Attractor reconstruction from filtered chaotic time series

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We present a method that allows one to decide whether an apparently chaotic time series has been filtered or not. For the case of a filtered time series we show that the parameters of the unknown filter can be extracted from the time series, and thereby we are able to reconstruct the original time series. It is demonstrated that our method works and provides reliable values of the fractal dimensions for systems that are described by maps or differential equations and for real experimental data.

I. INTRODUCTION

In the last few years, it has been shown that for many chaotic systems, the asymptotic time evolution can be described as a motion on a strange attractor.[1] A specific attractor is characterized by its entropy, its Liapunov exponents, and by a fractal dimension that is always smaller than the dimension Δ of the phase space of the system. It is known that the filtering of the time series influences the determination of the fractal dimension. Badii and Politi[2] have shown that the dimension of a low-dimensional chaotic system can be raised by changing the bandwidth of a filter in a measuring instrument. To demonstrate this they use a low-pass filter of first order, whose influence is described by the differential equation

$$\dot{Z}(t) = -\eta Z(t) + X(t) \; . \tag{1.1}$$

Here $X(t)$ and $Z(t)$ are the original and the filtered time series, respectively, and η is the bandwidth of the filter. The dimension of the phase space of the system when the filter is included is raised to $\Delta+1$ compared to a previous dimension of Δ in the absence of the filter. Moreover, the system also gets an addition negative Liapunov exponent $-\eta$. By use of the Kaplan-Yorke conjecture, it has been shown[2] that the fractal dimension of the filtered time series $Z(t)$ increases when $\lambda_- < -\eta \le 0$. λ_- denotes the negative Liapunov exponent that appears in the denominator of the Kaplan-Yorke formula.[3] Therefore, in order to have no effect on the time series, the filters inside the measuring instruments should have a large enough bandwidth such that $-\eta < \lambda_-$. Usually, however, we have no information on the Liapunov spectrum of the system under investigation. Thus it is difficult to choose the appropriate bandwidth of the measuring instruments. To solve this problem Mitschke, Möller, and Lange[4] have suggested using the derivatives $\dot{Z}(t)$ instead of the filtered time series $Z(t)$, to estimate the dimension. This procedure leads to reliable results in the case of very strong filters ($\eta \to 0$), as can be easily inferred from Eq. (1.1).

In this paper we want to propose a procedure for the investigation of filtered (low-passed) chaotic time series. This procedure has two important features that will be discussed in Secs. II and III, respectively. First, it indicates whether the signal is filtered or not. Second, it allows us, in most cases, to reconstruct the original time series by determining the filter parameters from the filtered time series. From these parameters (obtained from the filtered time series), we can find a good estimate of the information dimension D_1 of the attractor corresponding to the unfiltered time series. In Appendix 1 a we discuss the important case of purely recursive filters (the commonly used low-pass filters). The problem of nonpurely recursive filters (that include high-pass filters) is considered in Appendix 1 b. We will discuss our method using data obtained from several numerical simulation and experimental systems.

We would like to emphasize that the identification of the presence of the filter and the reconstruction of the original time series from a physically filtered signal (RC low-pass filter) of a hydrodynamic system are presented here for the first time.

II. THE INFLUENCE OF FILTERS ON THE TIME SERIES

The discretized version of the differential equation (1.1) is, according to Ref. 2,

$$Z_{n+1} = e^{-\eta} Z_n + X_{n+1} \; . \tag{2.1}$$

As an example of the application of the formula (2.1) we have investigated the effect of a filter on the chaotic time series $\{X_i\}_{i=1}^{N}$ corresponding to the logistic map $X_{n+1} = rX_n(1-X_n)$ with $r=4$. Figure 1 shows the return maps (Z_{n+1} versus Z_n) of the filtered times series $\{Z_i\}$ using the filter parameter $\eta=1$. In this figure we see that

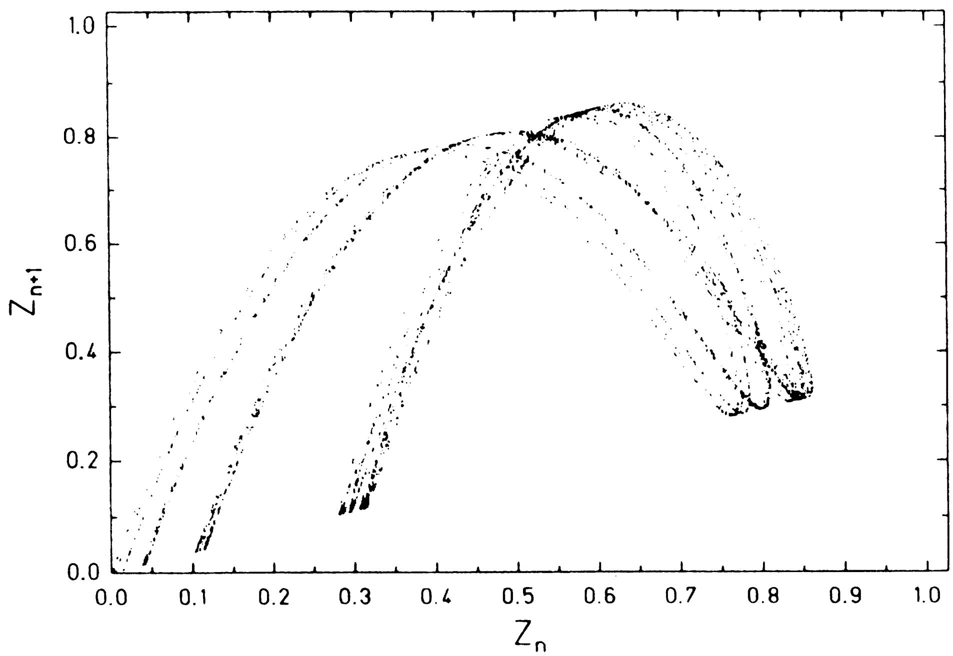


FIG. 1. The return map (Z_{n+1} vs Z_n) of the filtered time series of the quadratic map ($r=4$) for filter parameter $\eta=1$.

the filter has induced a self-similarity that is responsible for the increase of the dimension. This effect has been treated analytically for one-dimensional maps in a previous paper.[5] The dimension therefore increases, as predicted by the Kaplan-Yorke conjecture.[2] To estimate this increase of dimension from a given time series, we apply the fixed mass method:[6] First, we reconstruct the phase-space trajectory optimally from a given scalar time series of N points, thus obtaining vectors $\{\mathbf{x}_i\}$, $\mathbf{x}_i \in \mathrm{R}^m$.[7] Then n reference points $\mathbf{y}_i$ are chosen on the attractor and distributed according to the invariant measure. The distances $\delta=\delta_{j,k}$ between the points $\mathbf{y}_j$ and their kth nearest neighbors among the N points $\{\mathbf{x}_i\}$ are then computed. From these distances the information dimension D_1 is evaluated using the asymptotic relation

$$D_1 \sim \lim_{N\to\infty} \frac{\ln p}{\langle \ln\delta_k \rangle}\ , \tag{2.2}$$

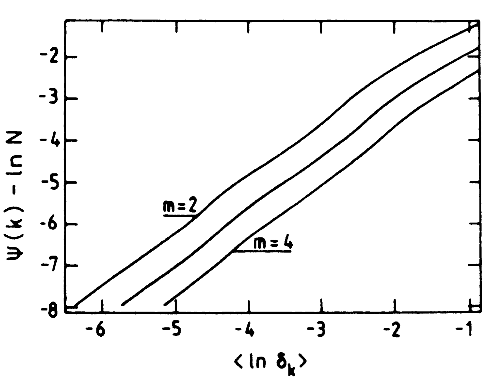


FIG. 2. $\psi(k)-\ln N$ is plotted vs $\langle \ln\delta_k \rangle$ of the filtered time series of the quadratic map ($r=4$) for $\eta=1$. Oscillations occur and grow with decreasing values of η.

where

$$\langle \ln\delta_k \rangle = \frac{1}{N} \sum_{j=i}^{N} \ln\delta_{j,k}$$

and $p=k/N$. In the case of very small values of k, formula (2.2) has to be replaced by

$$D_1 = \lim_{N\to\infty} \frac{\psi(k)-\ln N}{\langle \ln\delta_k \rangle}\ , \tag{2.3}$$

where $\psi(k)=d\ \ln\Gamma(k)/dk$ and $\Gamma(k)$ is the usual Γ function.[8] All the results described in this paper have been obtained with the help of formula (2.3).

Applying the method for the filtered time series $\{Z_i\}$ from the logistic map as mentioned before we see the appearance of oscillations, as shown in Fig. 2. Such oscillations are characteristic for attractors possessing strong self-similarity, as, e.g., the Zaslavasky attractor.[8] In our case, however, this is purely an effect of the self-similarity induced by filtering, since the attractor corresponding to the logistic map with $r=4$ does not have this property.[9]

The real attractor can properly be reconstructed from $X(t)$ or $\dot{X}(t)$, respectively. In both cases the dimension algorithm (2.3) leads to the same fractal dimension from an unfiltered time series. But, if we investigate a low-pass filtered signal $Z(t)$, the fractal dimensions obtained from $\dot{Z}(t)$ or $Z(t)$ are different [the fractal dimension obtained from $\dot{Z}(t)$ always being smaller than that obtained from $Z(t)$]. As we see from Eq. (1.1), the fractal dimension obtained from $\dot{Z}(t)$ is equal to that of $X(t)$ only if $\eta\to 0$ (limit of very strong filter). One can thus decide if a series is filtered or not by comparing the fractal dimensions of $\dot{Z}(t)$ and $Z(t)$. As an example we consider in Fig. 3 a filtered time series (X component) of the Rössler systems[10] with $\eta=0.2$. The decrease in the dimension using the derivatives $\dot{Z}(t)$ is clearly visible although incom-

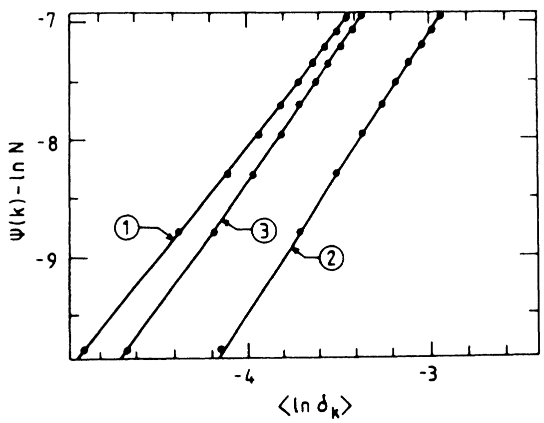


FIG. 3. $\psi(k)-\ln N$ is plotted vs $\langle \ln\delta_k \rangle$ using the embedding dimension $m=5$ for the X component of the Rössler differential equation ($a=0.2$, $b=0.15$, $c=10$) in the case of the unfiltered time series (curve 1), a filtered time series with $\eta=0.2$ (curve 2) and its derivatives (curve 3). In these three cases the dimensions are $D_1=1.99$ (curve 1), $D_2=2.35$ (curve 2), and $D_3=2.17$ (curve 3).

plete. Together with the optimal reconstruction of the phase-space trajectory from the time series,[7] this serves as a strong indication that the time series has been filtered.

III. RECONSTRUCTION OF ATTRACTORS FROM FILTERED TIME SERIES

In the case of a pure recursive filter as in Eq. (2.1), the recovery of the original time series is possible by simply inverting the filter, i.e.,

$$X_{n+1}=Z_{n+1}-e^{-\eta}Z_n \ . \tag{3.1}$$

If the value of the filter parameter η is unknown, an ansatz for the back transformation is

$$\tilde{X}_{n+1}=Z_{n+1}-e^{-\alpha}Z_n \ . \tag{3.2}$$

Here, of course, the original time series $\{X_i\}$ is obtained for $\alpha=\eta$. By combining Eqs. (3.1) and (3.2), we find a connection between the time series $\{\tilde{X}_i\}$, $\{X_i\}$, and $\{Z_i\}$ as

$$\tilde{X}_{n+1}=X_{n+1}+(e^{-\eta}-e^{-\alpha})Z_n \ . \tag{3.3}$$

As a first approximation we ignore the dependence of Z_n on X_n. Then we may consider the term $\beta=(e^{-\eta}-e^{-\alpha})Z_n$ as a (in general colored) noise with an amplitude proportional to $\gamma=(e^{-\eta}-e^{-\alpha})$ due to the finite-dimensional system represented by Z_n. We therefore expect a "kink" to appear in a plot of $\psi(k)-\ln N$ vs $\langle \ln\delta_k \rangle$ analogous to the "kink" we expect in the presence of infinite-dimensional noise.[11] In contradistinction to the situation there, here we expect that the slope of $\psi(k)-\ln N$ versus $\langle \ln\delta_k \rangle$ converges for small values of $\langle \ln\delta_k \rangle$ to the dimension of the system plus filter if the embedding dimension m increases.

To demonstrate this, Fig. 4 shows $\psi(k)-\ln N$ versus $\langle \ln\delta_k \rangle$ for the time series $\{\tilde{X}_i\}$, which has been recon-

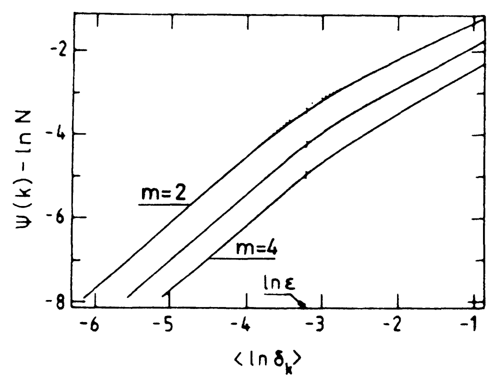


FIG. 4. $\psi(k)-\ln N$ is plotted vs $\langle \ln\delta_k \rangle$ for the inverse transformed time series $\{\tilde{X}_i\}$ of the filtered time series $\{Z_i\}$ ($\eta=1$) corresponding to the logistic map ($r=4$), using Eq. (3.2) with $\alpha=0.9$. The embedding dimension varies from $m=2$ to 4.

structed with $\alpha=0.9$ from the filtered time series of the logistic map considered earlier. Clearly a "kink" arises and the slope is as predicted 1 above a critical value of ϵ and 1.6 below that value. The "kink" position moves in the direction of the small value ϵ when α approaches the value $\eta=1$. There it disappears completely. With a knowledge of the "kink" position ϵ the estimation of the filter parameter η would be easy, provided that the value of Z_n in the expression $\epsilon\sim\beta=(e^{-\eta}-e^{-\alpha})Z_n$ could be replaced, e.g., by its mean value $\bar{Z}=(1/N)\sum_i Z_i$. Since this generally is not possible we have to invert Eq. (3.2) for two different values α and α'. From the corresponding "kink" positions $\epsilon\sim\beta=(e^{-\eta}-e^{-\alpha})Z_n$ and $\epsilon'\sim\beta'==(e^{-\eta}-e^{-\alpha'})Z_n$ we can eliminate Z_n and obtain η from

$$e^{-\eta}=\frac{\epsilon e^{-\alpha'}-\epsilon' e^{-\alpha}}{\epsilon-\epsilon'} \ . \tag{3.4}$$

The "kink" position seems to be most easily determined by looking at the crossing point of the straight lines that are defined by the clear scaling behavior above and below the "kink" position in the case of the logistic map (Fig. 4). This method works quite well for maps but in many cases it proves to be advantageous to consider a diagram of the derivative $d[\psi(k)-\ln N]/d\langle \ln\delta_k \rangle$ versus $\langle \ln\delta_k \rangle$. Figure 5 shows this for the aforementioned logistic map. Here an alternative location of the "kink" position would be the inflection point.

We now demonstrate the application of Eq. (3.4) using two examples. First we consider the filtered logistic map ($\eta=1$) that yields an information dimension $D_1=1.6$ which is larger than that of the unfiltered system. Figure 6(a) shows the "kink" position in the plot $\psi(k)-\ln N$ versus $\langle \ln\delta_k \rangle$ for $\alpha=0.7$ and $\alpha'=0.8$. The resulting approximation $\tilde{\eta}=1.02$ for the filter parameter calculated from the "kink" positions $\epsilon\approx0.135$ and $\epsilon'\approx0.088$ differs from the true value $\eta=1$ by 2%. The second example is obtained by considering the filtered X component of the Hénon map[12] with parameters $a=1.4$, $b=0.3$. We choose a filter parameter $\eta=0.2$. The numerically obtained information dimension of the filtered time series is

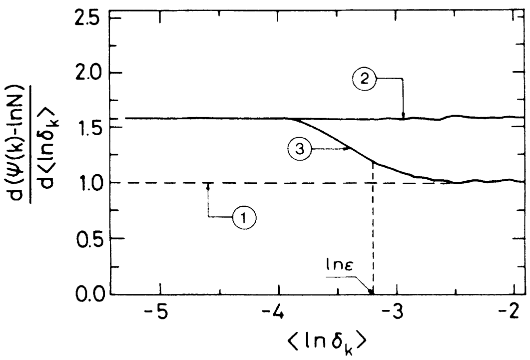


FIG. 5. $d[\psi(k)-\ln N]/d\langle \ln\delta_k\rangle$ is plotted vs $\langle \ln\delta_k\rangle$ for the unfiltered time series $\{X_t\}$ (curve 1), for the filtered time series $\{Z_t\}$ with $\eta=1$ (curve 2), and for the transformed time series $\{\tilde{X}_t\}$ [using Eq. (3.2)] with $\alpha=0.9$ (curve 3) of the logistic map ($r=4$). The embedding dimension in all cases is $m=2$. $\ln\epsilon$ is the position of the inflection point.

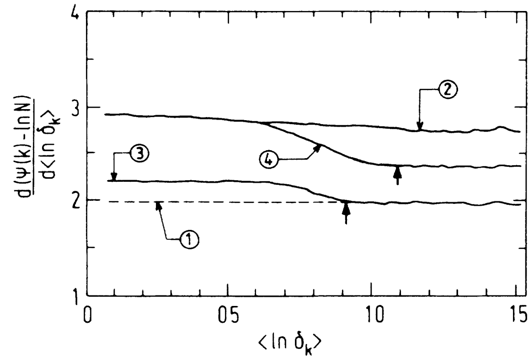


FIG. 7. $d[\psi(k)-\ln N]/d\langle \ln\delta_k\rangle$ is plotted vs $\langle \ln\delta_k\rangle$ (embedding dimension $m=5$) for the unfiltered time series of the X component of the Rössler system with $a=0.2$, $b=0.15$, and $c=10$ (curve 1), the filtered time series with $\eta=0.01$ (curve 2), and for the inverse transformed time series [using Eq. (3.2)] with $\alpha=0.1$ (curve 3) and with $\alpha=0.3$ (curve 4). In the first two cases the information dimensions are $D_1=1.99$ and $D_1=2.94$, respectively.

$D_1=1.98$. In this case the plot $\psi(k)-\ln N$ versus $\langle \ln\delta_k\rangle$ for $\alpha=0.4$ and $\alpha'=0.3$ leads to $\tilde{\eta}=0.192$, which differs from the original filter parameter $\eta=0.2$ by 4% [Fig. 6(b)]. The deviation from the exact value of η is due to the uncertainty of the values of $\epsilon\approx0.122$ and $\epsilon'\approx0.223$.

Unfortunately, two problems arise for time-continuous systems and for experimental data: First, in many cases X_n and Z_n are strongly correlated. This fact induces an indeterminacy in the amplitude of the "noise" term β. Furthermore, we cannot then expect the measured dimensions above and below the "kink" to be the exact di-

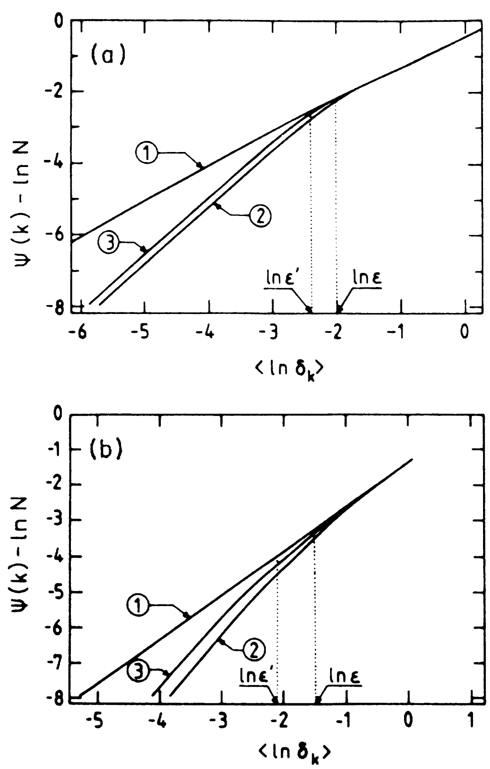


FIG. 6. Plots $\psi(k)-\ln N$ vs $\langle \ln\delta_k\rangle$ for the unfiltered time series (curve 1) and for two differently backtransformed time series (curves 2 and 3) for (a) the logistic map ($r=4$, $\eta=1$, embedding dimension $m=2$) with transformation parameters $\alpha=0.7$ (curve 2) and $\alpha'=0.8$ (curve 3) and (b) the X component of the paradigmatic Hénon map ($\eta=0.2$, embedding dimension $m=3$) with transformation parameters $\alpha=0.4$ (curve 2) and $\alpha'=0.3$ (curve 3), respectively.

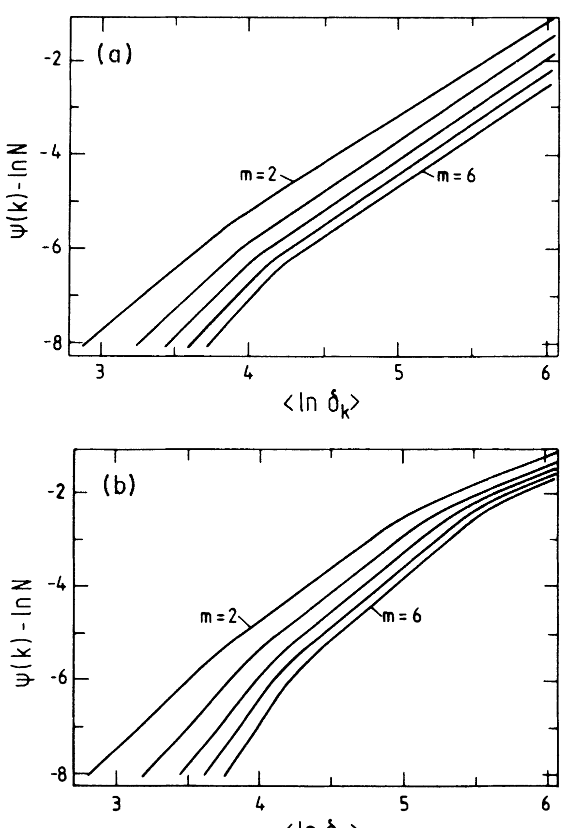


FIG. 8. Plots $\psi(k)-\ln N$ is plotted vs $\langle \ln\delta_k\rangle$ (embedding dimension $m=2,\ldots,6$) for the unfiltered (a) and filtered (b) time series of the local velocity from the Taylor-Couette system. The values of the information dimension are (a) $D_1=2.17$ and (b) $D_1=2.5$ respectively.

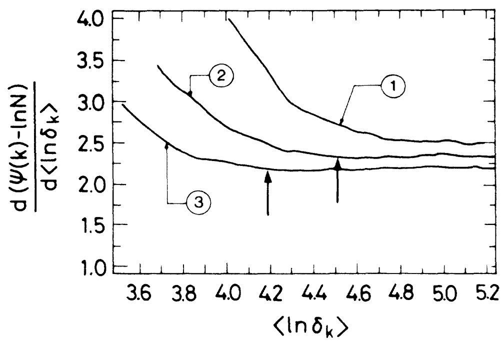


FIG. 9. $d[\psi(k)-\ln N]/d\langle \ln\delta_k \rangle$ is plotted vs $\langle \ln\delta_k \rangle$ for the filtered time series (curve 1) and for the inverse transformed time series [using Eq. (3.2)] with $\alpha=1$ (curve 2) and $\alpha=0.15$ (curve 3) of the local velocity from the Taylor-Couette system.

mensions of the unfiltered and filtered system, respectively. Actually these effects can be demonstrated with data from the Rössler system,[10] with the parameters being $a=0.2$, $b=0.15$, and $c=10$. The value of the information dimension of the unfiltered time series is $D_1=1.99$. The filter parameter is taken as $\eta=0.01$ and the dimension algorithm (2.3) leads to an information dimension $D_1=2.94$ for the filtered time series.[13] Figure 7 shows the plot $d[\psi(k)-\ln N]/d\langle \ln\delta_k \rangle$ versus $\langle \ln\delta_k \rangle$ using the embedding dimension $m=5$ for the unfiltered time series (curve 1), for the filtered time series (curve 2), and for the transformed time series [using Eq. (3.2)] with $\alpha=0.1$ (curve 3) and $\alpha=0.3$ (curve 4). This plot shows a lowering of the dimension on the right-hand side of the arrows when the parameter α approaches the filter parameter η. In addition, we notice that the region to the right of the arrow, where the dimension is lowered, is increased when α tends towards η. To the left of the arrow, and for small values of $\langle \ln\delta_k \rangle$, we observe that the dimension of curve 3 is lower than that of curve 4 as we mentioned before.

A second problem arises from the fact that in experimental situations one generally has background noise such as that of a "kink" caused by the deterministic noise induced by the filter which can be hidden by the background noise. In these cases we consider the deviation of the scaling behavior (arrows in Fig. 7) as significant and plot this scale ϵ versus the reconstruction parameter α. Determining the minimum of $\epsilon(\alpha)$ as ϵ becomes small for $\alpha\to\eta$ is for practical cases a useful method for the computation of the filter parameter.

We have applied this procedure to the experimental case, i.e., to the rotational Taylor-Couette flow. The apparatus and the measuring techniques that have been used are described in detail in Refs. 14 and 15. The flow we observed was a ten-vortex state with a gap height-to-width ratio of 16. This gives a vortex wave length of $3.2d$, where d is the gap width. The scenario observed was a breakup of a torus at Reynolds number 630. The local-velocity data were filtered by a first-order RC low-pass filter with a cutoff frequency of 1 Hz. This corresponds to $\eta=0.15$ for a sampling frequency of 41.66 Hz. For small values of $\langle \ln\delta_k \rangle$ we see the effect of the white noise in the plot of $\psi(k)-\ln N$ versus $\langle \ln\delta_k \rangle$ both for unfiltered [Fig. 8(a)] and for filtered [Fig. 8(b)] time series. Let us imagine now that we only have access to the filtered time series. To determine whether this time series is filtered or not, we compare the information dimension D_1 of this time series and its derivative as explained in Sec. II. We get the derivative of the given time

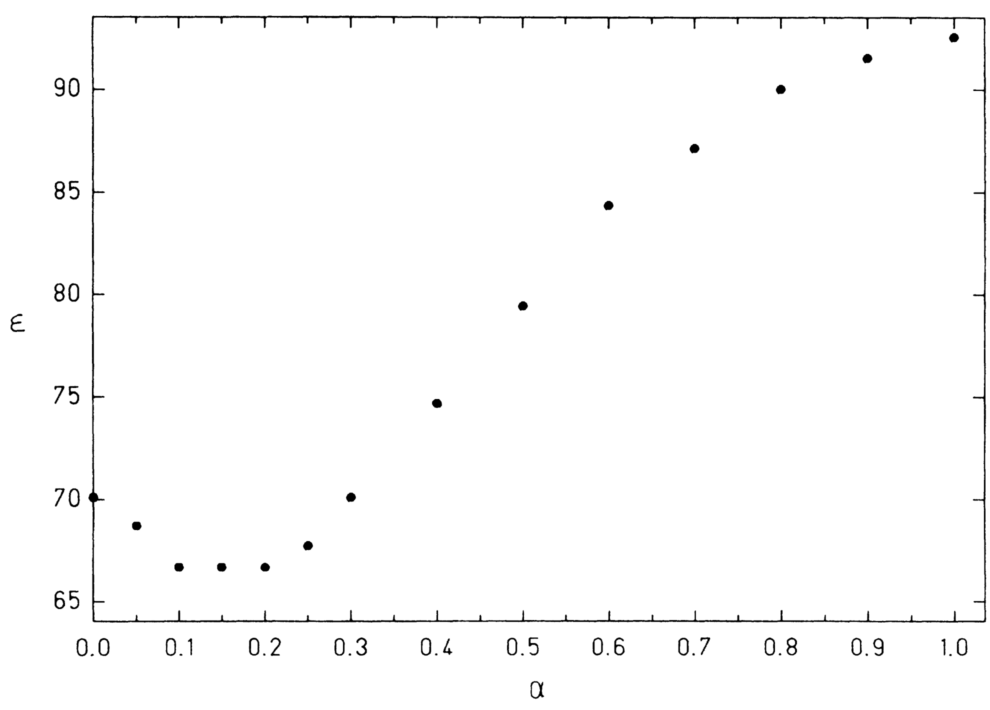
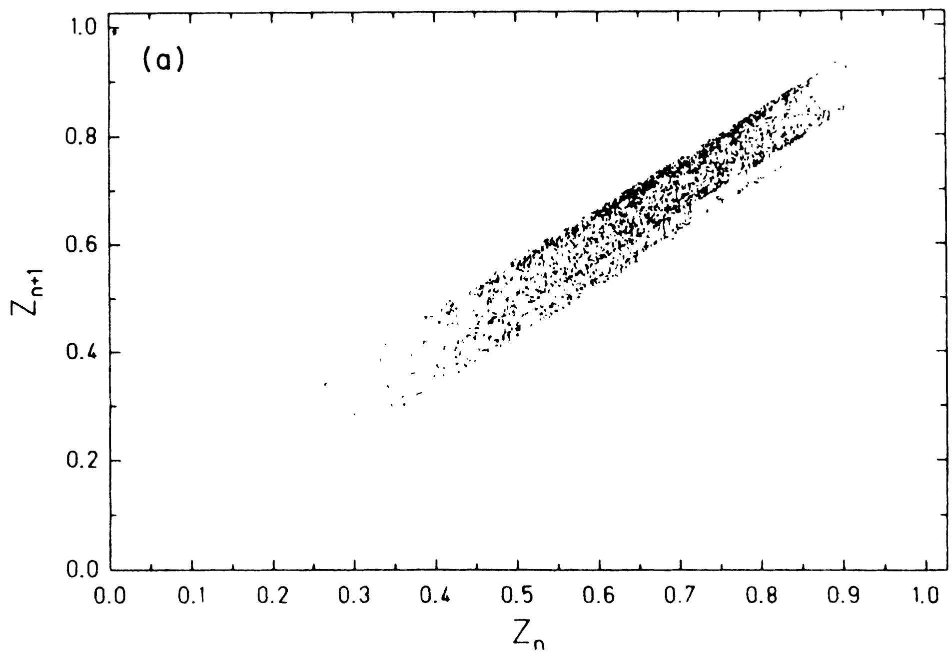


FIG. 10. The "kink" position ϵ (corresponding to the arrows in Fig. 9) plotted vs α for the data from the Taylor-Couette system.

series by choosing $\alpha=0$ in Eq. (3.2). The information dimension of the time series of the derivative is found to be $D_1=2.19$. This strongly differs from the value $D_1=2.50$ for the given time series. This shows that the investigated time series has been filtered. We plot $d[\psi(k)-\ln N]/d\langle \ln\delta_k \rangle$ versus $\langle \ln\delta_k \rangle$ (Fig. 9) and observe that the curves of the filtered time series (curve 1), and the transformed time series [using Eq. (3.2)] with $\alpha=1$ (curve 2) and $\alpha=0.15$ (curve 3) converge for small values of $\langle \ln\delta_k \rangle$. The reason for this behavior is the noise inherent in the experimental data. Curve 3 in Fig. 9 especially does not show the two-plateau pattern as found in curves 3 and 4 in Fig. 7. Hence we are not able to locate the position of the "kink" as the point of inflection of the curve.

However, we obtain an approximation of the filter parameter by plotting the "kink" position versus α (arrows in Fig. 9). As shown in Fig. 10 we find that the filter pa-



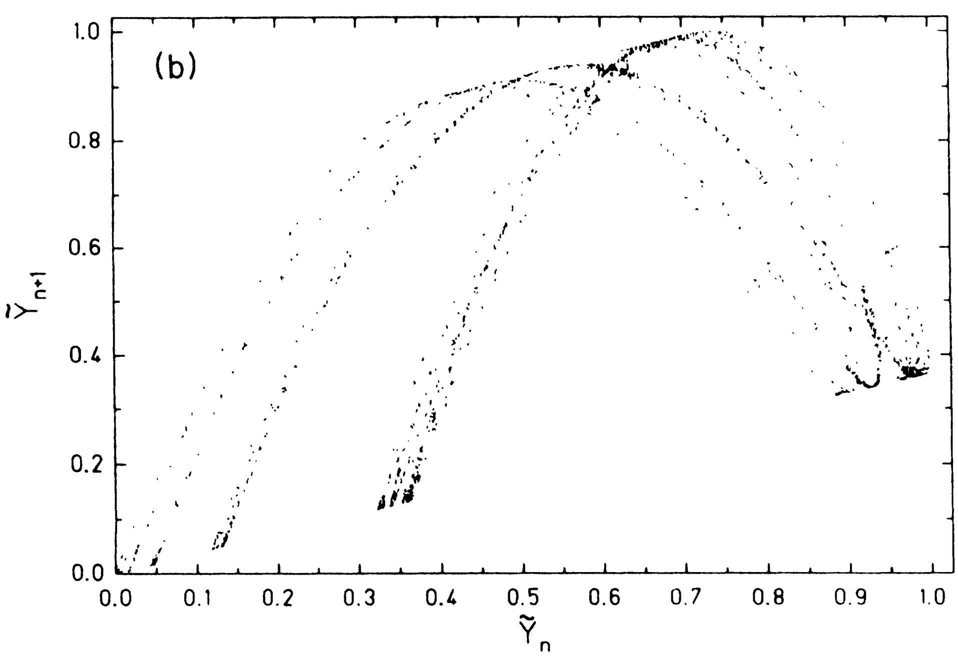


FIG. 11. The return maps of the filtered time series $\{Z_i\}$ with a purely recursive low-pass filter with two poles for the logistic map ($r=4$) for (a) the filtered case (parameters $a_0=0.9$, $b_1=1.27$ and $b_2=-0.33$), (b) for the inverse transformed time series $\{\tilde{Y}_i\}$ with $\alpha=\eta=0.1$, and (c) for the second inverse transformed time series $\{\tilde{X}_i\}$ with $\alpha=\eta'=1$.

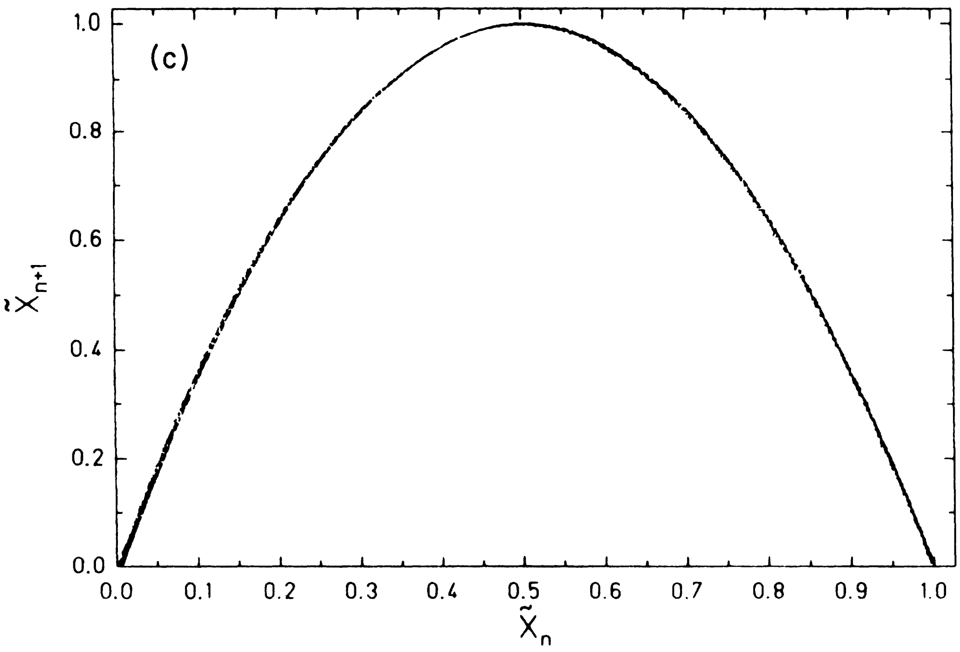


FIG. 11. *(Continued)*.

rameter is $0.1 \le \tilde{\eta} \le 0.2$ in coincidence with $\eta = 0.15$ of the physical RC low-pass filter. The effect of the retransformation on D_1 within this range is remarkable. We get the following results:

$$D_1 = \begin{cases} 2.17 \text{ for the unfiltered signal} \\ 2.50 \text{ for the filtered signal} \\ 2.18 \text{ for the retransformed signal .} \end{cases}$$

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APPENDIX

1. Generalization of the inversion procedure

a. The case of purely recursive filters of higher order

We now want to generalize our method to the investigation of filters of higher order. The general form of a purely recursive filter with l poles is

$$Z_n = a_0 X_n + b_1 Z_{n-1} + b_2 Z_{n-2} + \cdots + b_l Z_{n-l} \ . \tag{A1}$$

The transfer function $F(\omega)$ of any filter with l poles can be described by a product of transfer functions $F_i(\omega)$ $(i = 1, \ldots, l)$ as

$$F(\omega) = F_1(\omega) \cdots F_l(\omega) \ . \tag{A2}$$

If a time series is filtered with a purely recursive filter (l poles) it is possible to determine the corresponding filter parameters $\eta_1, \ldots, \eta_l$ by iterating our method. To illustrate this point we consider a purely recursive filter with two poles. Because of Eq. (A1) this filter has the general form

$$Z_n = a_0 X_n + b_1 Z_{n-1} + b_2 Z_{n-2} \ , \tag{A3}$$

which can be rewritten as a product of two filters with one pole each, and with cutoff frequencies η and η' respectively, as

$$Y_n = e^{-\eta} Y_{n-1} + a_0 X_n \ ,$$

$$Z_n = e^{-\eta'} Z_{n-1} + Y_n \ ,$$

which imply (A4)

$$Z_n = (e^{-\eta} + e^{-\eta'}) Z_{n-1} - e^{-\eta} e^{-\eta'} Z_{n-2} + a_0 X_n \ .$$

Using (A4) it turns out that a purely recursive filter with two poles can be written as a twofold iteration of a filter of the type of Eq. (2.1). Comparing (A3) and (A4) we obtain

$$b_1 = e^{-\eta} + e^{-\eta'}, \quad b_2 = e^{-\eta} e^{-\eta'} \ . \tag{A5}$$

This system of equations guarantees the existence of two cutoff frequencies η and η'. As an example we discuss the logistic map that has been filtered according to Eq. (A3) with $a_0 = 0.9$, $b_1 = 1.27$, and $b_2 = -0.33$ [Fig. 11(a)]. Using our "kink" method in an iterative fashion, the first approximate cutoff frequency is calculated to be $\tilde{\eta} = 0.11$. If the parameter α is varied in a neighborhood of $\tilde{\eta}$, the exact value $\eta = 0.1$ can be obtained. It follows that the information dimension D_1 is reduced from $D_1 = 2.4$ to 1.56 [Fig. 11(b)]. The same method is used to calculate the second cutoff frequency $\eta' = 1$, which reduces the di-

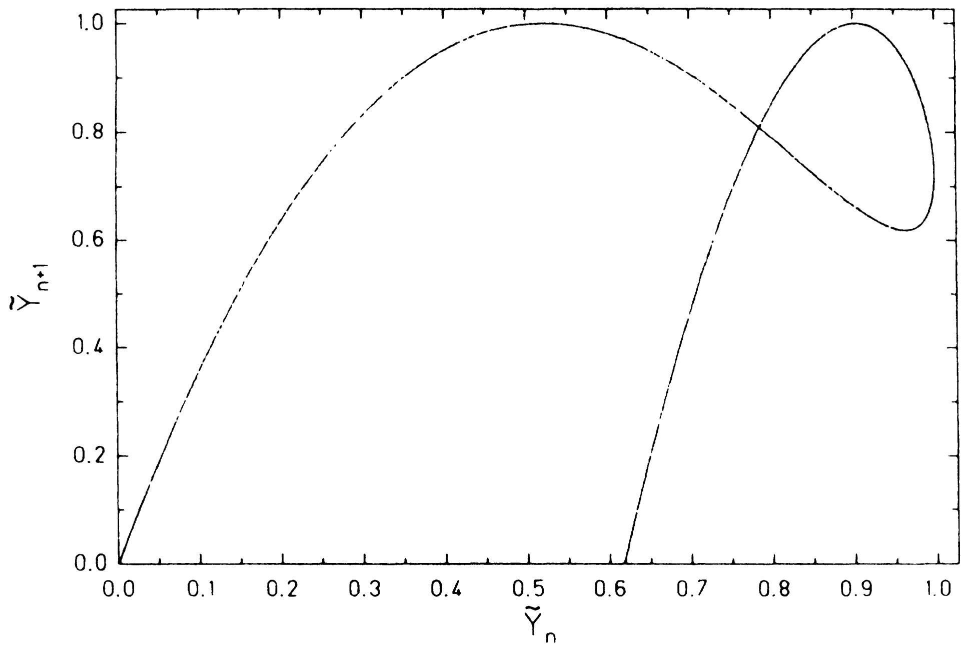

FIG. 12. The return map ($\tilde{Y}_{n+1}$ vs $\tilde{Y}_n$) for the time series $\{\tilde{Y}_i\}$ obtained by Eq. (A7) with $\alpha=0.1$. The filtered time series $\{Z_i\}$ is calculated from the time series $\{X_i\}$ corresponding to the logistic map ($r=4$) using the Eq. (A6) with the parameters $a_0=0.9$, $a_1=0.85$, and $b_1=0.9$.

mension to $D_1=1$. The logistic map for this example is shown in Fig. 11(c).

b. The case of nonpurely recursive filters

The "kink" method may also be applied to nonpurely recursive filters with one pole. The general form of a nonpurely recursive filter with one pole is defined as

$$Z_n=b_1Z_{n-1}+a_0X_n+a_1X_{n-1} . \tag{A6}$$

Here the value of Z_n is not only dependent on X_n, but also on X_{n-1}. For negative values of a_1 this is a high-pass filter. To illustrate the method, we have filtered the logistic map with a nonpurely recursive low-pass filter and with one pole using the parameters $a_0=0.90$, $a_1=0.85$, and $b_1=0.90$. First, we invert the purely recursive part

$$Y_n=Z_n-e^{-\alpha}Z_{n-1} . \tag{A7}$$

For $e^{-\alpha}=b_1$ the dimension is reduced to the exact value $D_1=1$ (Fig. 12). The value $\alpha=0.1$ can be calculated using the method described above. The original time series $\{X_n\}$ is not obtained at the end of this transformation. Instead, the time series $\{Y_n\}$ obtained now contains the nonpurely recursive part, which is, from Eq. (A6),

$$Y_n=a_0X_n+a_1X_{n-1} . \tag{A8}$$

The inversion of (A8) is

$$\tilde{X}_n=Y_n-B\tilde{X}_{n-1} , \tag{A9}$$

where B is a free parameter. Thus

$$\begin{aligned}
&\tilde{X}_1=Y_1=a_0X_1 ,\\
&\tilde{X}_2=Y_2-B\tilde{X}_1=a_0X_2+(a_1-Ba_0)X_1 ,\\
&\tilde{X}_3=Y_3-B\tilde{X}_2=a_0X_3+(a_1-Ba_0)(X_2-BX_1) ,\\
&\vdots\\
&\tilde{X}_n=Y_n-BX_{n-1}=a_0X_n+(a_1-Ba_0)\sum_{k=1}^{n-1}(-B)^{n-1-k}X_k .
\end{aligned} \tag{A10}$$

The term containing B in (A10) is the noise contribution. If $B=a_1/a_0$, we see that $\tilde{X}_n=a_0X_n$, i.e., we obtain, up to an irrelevant factor, the unfiltered time series. If B is a number in the vicinity of a_1/a_0, but is not equal to a_1/a_0, a "kink" appears in the plot of $\psi(k)-\ln N$ versus $\langle \ln\delta_k \rangle$. This "kink" vanishes as $B=a_1/a_0$. In order to find this value of a_1/a_0, one has to vary B until the "kink" disappears. This procedure can also be applied on a high-pass filter with one pole. It may also be applied on a nonpurely recursive low-pass filter with l poles

$$Z_{n+1}=a_0X_n+a_1X_{n-1}+b_1Z_{n-1}+\cdots+b_lZ_{n-l} . \tag{A11}$$

[1]H. G. Schuster, *Deterministic Chaos,* 2nd ed. (Physik Verlag, Weinheim, 1988).
[2]R. Badii and A. Politi, in *Dimensions and Entropies in Chaotic Systems,* edited by G. Mayer-Kress (Springer-Verlag, Berlin, 1986); R. Badii *et al.*, Phys. Rev. Lett. **60**, 979 (1988).
[3]P. Frederickson, J. C. Kaplan, E. D. Yorke, and J. A. Yorke, J. Diff. Equ. **49**, 185 (1983).
[4]F. Mitschke, M. Möller, and W. Lange, Phys. Rev. A **37**, 4518

(1988).
[5]A. Chennaoui, J. Liebler, and H. G. Schuster, J. Stat. Phys. (to be published).
[6]R. Badii and A. Politi, J. Stat. Phys. **40**, 725 (1985).
[7]F. Takens, in *Lecture Notes in Mathematics*, edited by D. A. Rand and L. S. Young (Springer, Berlin, 1981), Vol. 898, pp. 366–381; W. Liebert, K. Pawelzik, and H. G. Schuster (unpublished).
[8]P. Grassberger, Phys. Lett. **107A**, 101 (1985).
[9]R. Badii and A. Politi, Phys. Lett. **104A**, 303 (1984).
[10]O. E. Rössler, Phys. Lett. **57A**, 397 (1976).
[11]A. Mizrachi and I. Procaccia, Phys. Rev. A **29**, 975 (1984).
[12]M. Hénon, Comm. Math. Phys. **50**, 69 (1976).
[13]We have determined the Kaplan-Yorke dimension of the filtered time series. The filter causes the appearance of an additional negative Liapunov exponent $\lambda_{\rm add}=-\eta=-0.01$. The Rössler system possesses three Liapunov exponents $\lambda_1=0.09$, $\lambda_2=0.00$, and $\lambda_3=-9.8$ [A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, Physica **16D**, 285 (1985)]. Consequently, the Kaplan-Yorke dimension has to be formulated in the following way: $D_{\rm KY}=3+[(\lambda_1+\lambda_{\rm add})/|\lambda_3|]$. With that we get $D_{\rm KY}=3.008$.
[14]T. Mullin, K. A. Cliffe, and G. Pfister, Phys. Rev. Lett. **58**, 2212 (1987).
[15]G. Pfister, H. Schmidt, K. A. Cliffe, and T. Mullin, J. Fluid Mech. **19**, 1 (1988).