac{\partial\mathcal{L}}{\partial\phi_t}=\frac{-i}{2}\phi^*\ , \tag{8}$$

then the corresponding Lagrangian density may be written as

$$\mathcal{L}=\frac{i}{2}(\phi\phi_t^*-\phi^*\phi_t)-\frac{D}{2}|\nabla\phi|^2-A_2|\phi|^2-A_4|\phi|^4-A_6|\phi|^6\ , \tag{9}$$

and the Hamiltonian density is that of Eq. (6) with $m=0$. The ELE then takes the form of a NLSE

$$i\phi_t=D\Delta\phi-2A_2\phi-4A_4|\phi|^2\phi-6A_6|\phi|^4\phi\ . \tag{10}$$

Recently,[11] this formalism was used to describe a boson gas with two- and three-body interactions. Rotationally symmetric "bubblelike" soliton solutions satisfying an Emden-like reduced ODE have been found numerically. Furthermore, Gagnon and Winternitz[4] have published extensive studies for Eq. (10) using the MSR. A few reductions result in an Emden type of equation for the envelope of ϕ. All of them involve either cylindrical or spherical geometries with $k=1$ or 2 in Eq. (4), respectively.

In this context, a recent result[7,8] links a very general second-quantized Hamiltonian for strongly interacting many-body systems of bosons or fermions near criticality, i.e.,

$$H_3=\sum_{\mathbf{k},l}w_{\mathbf{k},l}q_{\mathbf{k}}^\dagger q_l+\sum_{\mathbf{k},l,\mathbf{m}}\Delta_{\mathbf{k},l,\mathbf{m}}q_{\mathbf{k}}^\dagger q_l^\dagger q_{\mathbf{m}}q_{\mathbf{k}+l-\mathbf{m}}\ , \tag{11}$$

with the NLSE in an exact way because of renormalization theory. Such a generic form of the Hamiltonian is frequently encountered in the physics of superconductors, superfluids, metals, crystal lattices,and magnetic spin systems, to mention but a few. Heisenberg's equation for the creators and annihilators is derived using Eq. (11) followed by the introduction of quantum fields. The equation of motion for the quantum field is then demonstrated to be a generalized NLSE. Therefore on using the MSR, one finds, for example, in cylindrical and spherical geometries, that the reduced ODE's are Emden-like. Through the connection with quantum many-body physics oscillating solutions may be interpreted as elementary excitations, whereas localized solutions lead to coherent structures such as solitons or vortices.

A more phenomenological approach to the kinetics of critical systems is to postulate a LG expansion of the free-energy density G as

$$G=\tfrac{1}{2}D(\nabla\phi)^2+A_2\phi^2+A_4\phi^4+A_6\phi^6\ . \tag{12}$$

The relaxation kinetics of the nonconserved order parameter ϕ is described[12] by the TDLGE

$$\phi_t+\nabla^2\phi=+b\phi+c\phi^3+d\phi^5\ , \tag{13}$$

where t is a scaled time variable and the parameters b, c, and d are also scaled from A_2, A_4, and A_6, respectively. Equation (13) (Ref. 9) and its steady-state analog[13] have also been the subject of analyses using MSR.[13] Two reductions correspond to cylindrical and spherical geometries, for which the reduced equation is of the Emden form.

Thus the generalized Emden equation appears in both Hamiltonian (Lagrangian) and free-energy descriptions of critical systems with real or complex order parameters. The types of reduction that lead to the Emden equation correspond to cylindrical and spherical geometries or, only for the vicinity of the critical or tricritical point, to fairly arbitrary ones as a result of the presence of degenerate symmetry variables.

II. SURVEY OF KNOWN SOLUTIONS

In this section we briefly review the known analytical solutions we are aware of, for Eq. (4).

A. Monomial $g(\phi)=\gamma\phi^n$

Without loss of generality γ may be put equal to unity as a result of scaling the independent or dependent variable.

1. *Special cases with* $n=0$, $n=1$, $k=0$

For special cases with $n=0$, $n=1$, and $k=0$, we have the following.

(a) $n=0$. This applies to Eq. (4) with $g(\phi)=1$. Putting $Z=d\phi/d\xi$ reduces it to a first-order linear ODE with an integrating factor ξ^k. Then, the following explicit solutions are found when $k\neq 1$, $k\neq -1$:

$$\phi=\frac{-\xi^2}{2(k+1)}+\frac{k_0\xi^{-k+1}}{(1-k)}+k_1 , \tag{14}$$

when $k=1$;

$$\phi=\frac{-\xi^2}{4}+k_0\ln\xi+k_2 , \tag{15}$$

when $k=-1$;

$$\phi=\frac{-\xi^2}{2}\ln\xi+\frac{\xi^2}{4}+\frac{k_0\xi^2}{2}+k_3 , \tag{16}$$

where k_0, k_1, k_2, and k_3 are integration constants.

(b) $n=1$. Here Eq. (4) can be cast in the form of the Bessel equation[22]

$$z^2W''+(1-2\alpha)zW'+[(\beta\gamma z^\gamma)^2+\alpha^2-\delta^2\gamma^2]W=0 ,$$

with the solution $W=z^\alpha Z_\delta(\beta z^\gamma)$ and Z_δ is any arbitrary solution of the standard Bessel equation. Hence the solution to Eq. (4) with $n=1$ is

$$\phi=\xi^{(1-k)/2}Z_{\mp(1-k)/2}(\mp\xi) . \tag{17}$$

(c) $k=0$. When $g(\phi)$ is a cubic polynomial this ODE has been studied in Ref. 13 while for a quintic polynomial it has been exhaustively investigated in Ref. 5. However, for a special case when $g(\phi)=\phi^n$ we may integrate directly to obtain for $n\neq -1$

$$2\sqrt{\xi}=\int\frac{d\phi}{[k_0-\phi^{n+1}/(n+1)]^{1/2}} \tag{18}$$

while for $n=-1$

$$\frac{-\xi}{2}=\int\exp(k_0-Z^2)dZ , \tag{19}$$

where $k_0-\ln\phi=Z^2$.

2. *Solution for* $n=0$, $k=2$, $\gamma=1$

Following Davis[3] a general solution is

$$\phi=C_1+\frac{C_2}{\xi}-\frac{\xi^2}{6} , \tag{20}$$

where C_1 and C_2 are arbitrary integration constants.

3. *Solution for* $n=1$, $k=2$, $\gamma=1$

For this case Davis[3] provides the general solution

$$\phi=C_1\frac{\sin\xi}{\xi}+C_2\frac{\cos\xi}{\xi} . \tag{21}$$

To the best of our knowledge there are no solutions for $n=2$.

4. *Solution for* $n=3$

A special solution for $k\neq 2$ and $\gamma<0$, but otherwise arbitrary, is found, for example, in Skierski, Grundland, and Tuszyński,[9]

$$\phi=\mp\sqrt{(2-k)/|\gamma|}\,\xi^{-1} . \tag{22}$$

The $k=3$ case has been studied by Fowler,[14] and he found for $\gamma>0$,

$$\phi(\xi)=\frac{1}{\xi}\left[\frac{2}{\gamma(2-K^2)}\right]^{1/2}\mathrm{dn}\left[\frac{\ln(\xi/\xi_0)}{(2-K^2)^{1/2}},K\right] , \tag{23}$$

where ξ_0 and K are arbitrary constants, K being the Jacobi modulus such that $0\leq K\leq 1$. Since dn is an elliptic function, ϕ represents an oscillating damped solution.

For $K=0$ and $K=1$ one obtains,[14] respectively,

$$\phi(\xi)=\frac{1}{\xi\sqrt{\gamma}},\quad \phi(\xi)=\left[\frac{8}{\xi}\right]^{1/2}\frac{\xi_0}{\xi^2+\xi_0^2} . \tag{24}$$

If $\frac{1}{2}<K\leq 1$ there is another solution given by[15]

$$\phi(\xi)=\frac{1}{\xi}\left[\frac{2K^2}{\gamma(2K^2-1)}\right]^{1/2}\mathrm{cn}\left[\frac{\ln(\xi/\xi_0)}{(2K^2-1)^{1/2}},K\right] . \tag{25}$$

For $\gamma<0$ a general solution may be expressed in terms of

$$\frac{1}{\xi}\mathrm{tn}(\lambda\ln\xi),\quad \frac{1}{\xi}\frac{\mathrm{sn}(\lambda\ln\xi)}{\mathrm{cd}(\lambda\ln\xi)},\quad \frac{1}{\xi}\mathrm{nc}(\lambda\ln\xi) ,$$

where λ is a suitably chosen constant. All these solutions, unfortunately have periodic singularities.

5. *Solution for* $n=5$

Davis[3] has given a particular solution for $k=2$ and $\gamma=1$ which depends on one arbitrary constant C_1 namely,

$$\phi=[3C_1/(\xi^2+3C_1^2)]^{1/2} . \tag{26}$$

For $k=2$ Gagnon and Winternitz[4] have provided a general solution using a sequence of two substitutions of independent and dependent variables and the type of solution obtained depends on the types of roots of the polynomial

$$P(W)=W^4+\frac{3}{4\gamma\lambda_0^2}W^2+C_2W$$
$$\equiv W(W-W_1)(W-W_2)(W-W_3) , \tag{27}$$

where λ_0 and C_2 are arbitrary integration constants. When $P(W)$ has a double root, say W_1, then the solution is

$$\phi=\left[\frac{\lambda_0}{\xi}\frac{\alpha+\beta\cos\tau}{\nu+\delta\cos\tau}\right]^{1/2} \quad \text{with } \gamma<0\ , \tag{28}$$

where α, β, ν, and δ are constants given by Gagnon and Winternitz[4] and τ is

$$\tau=(-8\gamma W_1^2\lambda_0^2-1)^{1/2}\ln\left[\frac{\xi}{\xi_0}\right] .$$

When $C_2=0$ and $\gamma<0$ Gagnon and Winternitz[4] recover Eq. (26) using their method. If $P(W)$ has four distinct real roots such that $W_1>0>W_2>W_3$ the solution becomes

$$\phi=\left[\frac{\lambda_0}{\xi}\frac{[W_3(W_2-W_1)-W_1(W_2-W_3)\mathrm{sn}^2(\tau,K)]}{[W_2-W_1-(W_2-W_3)\mathrm{sn}^2(\tau,K)]}\right]^{1/2} , \tag{29}$$

where $\gamma,\lambda_0<0$ and τ depends logarithmically on ξ.

Two more solutions are found when $P(W)$ has two real roots (W_0 and W_1) and two complex conjugate ones W_2 and W_3. They take the form

$$\phi=\left[\frac{\lambda_0}{\xi}\frac{[\epsilon W_0A+W_1B+(W_0A-\epsilon W_1B)\mathrm{cn}(\tau,K)]}{\epsilon A+B+(A-\epsilon B)\mathrm{cn}(\tau,K)}\right]^{1/2} . \tag{30}$$

Constants A, B, and K are given in Ref. 4 and τ is also a logarithmic function of ξ. It is required that $W_1>W_0=0$, $\gamma<0$, and $\epsilon=1$ or $W_1\geq 0$, $W_0=0$, $\gamma>0$, and $\epsilon=-1$.

Equations (29) and (30) represent functions which are singular at $\xi=0$, finite everywhere else, and exhibit damped oscillations. A similar behavior is shown in Eq. (30) with $\gamma<0$ and otherwise singular.

B. Polynomial $g(\phi)$

In the general case

$$-g(\phi)=\alpha\phi+\beta\phi^3+\gamma\phi^5\ , \tag{31}$$

where α, β, and γ are constants, the existence of particle-like (localized) solutions free of singularities at the origin has been proved[16–18] for both $k=1$ and 2, provided $\alpha>0$ and one of the following conditions is satisfied: (i) for $k=1$, $\gamma<0$; for $k=2$, $\gamma=0$ and $\beta<0$; or (ii) for $k=1$ and 2, $\gamma>0$ and $\beta<-(16\alpha\gamma/3)^{1/2}$.

More importantly, a Painlevé analysis was performed[5] for Eq. (4) and it was found that only the following special cases have the Painlevé property[26] (i.e., no movable critical points). (i) $k=0$, giving an equation which is easily integrated by quadratures and the solutions are expressed in terms of elliptic functions. (ii) $k=2$, with $\alpha=\beta=0$. This case has been analyzed by Gagnon and Winternitz.[4] (iii) $k=3$, with $\alpha=\gamma=0$. Fowler[14] provided a complete state of solutions for this equation.

A special solution for arbitrary k and $-g(\phi)=\delta\phi^2+\beta\phi^3$ ($\beta<0$, $\delta>0$) has recently been found[19] as

$$\phi(\xi)=\left[\frac{4}{3-k}\right]\left[\frac{-\delta}{\beta}\right]\left[1+\frac{2}{(3-k)^2}\left[\frac{-\delta^2}{\beta}\right]\xi^2\right]^{-1} . \tag{32}$$

For $k=2$ and $-g(\phi)=\alpha\phi+\gamma\phi^5$ a solution has been obtained[20] of the type

$$\phi(\xi)=\left[\left[\frac{\lambda}{\gamma}\right]^{1/4}\xi_0\right]^{-1/2}\left[\frac{\xi}{\xi_0}\right]^{1/2}Z(\tau)\ , \tag{33}$$

where λ, γ, and ξ_0 are real constants, $\tau=(\lambda/\gamma)^{1/4}\ln(\xi/\xi_0)$, and Z is an elliptic function satisfying the equation

$$\left[\frac{dZ}{d\tau}\right]^2+\mu Z^2+\frac{\gamma}{3}Z^6=C_1\ .$$

Series solutions of Eq. (4) have also been obtained subject to appropriate boundary conditions.[3] Clearly, constant solutions can be obtained which satisfy $g(\phi)=0$ and it appears that some of them play the role of "attractors" for damped oscillating or decaying solutions. This seems to be supported by a number of numerical studies elaborated on elsewhere.[3,21]

Equation (4) with $g(\phi)=\phi^n$ actually passes the Painlevé test for many values of k and n, e.g.,

$$(k,n)=(2,5),(3,3),(4,\tfrac{7}{3}),(5,2),(6,\tfrac{9}{5}),\ldots,$$

and hence systematic techniques are available to then find solutions.[26] Incidentally, it is of interest to note that the equations of motion of the form of Eq. (3) become conformally invariant[27] precisely when

$$n=\frac{k+3}{k-1}\ . \tag{34}$$

III. FURTHER RESULTS AND NEW METHOD OF CALCULATION

A. Solutions of the form $\phi=\mu\xi^s$

Here we seek solutions of the above form to Eq. (4) with $g(\phi)=\phi^n$ and $n\neq 1$. Substituting $\phi(\xi)$ into (4) gives

$$\mu s(s-1)\xi^{s-2}+k\mu s\xi^{s-2}+\mu^n\xi^{ns}=0\ . \tag{35}$$

Equation (35) requires that

$$s=\frac{2}{1-n},\quad \mu=\left[\frac{-2(1-k)}{(n-1)}-\frac{4}{(n-1)^2}\right]^{1/(n-1)} , \tag{36}$$

and it is easy to see that μ becomes zero when $k=(n+1)/(n-1)$. Clearly, this solution includes Eq. (22) as a special case. Also note that this solution asymptotically solves the generalized Emden-like equation, Eq. (4), with the polynomial $g(\phi)=\sum_{m=1}^{n}A_m\phi^m$ close to the critical point $\xi=0$.

B. A generalization of the Fowler method

We now consider Eq. (4) with an *a priori* arbitrary value of k and n and seek solutions in the form

$$\phi=\psi(\xi)\xi^{-s} \quad \text{with } \xi=\exp(\lambda\omega), \quad \lambda\neq 0$$

choosing s so that, after substituting in Eq. (4), the coefficient of $d\psi/d\omega$ is zero. This results in the relation, $1+2s=k$, and the ODE becomes

$$\frac{d^2\psi}{d\omega^2}+A\psi+B\psi^n\exp[\lambda\omega(2+s-sn)]=0 \ , \tag{37}$$

where $A=\lambda^2[s(s+1)-sk]$ and $B=\lambda^2$. In order to make Eq. (37) autonomous we demand that the argument of the exponential vanish, which means $2+s-sn=0$. This latter condition, together with $k=1+2s$, results in Eq. (34). This method generates an infinite number of sets of solutions for the Emden equation, provided only that Eq. (34) holds. In particular, the two very special cases when either $k=2$, $n=5$, or $k=3$, $n=3$, have already been discussed in Sec. II following Refs. 4 and 14, respectively. By multiplying both sides of Eq. (36) by $d\psi/d\omega$ and integrating gives the implicit form

$$\int\frac{d\psi}{\left[\left[\frac{2}{n-1}\right]^2\psi^2+\frac{-2}{(n+1)}\psi^{n+1}+C\right]^{1/2}}=\mp\ln(\xi/\xi_0) \quad \text{for } n\neq -1 \tag{38}$$

or

$$\int\frac{d\psi}{(\psi^2-2\ln\psi+C)^{1/2}}=\mp\ln(\xi/\xi_0) \quad \text{for } n=-1 \ , \tag{39}$$

where C and ξ_0 are integration constants.

C. A new method

Here we use Eq. (4) with $g(\phi)=\phi^n$ where $n\neq 1,0$. We first multiply Eq. (4) through by ξ^k to obtain

$$\frac{d}{d\xi}(\xi^k\phi')=-\xi^k\phi^n \ . \tag{40}$$

Putting $e^V\equiv\xi^k\phi^n$ we differentiate to obtain

$$\frac{dV}{d\xi}=\frac{k}{\xi}+\frac{n\eta\phi^{n-1}}{e^V} \quad \text{where } \eta\equiv\xi^k\phi' \ . \tag{41}$$

Solving Eq. (41) for η we obtain

$$\eta=\frac{e^V}{n\phi^{n-1}}\left[\frac{dV}{d\xi}-\frac{k}{\xi}\right] . \tag{42}$$

Substituting η from Eq. (42) into the left-hand side of Eq. (40), and using the definition of V, produces

$$\frac{d}{d\xi}\left[\frac{e^V}{n\phi^{n-1}}\left[\frac{dV}{d\xi}-\frac{k}{\xi}\right]\right]=-e^V \ , \tag{43}$$

which on differentiation gives

$$\frac{dV}{d\xi}\left[\frac{dV}{d\xi}-\frac{k}{\xi}\right]+\frac{d^2V}{d\xi^2}+\frac{k}{\xi^2}+\left[(1-n)\frac{\phi'}{\phi}\left[\frac{dV}{d\xi}-\frac{k}{\xi}\right]\right]=-n\phi^{n-1} \ . \tag{44}$$

We now use Eq. (41) and differentiate again to obtain

$$\frac{d^2V}{d\xi^2}=-\frac{k}{\xi^2}+n\phi''\phi^{-1}+n\phi'(-1)\phi^{-2}\phi' \ . \tag{45}$$

By substituting for ϕ'' from our original Eq. (4) and ϕ'/ϕ from Eq. (41) we can obtain a relationship between the second and first derivatives of V, the independent variable ξ, and the entity $n\phi^{n-1}$. If the value of $n\phi^{n-1}$, obtained in this way, is substituted back into Eq. (44), with ϕ'/ϕ from Eq. (41), we simply obtain an equation for zero. Thus, we must evaluate the third derivative of V from Eq. (45), namely,

$$\frac{d^3V}{d\xi^3}=\left[\frac{2k}{\xi^3}+\frac{nk}{\xi^2}\left[\frac{\phi'}{\phi}\right]+\frac{k^2n}{\xi^2}\left[\frac{\phi'}{\phi}\right]+\frac{3nk}{\xi}\left[\frac{\phi'}{\phi}\right]^2+2n\left[\frac{\phi'}{\phi}\right]^3\right]+n\phi^{n-1}\left[\frac{k}{\xi}+(-n+3)\left[\frac{\phi'}{\phi}\right]\right] . \tag{46}$$

At this point $n\phi^{n-1}$ is determined from Eq. (46), where (ϕ'/ϕ) is replaced by $(1/n)(dV/d\xi-k/\xi)$ from Eq. (41) and substituted into Eq. (44). This results in the following rather complicated relationship:

$$\left\{\frac{dV}{d\xi}\left[\frac{dV}{d\xi}-\frac{k}{\xi}\right]+\left[\frac{d^2V}{d\xi^2}+\frac{k}{\xi^2}\right]+\left[\frac{(1-n)}{n}\left[\frac{dV}{d\xi}-\frac{k}{\xi}\right]^2\right]\right\}\left[\frac{(-n+3)}{n}\left[\frac{dV}{d\xi}-\frac{k}{\xi}\right]+\frac{k}{\xi}\right]$$
$$=\frac{-d^3V}{d\xi^3}+\frac{2k}{\xi^3}+\frac{(k+k^2)}{\xi^2}\left[\frac{dV}{d\xi}-\frac{k}{\xi}\right]+\frac{3k}{n\xi}\left[\frac{dV}{d\xi}-\frac{k}{\xi}\right]^2+\frac{2}{n^2}\left[\frac{dV}{d\xi}-\frac{k}{\xi}\right]^3 . \tag{47}$$

The substitution

$$p=\frac{dV}{d\xi}-\frac{k}{\xi} \tag{48}$$

reduces Eq. (47) to

$$\frac{d^2p}{d\xi^2}+\frac{dp}{d\xi}\left[\frac{(3-n)p}{n}+\frac{k}{\xi}\right]-p\frac{k}{\xi^2}+p^2\frac{k(1-n)}{n\xi}+p^3\frac{(1-n)}{n^2}=0 \ . \tag{49}$$

A further substitution

$$p=\frac{u}{\xi} \tag{50}$$

transforms Eq. (49) into the equation

$$\xi^2\frac{d^2u}{d\xi^2}+\xi\frac{du}{d\xi}\left[k-2+\frac{(3-n)u}{n}\right]+u(2-2k)+u^2\left[-\left[\frac{3-n}{n}\right]+\frac{k(1-n)}{n}\right]+u^3\left[\frac{1-n}{n^2}\right]=0\ . \tag{51}$$

A Euler-like change of independent variable, namely,

$$\xi=\exp Z\ , \tag{52}$$

gives our final equation

$$\frac{d^2u}{dZ^2}+\frac{du}{dZ}\left[k-3+\frac{(3-n)}{n}u\right]+u(2-2k)+u^2\left[\frac{(n-3)}{n}+\frac{k(1-n)}{n}\right]+u^3\left[\frac{1-n}{n^2}\right]=0\ , \tag{53}$$

where u only appears with powers 1 to 3 inclusive and the original power n appears as part of a coefficient with the other parameter k. Equation (53) is, first of all, an autonomous equation and, in fact, is a particular form of the damped anharmonic oscillator equation. From a solution $u(Z)$ of Eq. (53) we generate a solution of Eq. (4) by using Eqs. (52), (48), and the definition of V to obtain

$$\phi(\xi)=\frac{1}{K_0^{1/n}}\exp\left[\frac{1}{n}\int_{\xi_0}^{\xi}\frac{u(\ln\xi')}{\xi'}d\xi'\right]\ , \tag{54}$$

where K_0 is an integration constant from the integration of Eq. (48) (which is not arbitrary). It is easy to check that ϕ in Eq. (53) may be a solution of Eq. (4) by direct substitution. Thus the remaining task is to find as many solutions of Eq. (53) as possible. In fact, Eq. (54) involves an integration constant K_0 which is not free but may be subject to a stringent consistency check when inserted in Eq. (4). Another point worth making is that, if $u(Z)$ is a solution of Eq. (53), so is $u(Z+Z_0)$ where Z_0 is an arbitrary constant.

Equations of type Eq. (53) frequently have the Painlevé property and can be solved. We have attempted to find conditions on Eq. (53) for which it would possess this property using standard approaches.[25,26] However, due to the fact that the coefficients appearing in Eq. (53) are interrelated, we have been unable to find a set of values of n and k, other than those mentioned earlier, for which Eq. (53) reduces to one of the Painlevé types. We therefore try another approach to find special solutions.

1. Ansatz solutions

We solve here Eq. (53) in the form

$$u''+u'(b_0+b_1u)+a_1u+a_2u^2+a_3u^3=0\ . \tag{55}$$

The method we use was developed by Otwinowski, Paul, and Laidlaw,[23] and is based on seeking kinklike solutions to Eq. (55) which simultaneously satisfy

$$u'=Au^2+Bu+C\ , \tag{56}$$

where A, B, and C are constants to be determined self-consistently. We first differentiate Eq. (56) with respect to Z to obtain u''. This, together with u' from Eq. (56), is inserted into Eq. (55) and equal powers of u are equated for compatability. This results in four simultaneous equations on the set of parameters. One of these conditions yields that either $C=0$ or $B=-b_0$.

If $C=0$, then this results in four separate cases, all with $C=0$:

(i) $A=\dfrac{n-1}{2n},\quad B=2\ ,$

(ii) $A=\dfrac{-1}{n},\quad B=2\ ,$

(iii) $A=\dfrac{n-1}{2n},\quad B=1-k\ ,$

(iv) $A=\dfrac{-1}{n},\quad B=1-k\ .$

On the other hand, if $B=-b_0$ (C is not necessarily zero), then we find the following two cases:

(v) $A=\dfrac{n-1}{2n},\quad B=-b_0,\quad C=n(k-1)\ ,$

(vi) $A=\dfrac{-1}{n},\quad B=-b_0,\quad C=\dfrac{2n(k-1)}{1-n}\ .$

In each of these cases a substitution into an outstanding equation gives a relationship between n and k as follows.

(i) $n=(3+k)/(k-1)$ for $k\neq1$.

(ii) $n=0$ or $k=-1$ (the case $n=0$ having been solved earlier).

(iii) $k=0$ or $n=-1$ (the former already discussed).

(iv) This is true for *any* n and any k subject only to $n\neq0$ ($n=0$ has been discussed earlier).

(v) $k\neq1$ and $n=(3-k)/(k-1)$.

(vi) $n=0$ (considered earlier).

Hence, there are only five cases to consider. We investigate each of these cases in turn and integrate Eq. (56) to give

$$\text{(i)}\quad u=\frac{(3+k)e^{2Z}}{K_1-e^{2Z}},\quad k\neq1,\quad n=\frac{k+3}{k-1}\ , \tag{57a}$$

$$\text{(ii)}\quad u=\frac{2ne^{2Z}}{e^{2Z}-nK_1},\quad k=-1\ , \tag{57b}$$

$$\text{(iii)}\quad u=\frac{(k-1)e^{Z(1-k)}}{e^{Z(1-k)}-K_1},\quad n=-1\ , \tag{57c}$$

$$\text{(iv a)}\quad u=\frac{n(1-k)e^{Z(1-k)}}{K_1+e^{Z(1-k)}} \tag{57d}$$

for $k \neq 1$, $n \neq 0$ (otherwise any n and k) ,

or

$$\text{(iv b)}\quad u=\frac{n}{Z+K_1}\quad \text{for } k=1,\ n\neq 0 \tag{57e}$$

and

$$\text{(v)}\quad u=\frac{n}{(1-n)}\left[(3-k)+|k-1|\tanh\left[\frac{|k-1|(Z-Z_0)}{2}\right]\right]$$

provided $n\neq 0,\ k\neq 1,\ n=\frac{3-k}{k-1}$ (57f)

where K_1 and Z_0 are arbitrary constants. We now calculate the final result for each of the cases above. From each of Eqs. (57a)–(57e), the final result is in the form

$$\phi(\xi)=K_0^{-1/n}(\mu_1+\mu_2\xi^{\lambda})^{\gamma}\ , \tag{58}$$

where $\gamma=\mu_3/\mu_2\lambda n$. Here μ_1, μ_2, μ_3, λ, and γ are fixed constants, except μ_1, which is proportional to an arbitrary integration constant K_1. That is,

(i) $\lambda=2,\ \mu_1=-K_1,\ \mu_2=-1,\ \mu_3=-(3+k)$,

(ii) $\lambda=2,\ \mu_1=-nK_1,\ \mu_2=+1,\ \mu_3=2n$,

(iii) $\lambda=1-k,\ \mu_1=-K_1,\ \mu_2=+1,\ \mu_3=k-1$,

(iv a) $\lambda=1-k,\ \mu_1=+K_1,\ \mu_2=+1,\ \mu_3=n(1-k)$.

In case (iv b) the solution may be obtained separately to give

$$\phi(\xi)=\frac{1}{K_0^{1/n}}\left[\frac{\ln\xi+K_1}{\ln\xi_0+K_1}\right] . \tag{59}$$

For case (v) we find

$$\phi=(K_0')^{-1/n}\left[\frac{\xi}{\xi_0}\right]^{(3-k)/(1-n)}\left[\left[\frac{\xi}{x_0}\right]^{|k-1|/2}-\left[\frac{\xi}{x_0}\right]^{-|k-1|/2}\right]^{2/(1-n)}, \tag{60}$$

where

$$K_0'=K_0\left[\left[\frac{\xi_0}{x_0}\right]^{|k-1|/2}-\left[\frac{\xi_0}{x_0}\right]^{-|k-1|/2}\right]^{2n/(1-n)}$$

and $Z_0=lnx_0$.

The solutions we have now derived are still subject to verification in the original equation. Upon inspection we find that case (i) is allowed unless $k=-1$ and $n=-1$ (assuming u is not a constant). Cases (ii), (iii), and (iv) are *not* acceptable under any circumstances. Case (v) is, however, acceptable and Eq. (60) represents a new solution.

2. *The Fowler solution after our transformations*

The power of our method is best seen by considering Eq. (53) with a special set of values k and n, namely, $n=3$ and $k=3$, which reduces it to an elliptic form on subsequent integration. Thus the Fowler solution is recovered in a straightforward way.

3. *Constant solutions*

Equation (53) also possesses the three constant solutions

$$u_0=0,\quad u_1=\frac{2n}{1-n},\quad u_2=n(1-k)\ . \tag{61}$$

However, before substituting these solutions into Eq. (54), in order to obtain $\phi(\xi)$, we must, once again, make sure that they do not violate self-consistency requirements. In fact, $u_0=0$ and $u_2=n(1-k)$ both violate this condition always. However, the solution $u_1=2n/(1-n)$ only violates this condition if $k=(n+1)/(n-1)$. Using Eq. (54) then gives

$$\phi(\xi)=K_0^{-1/n}(\xi/\xi_0)^{2/(1-n)}\ ,$$

and K_0 must have a particular value which agrees exactly with Eq. (22) and the restriction on k above it.

4. *The Leach method*

It has been pointed out by Leach, Feix, and Bouquet[24] that the equation

$$y''+3a(x)yy'+b(x)y'+a^2(x)y^3+c(x)y^2+d(x)y+e(x)=0 \tag{62}$$

may be transformed into a linear second-order equation by means of a point transformation. Identifying Eq. (62) with our Eq. (53) we see that the latter will be of this form when $n=-3$ (or $n=0$, a case discussed earlier). In principle, therefore, we can find solutions of the original Emden equation using our special transformation.

IV. CONCLUSIONS

In this paper we have surveyed a generalized Emden equation, which plays a very important role in the kinetics of multidimensional critical systems. Those reductions of the order-parameter equations that lead to Emden-type equations involve either stationary or time-dependent unidirectional, cylindrical, spherical, or even, in some cases, so-called degenerate variable structures. Only the cases with $n=3$, $k=3$, and $n=5$, $k=2$, have been completely solved until now. However, the method of Fowler has been shown to produce a reduction to quadratures in the more general case when $n=(k+3)/(k=1)$. In the other cases the results are rather fragmentary. Based on the former cases one might expect three general types of behavior, namely, singular, decaying, or damped oscillatory. For polynomial nonlinearities in the equation, linearization around stable mean-field solutions is probably the most viable general approach at the moment.

Finally, a new method has been presented which involves a substitution of independent and dependent variables. The resultant equation Eq. (53), is apparently not of the Painlevé type. We have examined the Painlevé property of this equation by comparing it with a detailed analysis provided by Murphy.[25] The problem appears to be that the coefficients appearing in it, although constant, are not arbitrary and there are a series of relationships between them. Furthermore, it does not appear to be transformable into the Leach type of equation by, for example, scaling dependent and independent variables with constants, a scaling of the type $y \rightarrow y\alpha(x)$, changing the independent variable to some arbitrary function $V(x)$ or indeed a linear fractional transformation. In spite of this difficulty we have generated new solutions to the Emden-like equation by finding several special solutions of the transformed equation, such as constant solutions or kink-like solutions.

Through the discussed link with quantum many-body Hamiltonians the various solutions of our field equations may be attributed to a particular physical interpretation. The localized solutions represent the formation of a coherent structure in the system, such as, for example, solitons in charge- or spin-density wave systems, or for two-dimensional cases, as vortices in superconductors and superfluids. The oscillatory solutions represent elementary excitations which, in most of the cases, are of finite lifetime due to the presence of damping.

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