Exact quantum statistics of a nonlinear dissipative oscillator evolving from an arbitrary state

V. Peřinová and A. Lukš
Joint Laboratory of Optics, Palacký University, 771 46 Olomouc, Czechoslovakia
(Received 7 August 1989)

The statistical properties of the third-order nonlinear dissipative oscillator, which evolves from any state, are derived on the basis of the exact solution to the master equation. Some important features of the nonlinear oscillator model, such as the recurrences of the initial state, are related to the properties of quasidistributions connected with the phase of the complex field amplitude.

I. INTRODUCTION

The one-dimensional quantum anharmonic oscillator plays a significant role in applications. For instance, the nonlinear oscillator may be useful even in understanding the classical phenomena of light propagation.[1] Various modifications of this model are closely related, although they differ from one another by either the interpretation or the ordering of the position and momentum operators, which can be considered to correspond to either the real physical space or a more general phase space.

Optics provide the applications with this nonlinear oscillator model that has basically the same steady states as the linear oscillator, but the eigenfrequencies changed.[2,3] This nonlinear oscillator is interesting not only from the viewpoint of the quantum statistical physics,[4] but it also develops in the context of nonlinear quantum optics with all its requirements and simplifying assumptions.

In Refs. 5 and 6 dissipation was included by coupling the oscillator to a zero-temperature heat bath and in Ref. 6 a most general input state was considered. In Ref. 7 the assumption of zero temperature was removed, but the initial coherent state continued to be assumed. The involved quantitative results confirmed the qualitative predictions that some interesting quantum features can be destroyed by dissipation. The quantum coherence effects are explained in terms of the interference in phase space. In this context it is obvious that many authors prefer not to include dissipation on studying the nonlinear oscillator evolving from more general initial states, but their assumptions lead to interesting numerical results.[8]

In this paper we present the statistical characteristics of the third-order nonlinear dissipative oscillator responsive to the requirements of quantum optics and established without any restrictions on the initial state. Another interpretation of the quantum coherence is provided.

II. QUANTUM DYNAMICS

Modeling dissipation by coupling the third-order nonlinear oscillator to a reservoir of oscillators, we can write the Hamiltonian of the considered system in the form[6]

$$\hat{H}=\hbar\left[\omega(\hat{a}^{\dagger}\hat{a}+\tfrac{1}{2})+\kappa\hat{a}^{\dagger 2}\hat{a}^{2}+\sum_j \psi_j(\hat{b}_j^{\dagger}\hat{b}_j+\tfrac{1}{2}) + \sum_j(\kappa_j\hat{b}_j\hat{a}^{\dagger}+\kappa_j^{*}\hat{b}_j^{\dagger}\hat{a})\right] . \quad (1)$$

Here $\hat{a}$ ($\hat{a}^{\dagger}$) is the photon annihilation (creation) operator, ω is the frequency of light, κ is a real constant for the intensity dependence, $\hat{b}_j$ ($\hat{b}_j^{\dagger}$) are the boson annihilation (creation) operators of the reservoir oscillators with the frequencies ψ_j, and κ_j are the coupling constants of the interaction with the reservoir.

In the standard treatments of the quantum theory of damping[9] the master equation for the reduced density operator $\hat{\rho}_r$ in the interaction picture $[\hat{a}\rightarrow\exp(i\omega t)\hat{a}]$ can be derived

$$\frac{\partial}{\partial t}\hat{\rho}_r=-i\kappa(\hat{a}^{\dagger 2}\hat{a}^{2}\hat{\rho}_r-\hat{\rho}_r\hat{a}^{\dagger 2}\hat{a}^{2}) +\frac{\gamma}{2}(2\hat{a}\hat{\rho}_r\hat{a}^{\dagger}-\hat{a}^{\dagger}\hat{a}\hat{\rho}_r-\hat{\rho}_r\hat{a}^{\dagger}\hat{a}) +\gamma\bar{n}(\hat{a}^{\dagger}\hat{\rho}_r\hat{a}+\hat{a}\hat{\rho}_r\hat{a}^{\dagger}-\hat{a}^{\dagger}\hat{a}\hat{\rho}_r-\hat{\rho}_r\hat{a}\hat{a}^{\dagger}) , \quad (2)$$

where γ is the damping constant and $\bar{n}$ reflects the thermal properties of the reservoir. Considering the classical-quantum correspondence[10]

$$\tilde{C}\phi_{\mathcal{A}}=\pi^{-1}\hat{\rho}_r \quad (3)$$

related to the quantum-classical correspondence $C^{-1}(\hat{a}^{k}\hat{a}^{\dagger l})=\alpha^k\alpha^{*l}$, where the complex amplitude α corresponds to the operator $\exp(i\omega t)\hat{a}$, we obtain the generalized Fokker-Planck equation for the quasidistribution $\phi_{\mathcal{A}}(\alpha,t)$:

$$\frac{\partial}{\partial t}\phi_{\mathcal{A}}(\alpha,t)=\left[i\kappa\left[2\alpha|\alpha|^2\frac{\partial}{\partial\alpha}+\alpha^2\frac{\partial^2}{\partial\alpha^2}-\text{c.c.}\right] +\gamma+\frac{\gamma}{2}\left[\alpha\frac{\partial}{\partial\alpha}+\alpha^{*}\frac{\partial}{\partial\alpha^{*}}\right] +\gamma(\bar{n}+1)\frac{\partial^2}{\partial\alpha\partial\alpha^{*}}\right]\phi_{\mathcal{A}}(\alpha,t) . \quad (4)$$

We assume that the initial state is described by the quasidistribution

$$\phi_{\mathcal{A}}(\alpha,0)=\exp(-|\alpha|^2)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\alpha^m\alpha^{*n}f_{mn}(0)\ . \qquad (5)$$

Expressing the quasidistribution $\phi_{\mathcal{A}}(\alpha,t)$ in the form

$$\phi_{\mathcal{A}}(\alpha,t)=\exp(-|\alpha|^2)\phi'_{\mathcal{A}}(\alpha,t) \qquad (6)$$

and using the rules

$$\begin{aligned}\pi^{-1}\tilde{C}^{-1}(\hat{a}^{\dagger}\hat{\rho}_r)&=\exp(-|\alpha|^2)\alpha^*\phi'_{\mathcal{A}}\ ,\\ \pi^{-1}\tilde{C}^{-1}(\hat{\rho}_r\hat{a}^{\dagger})&=\exp(-|\alpha|^2)\frac{\partial}{\partial\alpha}\phi'_{\mathcal{A}}\ ,\\ \pi^{-1}\tilde{C}^{-1}(\hat{a}\hat{\rho}_r)&=\exp(-|\alpha|^2)\frac{\partial}{\partial\alpha^*}\phi'_{\mathcal{A}}\ ,\\ \pi^{-1}\tilde{C}^{-1}(\hat{\rho}_r\hat{a})&=\exp(-|\alpha|^2)\alpha\phi'_{\mathcal{A}}\ ,\end{aligned} \qquad (7)$$

Eq. (2) becomes the equation for $\phi'_{\mathcal{A}}(\alpha,t)$:

$$\begin{aligned}\frac{\partial}{\partial t}\phi'_{\mathcal{A}}(\alpha,t)=\Bigg[\left[i\kappa\alpha^2\frac{\partial^2}{\partial\alpha^2}+\text{c.c.}\right]&+\gamma\bar{n}(-1+|\alpha|^2)\\ &-\gamma(\bar{n}+\tfrac{1}{2})\left[\alpha\frac{\partial}{\partial\alpha}+\text{c.c.}\right]\\ &+\gamma(\bar{n}+1)\frac{\partial^2}{\partial\alpha\partial\alpha^*}\Bigg]\phi'_{\mathcal{A}}(\alpha,t)\ . \end{aligned} \qquad (8)$$

Allowing the solution of Eq. (8) to have the form[6]

$$\phi'_{\mathcal{A}}(\alpha,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\alpha^m\alpha^{*n}f_{mn}(t)\ , \qquad (9)$$

we get the set of differential-difference equations

$$\begin{aligned}\frac{d}{dt}f_{mn}(t)=[-\gamma\bar{n}-\gamma(\bar{n}+\tfrac{1}{2})(m+n)+i\kappa(m-n)(m+n-1)]f_{mn}(t)+\gamma\bar{n}f_{m-1,n-1}(t)\\ +\gamma(\bar{n}+1)(m+1)(n+1)f_{m+1,n+1}(t),m,n=0,1,\ldots,\infty\ ,\end{aligned} \qquad (10)$$

for $f_{mn}(t)$ subject to the initial conditions

$$f_{mn}(t)\Big|_{t=0}=f_{mn}(0)\ . \qquad (11)$$

Except for $\bar{n}=0$ the system (10) comprises the term $\gamma\bar{n}f_{m-1,n-1}(t)$, which represented an obstacle to obtaining an exact solution.[6] The assumption

$$f_{mn}(t)=\sum_{j=0}^{\min(m,n)}\frac{g^j_{m-n}(t)}{j!}h_{m-j,n-j}(t) \qquad (12)$$

leads to the quasidistribution $\phi_{\mathcal{A}}(\alpha,t)$ of the form

$$\begin{aligned}\phi_{\mathcal{A}}(\alpha,t)=&\exp(-|\alpha|^2)\\ &\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\alpha^m\alpha^{*n}\exp[|\alpha|^2g_{m-n}(t)]h_{mn}(t)\ ,\end{aligned} \qquad (13)$$

introduced in Ref. 7. Now we can express

$$\frac{d}{dt}h_{mn}(t)+h_{m-1,n-1}(t)\frac{d}{dt}g_{m-n}(t) \qquad (14)$$

in a form analogous to (10), but with the coefficients dependent on $g_{m-n}(t)$. The comparison of the coefficients at $h_{m-1,n-1}(t)$ provides the equation for the suitable functions $g_{m-n}(t)$:

$$\begin{aligned}\frac{d}{dt}g_{m-n}(t)=&\gamma(\bar{n}+1)g^2_{m-n}(t)\\ &+[-\gamma(2\bar{n}+1)+2i\kappa(m-n)]\\ &\times g_{m-n}(t)+\gamma\bar{n}\ ,\end{aligned} \qquad (15)$$

solved under the initial condition $g_{m-n}(0)=0$, and renders it possible to express $(d/dt)h_{mn}(t)$ in the form without the term $h_{m-1,n-1}(t)$,

$$\begin{aligned}\frac{d}{dt}h_{mn}(t)=&[A_{mn}+B_{mn}g_{m-n}(t)]h_{mn}(t)\\ &+C_{mn}h_{m+1,n+1}(t)\ ,\end{aligned} \qquad (16)$$

where A_{mn},B_{mn},C_{mn} are defined as follows:

$$\begin{aligned}A_{mn}&=i\kappa(m-n)(m+n-1)-\frac{\gamma}{2}(m+n)\\ &\quad-\gamma\bar{n}(m+n+1)\ ,\\ B_{mn}&=\gamma(\bar{n}+1)(m+n+1)\ ,\\ C_{mn}&=\gamma(\bar{n}+1)(m+1)(n+1)\ .\end{aligned} \qquad (17)$$

The functions $h_{mn}(t)$ obey the initial conditions

$$h_{mn}(t)\Big|_{t=0}=h_{mn}(0)=f_{mn}(0)\ . \qquad (18)$$

The Cauchy problem (15) is solved by the function[7]

$$g_{m-n}(t)=\frac{2\bar{n}}{\Omega+\Delta\coth\left[\frac{\gamma\Delta t}{2}\right]}\ , \qquad (19)$$

where

$$\begin{aligned}\Omega\equiv\Omega_{m-n}&=1+2\bar{n}-i\frac{2\kappa}{\gamma}(m-n)\ ,\\ \Delta\equiv\Delta_{m-n}&=[\Omega^2-4\bar{n}(\bar{n}+1)]^{1/2}\ .\end{aligned} \qquad (20)$$

For $\gamma=0$ it holds that

$$g_{m-n}(t)=\exp[2i\kappa(m-n)t]\ , \qquad (21)$$

for $m=n$ formula (19) simplifies to

$$g_0(t)=\frac{\bar{n}_t}{\bar{n}_t+1},\quad \bar{n}_t=\bar{n}[1-\exp(-\gamma t)]\ . \qquad (22)$$

Seeking for the solution of the set (16), we make the substitution

$$h_{mn}(t)=\exp[A_{mn}t+B_{mn}\int_0^t g_{m-n}(t')dt']h'_{mn}(t)\ . \tag{23}$$

The functions $h'_{mn}(t)$ satisfy the relations

$$\frac{d}{dt}h'_{mn}(t)=C_{mn}e_{m-n}(t)h'_{m+1,n+1}(t)\ , \tag{24}$$

where

$$e_{m-n}(t)=\exp\left[[2i\kappa(m-n)-\gamma-2\gamma\bar{n}]t+2\gamma(\bar{n}+1)\int_0^t g_{m-n}(t')dt'\right]\ , \tag{25}$$

and they obey the initial conditions

$$h'_{mn}(t)\Big|_{t=0}=h'_{mn}(0)=h_{mn}(0)=f_{mn}(0)\ . \tag{26}$$

Now, we assume that $m\geq n$ without loss of generality. Taking account of the form of the relations (24), we see that we can assume $h_{mn}(t)\equiv 0$ for $m>m'$, $n>n'$, and $h'_{m'n'}(0)=1$ for fixed m',n' $(m'\geq n')$ and this initial condition leads to a steady component of solution to (24), $h'_{m'n'}(t)\equiv 1$. In this case the relations (22) become recursive,

$$\frac{d}{dt}h'_{m'-1,n'-1}(t)=C_{m'-1,n'-1}e_{m-n}(t)h'_{m'n'}(t) \tag{27}$$

and we can solve them step by step obtaining first

$$h'_{m'-1,n'-1}(t)=C_{m'-1,n'-1}\int_0^t e_{m-n}(t')dt'\ , \tag{28}$$

secondly,

$$h'_{m'-2,n'-2}(t)=C_{m'-2,n'-2}C_{m'-1,n'-1}\frac{1}{2!}\times\left[\int_0^t e_{m-n}(t')dt'\right]^2\ , \tag{29}$$

etc. The method of mathematical induction provides

$$h'_{m'-k,n'-k}(t)=\prod_{j=1}^{k}C_{m'-j,n'-j}\frac{1}{k!}\left[\int_0^t e_{m-n}(t')dt'\right]^k\ , \tag{30}$$

which may be written in the form

$$h'_{mn}(t)=\prod_{j=1}^{n'-n}C_{m'-j,n'-j}\frac{1}{(n'-n)!}\left[\int_0^t e_{m-n}(t')dt'\right]^{n'-n}\ . \tag{31}$$

A general solution of the set of differential-difference equations (24) reads

$$h'_{mn}(t)=\sum_{n'=n}^{\infty}\frac{1}{(n'-n)!}\left[\frac{(\bar{n}+1)}{\bar{n}}g_{m-n}(t)\right]^{n'-n}\times\frac{m'!n'!}{m!n!}h_{m'n'}(0)\ , \tag{32}$$

where we used the relation

$$\int_0^t e_{m-n}(t')dt'=\frac{1}{\gamma\bar{n}}g_{m-n}(t)\ . \tag{33}$$

As a consequence of (32), the functions $h_{mn}(t)$ are expressed in the form

$$h_{mn}(t)=\exp\left[A_{mn}t+B_{mn}\int_0^t g_{m-n}(t')dt'\right]\times\sum_{l=0}^{\infty}\frac{1}{l!}\left[\frac{(\bar{n}+1)}{\bar{n}}g_{m-n}(t)\right]^{l}\times\frac{(m+l)!(n+l)!}{m!n!}h_{m+l,n+l}(0)\ . \tag{34}$$

Denoting

$$E_{m-n}(t)=\exp\left[\left[i\kappa(m-n)-\frac{\gamma}{2}-\gamma\bar{n}\right]t+\gamma(\bar{n}+1)\int_0^t g_{m-n}(t')dt'\right]=\frac{\Delta}{\Omega\sinh\left[\frac{\gamma\Delta t}{2}\right]+\Delta\cosh\left[\frac{\gamma\Delta t}{2}\right]} \tag{35}$$

and using the relation (33), we rewrite (34) as

$$h_{mn}(t)=\exp\left[\left[-2i\kappa(m-n)+\frac{\gamma}{2}\right]t\right]E_{m-n}^{m+n+1}(t)\times\sum_{l=0}^{\infty}\frac{1}{l!}\left[\frac{(\bar{n}+1)}{\bar{n}}g_{m-n}(t)\right]^{l}\frac{(m+l)!(n+l)!}{m!n!}\times h_{m+l,n+l}(0)\ . \tag{36}$$

With the aid of (34) the relation (12) becomes

$$f_{mn}(t)=\exp\left[\left[-2i\kappa(m-n)+\frac{\gamma}{2}\right]t\right]E_{m-n}^{m+n+1}(t)\times\sum_{l=0}^{\infty}\frac{1}{l!}\left[\frac{(\bar{n}+1)}{\bar{n}}g_{m-n}(t)\right]^{l}\frac{(m+l)!(n+l)!}{m!n!}f_{m+l,n+l}(0)F\left[-m,-n,l+1;\frac{4\bar{n}(\bar{n}+1)}{\Delta^2}\sinh^2\left[\frac{\gamma\Delta t}{2}\right]\right]\ , \tag{37}$$

where $F(-m,-n,l+1;x)$ is the hypergeometric function. Let us note that for $\bar{n}=0$ we arrive at the formula (12) from Ref. 6 for $f_{mn}(t)$ using the limit value

$$\lim_{\bar{n}\to 0}\frac{1}{\bar{n}}g_{m-n}(t)=\frac{\gamma}{\gamma-2i\kappa(m-n)}(1-\exp\{[-\gamma+2i\kappa(m-n)]t\}) . \tag{38}$$

In the case $m<n$ we use the fact that $f_{mn}(t)=[f_{nm}(t)]^*$.

III. PHOTON STATISTICS AND SQUEEZING

The quasidistribution $\phi_{\mathcal{A}}(\alpha,t)$ plays a substantial role when determining the photon statistics and squeezing properties of the radiation under consideration. Starting from its form (13), we establish the antinormal characteristic function

$$C_{\mathcal{A}}(\beta,t)=\int\phi_{\mathcal{A}}(\alpha,t)\exp(\beta\alpha^*-\beta^*\alpha)d^2\alpha=\pi\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!}{m!}\left[\frac{1}{1-g_{m-n}(t)}\right]^{m+1}h_{mn}(t)\beta^{m-n}$$
$$\times\exp\left[-\frac{|\beta|^2}{1-g_{m-n}(t)}\right]L_n^{m-n}\left[\frac{|\beta|^2}{1-g_{m-n}(t)}\right] , \tag{39}$$

where the Laguerre polynomials are used,

$$L_n^l(x)=\Gamma^2(n+l+1)\sum_{j=0}^{n}\frac{(-x)^j}{j!(n-j)!\Gamma(j+l+1)} . \tag{40}$$

Using the identity

$$\int\exp(-C|\beta|^2)\beta^{m-n+s}\beta^{*s}d^2\beta=\pi\delta_{mn}s!C^{-s-1} , \tag{41}$$

we may express the normal generating function

$$C_{\mathcal{N}}^{(W)}(\bar{\lambda},t)=\frac{1}{\pi\bar{\lambda}}\int\exp\left[-|\beta|^2\left[\frac{1}{\bar{\lambda}}-1\right]\right]C_{\mathcal{A}}(\beta,t)d^2\beta$$
$$=\pi\sum_{m=0}^{\infty}m!\frac{(1-\bar{\lambda})^m}{[1-g_0(t)(1-\bar{\lambda})]^{m+1}}h_{mm}(t) . \tag{42}$$

In accordance with the corresponding results in Ref. 6 this function depends on the diagonal elements $g_0(t),h_{mm}(t)$, only and as is evident from (19) and (36) the dependence on κ is ruled out.

On the basis of the relation[9]

$$C_{\mathcal{N}}^{(W)}(\bar{\lambda},t)=\sum_{n=0}^{\infty}(1-\bar{\lambda})^n p(n,t) \tag{43}$$

we obtain the photon number distribution

$$p(n,t)=\pi n!\sum_{m=0}^{n}\frac{[g_0(t)]^{n-m}}{(n-m)!}h_{mm}(t)=\pi n!f_{nn}(t) . \tag{44}$$

The factorial moments of $p(n,t)$ can be determined with the aid of (44) because it holds that

$$\langle W^k\rangle_{\mathcal{N}}=\sum_{n=k}^{\infty}\frac{n!}{(n-k)!}p(n,t) , \tag{45}$$

a consequence of the relation (43). The formulas (44) and (45) yield, after some rearrangement, the formulas (48) and (49) in Ref. 6, respectively. Formula (44) implies $\pi\sum_n n!f_{nn}(t)=1$.

The quasidistribution $\phi_{\mathcal{A}}(\alpha,t)$ enables us to calculate the expectation values of the antinormally ordered field operators[6]

$$\langle\hat{a}^k\hat{a}^{\dagger l}\rangle=\langle\alpha^k\alpha^{*l}\rangle_{\mathcal{A}}=\int\alpha^k\alpha^{*l}\phi_{\mathcal{A}}(\alpha,t)d^2\alpha$$
$$=\pi\sum_{n=0}^{\infty}(n+l)!\left[\frac{1}{1-g_{l-k}(t)}\right]^{n+l+1}\times h_{n+l-k,n}(t)$$
$$=\pi\sum_{n=0}^{\infty}(n+l)!f_{n+l-k,n}(t),\quad k\le l , \tag{46}$$

while for $k>l$ we consider the complex conjugate quantity. Respecting the dependence (36), we can rewrite (46) in the following way:

$$\langle\hat{a}^k\hat{a}^{\dagger l}\rangle=\pi\exp\left[\left[-2i\kappa(l-k)+\frac{\gamma}{2}\right]t\right]E_{l-k}^{l-k+1}(t)\left[\frac{1}{1-g_{l-k}(t)}\right]^{l+1}$$
$$\times\sum_{n=0}^{\infty}(n+l-k)!n!f_{n+l-k,n}(0)\sum_{j=0}^{n}\frac{(n-j+l)!}{j!(n-j)!(n-j+l-k)!}\left[\frac{(\bar{n}+1)}{\bar{n}}g_{l-k}(t)\right]^j\left[\frac{E_{l-k}^2(t)}{1-g_{l-k}(t)}\right]^{n-j} . \tag{47}$$

The formula (47) simplifies in cases $k=0,l\neq0$ and $k=1,l=1$ to the form

$$\langle\hat{a}^{\dagger l}\rangle=\pi\exp\left[\left[-2i\kappa l+\frac{\gamma}{2}\right]t\right]\left[\frac{E_l(t)}{1-g_l(t)}\right]^{l+1}\sum_{n=0}^{\infty}(n+l)!f_{n+l,n}(0)G_l^n(t) , \tag{48}$$

and

$$\langle \hat{a}\hat{a}^\dagger \rangle = \pi \exp\left[\frac{\gamma t}{2}\right] \frac{E_0(t)}{[1-g_0(t)]^2} \sum_{n=0}^{\infty} n! f_{nn}(0) \left[n \frac{E_0^2(t)}{1-g_0(t)} G_0^{n-1}(t) + G_0^n(t) \right]$$
$$= \exp(-\gamma t)[\langle \hat{a}(0)\hat{a}^\dagger(0)\rangle - 1] + \bar{n}_t + 1 \;, \tag{49}$$

where

$$G_l(t) = \frac{\bar{n}+1}{\bar{n}} g_l(t) + \frac{E_l^2(t)}{1-g_l(t)}, \quad 1=0,1,\ldots,\infty \;, \tag{50}$$

respectively. In particular $G_0(t)=1$.

Considering the quite reservoir case ($\bar{n}=0$), we use (38) and the following limit relations:

$$\lim_{\bar{n}\to 0} g_l(t) = 0 \;,$$
$$\lim_{\bar{n}\to 0} E_l(t) = \exp\left[\left[-\frac{\gamma}{2} + i\kappa l\right] t\right] \;, \tag{51}$$
$$\lim_{\bar{n}\to 0} G_l(t) = \frac{\gamma - 2i\kappa l \exp[(-\gamma + 2i\kappa l)t]}{\gamma - 2i\kappa l} \;.$$

Using the moments (48) and (49), we determine the squeezing properties of the third-order nonlinear dissipative oscillator. Defining the operators $\hat{Q}$ and $\hat{P}$ in terms of the operators $\hat{a}, \hat{a}^\dagger$,

$$\hat{Q} = \hat{a} + \hat{a}^\dagger, \quad \hat{P} = -i(\hat{a} - \hat{a}^\dagger), \quad [\hat{Q}, \hat{P}] = 2i\hat{1} \;, \tag{52}$$

we can deduce squeezing either from the variances

$$\langle (\Delta \hat{\underset{P}{Q}})^2 \rangle = -1 \pm 2[\langle \Delta\hat{a}\Delta\hat{a}^\dagger \rangle \pm \mathrm{Re}\langle (\Delta\hat{a})^2 \rangle] \;, \tag{53}$$

or from the quantity

$$\lambda = -1 + 2[\langle \Delta\hat{a}\Delta\hat{a}^\dagger \rangle - |\langle (\Delta\hat{a})^2 \rangle|^2] \;. \tag{54}$$

According to the standard definition of squeezing,[11] this phenomenon is observed if $\min(\langle (\Delta\hat{Q})^2\rangle, \langle (\Delta\hat{P})^2\rangle) < 1$; the principal squeezing[12,13] occurs under the condition $\lambda < 1$.

The principal squeezing definition may be advantageous in the case of the nonlinear oscillator because the free-field frequency is here modified by the self-interaction and depends on the intensity of the field. Thus the quantity λ, which is phase independent, is also independent of this effect.

IV. QUANTUM COHERENCE

The quasidistribution $\phi_{\mathcal{A}}(\alpha,t)$ itself illustrates a number of important features of the third-order nonlinear oscillator model. It is familiar that without dissipation the initial state repeats after a certain time interval, viz., the period. The period attained in our case differs from that of the model in Ref. 7 due to the ordering of the field operators. We will show that our period is half theirs.

In Ref. 5 the role of higher-order derivatives in the equation of motion for the Q function (equal to $\pi\phi_{\mathcal{A}}$) with respect to the revivals of the initial state was explained and in Ref. 7 it was pointed out to the interference in the phase space as a possible source of this quantum effect. In this section we would like to expose once again the origin of the recurrences of the initial state with the aid of an analysis of the quasidistribution $\phi_{\mathcal{A}}(\alpha,t)$ oriented to its relation to the density matrix elements in the number-state basis.

Remembering the third-order nonlinear oscillator models, we introduce their interaction-picture interaction Hamiltonians

$$\hat{H} = \hbar\kappa \hat{a}^{\dagger 2}\hat{a}^2 \;, \tag{55a}$$
$$\hat{H} = \hbar\kappa(\hat{a}^\dagger\hat{a})^2 \;. \tag{55b}$$

Without dissipation the functions $f_{mn}(t)$ read

$$f_{mn}(t) = \exp[i\kappa(m-n)(m+n-1)t] f_{mn}(0) \;, \tag{56a}$$
$$f_{mn}(t) = \exp[i\kappa(m-n)(m+n)t] f_{mn}(0) \;, \tag{56b}$$

and they determine the quasidistributions $\phi_{\mathcal{A}}(\alpha,t)$ with respect to the formulas (6) and (9). From (56a) and (56b) it follows that the functions $f_{mn}(t)$ have the shortest common period $t_a = \pi/\kappa$, $t_b = 2\pi/\kappa$, respectively. This period is possessed, for instance, by the function

$$f_{20}(t) = \exp(2i\kappa t) f_{20}(0) \;, \tag{57a}$$
$$f_{10}(t) = \exp(i\kappa t) f_{10}(0) \;, \tag{57b}$$

in the model (55a) and (55b), respectively. All relevant quantum statistics repeat after this time interval. Recalling the respected role of the classical dynamics in the previous considerations,[3,5,7] we present the classical equation for the complex amplitude

$$\alpha^*(t) = \exp(i2\kappa|\alpha(0)|^2 t)\alpha^*(0) \;. \tag{58}$$

We observe that the rotational shear occurring in the classical description can be reduced and/or eliminated by the assumption of discreet values for the intensity $|\alpha(t)|^2$ and we establish the corresponding periodicity to the case b for $|\alpha(t)|^2 = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$.

To investigate the periodicity we express the quasidistribution $\phi_{\mathcal{A}}(\alpha,t)$ in terms of the density-matrix elements[9]

$$\phi_{\mathcal{A}}(\alpha,t) = \frac{1}{\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \rho_{nm}(t) \frac{\alpha^m \alpha^{*n}}{(m!n!)^{1/2}} \exp(-|\alpha|^2) \;, \tag{59}$$

where[6]

$$\rho_{nm}(t) = (n!m!)^{1/2} \pi f_{mn}(t) \;. \tag{60}$$

For the trigonometric form $\alpha = r\exp(i\varphi)$ of the complex amplitude the formula (59) reads

$$\phi_{\mathcal{A}}(r,\varphi,t)=r\phi_{\mathcal{A}}(r\exp(i\varphi),t)$$
$$=\frac{1}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\rho_{nm}(t)\frac{r^{m+n+1}}{(m!n!)^{1/2}}\times\exp(-r^2)\exp[i(m-n)\varphi] \ . \quad (61)$$

Putting $m=l+k$, $n=l-k$, where $l=0,\frac{1}{2},1,\frac{3}{2},2,\ldots$, and $k=-l,-l+1,\ldots,l-1,l$, we rewrite (61) as

$$\phi_{\mathcal{A}}(r,\varphi,t)=\sum_l \frac{2}{\Gamma(l+1)}r^{2l+1}\exp(-r^2)\phi_l(\varphi,t) \ , \quad (62)$$

where

$$\phi_l(\varphi,t)=\frac{\Gamma(l+1)}{2\pi}\sum_{k=-l}^{l}\frac{\rho_{l-k,l+k}(t)}{[(l-k)!(l+k)!]^{1/2}}\exp(i2k\varphi) \ , \quad (63)$$

or, equivalently,

$$\phi_l(\varphi,t)=\frac{\Gamma(l+1)}{2}\sum_{k=-l}^{l}f_{l+k,l-k}(t)\exp(i2k\varphi) \ . \quad (64)$$

Let us note that it holds

$$\phi_l(\varphi+\pi,t)=(-1)^{2l}\phi_l(\varphi,t) \ , \quad (65)$$

because

$$\exp(i2k\pi)=\begin{cases}\exp(\pm i\pi)=-1=(-1)^{2l}, & l=\frac{1}{2},\frac{3}{2},\ldots,\\ 1=(-1)^{2l}, & l=0,1,2,\ldots\ .\end{cases} \quad (66)$$

Since it is valid that

$$\sum_l\int_0^{2\pi}\phi_l(\varphi,t)d\varphi=1 \ , \quad (67)$$

the functions $\phi_l(\varphi,t)$, $l=0,\frac{1}{2},1,\frac{3}{2},2,\ldots$, form a quasidistribution. Substituting the quasidistribution $\phi_{\mathcal{A}}(r,\varphi,t)$ in the form (62) into the equation [cf. the generalized Fokker-Planck equation (4) for $\gamma=0$, $\bar{n}=0$ in the polar coordinates]

$$\frac{\partial}{\partial t}\phi_{\mathcal{A}}(r,\varphi,t)=\frac{\partial}{\partial\varphi}\left[\kappa\left[2r^2-1+r\frac{\partial}{\partial r}\right]\right]\phi_{\mathcal{A}}(r,\varphi,t) \ , \quad (68a)$$

$$\frac{\partial}{\partial t}\phi_{\mathcal{A}}(r,\varphi,t)=\frac{\partial}{\partial\varphi}\left[\kappa\left[2r^2+r\frac{\partial}{\partial r}\right]\right]\phi_{\mathcal{A}}(r,\varphi,t) \ , \quad (68b)$$

we arrive at the set of equations for the functions $\phi_l(\varphi,t)$,

$$\frac{\partial}{\partial t}\phi_l(\varphi,t)=\frac{\partial}{\partial\varphi}[\kappa(2l-1)]\phi_l(\varphi,t) \ , \quad (69a)$$

$$\frac{\partial}{\partial t}\phi_l(\varphi,t)=\frac{\partial}{\partial\varphi}[\kappa(2l)]\phi_l(\varphi,t) \ . \quad (69b)$$

Of course, it has the solution

$$\phi_l(\varphi,t)=\phi_l(\varphi+\kappa(2l-1)t,0) \ , \quad (70a)$$

$$\phi_l(\varphi,t)=\phi_l(\varphi+\kappa(2l)t,0) \ , \quad (70b)$$

respectively, which can be verified easily by substituting (56a) and (56b) in (63).

The set of equations (69b) for the functions $\phi_l(\varphi,t)$ can be derived on substituting the series

$$\phi(r,\varphi,t)=\sum_l\delta(r-l^{1/2})\phi_l(\varphi,t) \ , \quad (71)$$

where δ is the Dirac δ function, in the classical Liouville equation

$$\frac{\partial}{\partial t}\phi(r,\varphi,t)=\kappa 2r^2\frac{\partial}{\partial\varphi}\phi(r,\varphi,t) \ . \quad (72)$$

Because of the following:

$$\int_0^{\infty}\int_0^{2\pi}\phi(r,\varphi,t)d\varphi\, dr=1 \ , \quad (73)$$

the function $\phi(r,\varphi,t)$ given in (71) is a quasidistribution. It is evident that the second-order term in Eq. (68a) and (68b) occurs due to the presence of the smooth functions $2[\Gamma(l+1)]^{-1}r^{2l+1}\exp(-r^2)$ in the series (62) instead of the singularities $\delta(r-l^{1/2})$ in the series (71).

The equations of motion (69a) and (69b) evoke the picture of the quasidistributions $\phi_l(\varphi,t)$ rotating in a clockwise direction with the angular velocity $\kappa(2l-1),\kappa(2l)$, respectively. In the case (55a) the quasidistribution $\phi_l(\varphi,t)$ for $l=\frac{1}{2}$ is not moving and for $l=1$ it is rotating with the angular velocity κ but according to the formula (65) it consists of two identical parts, hence the circular frequency is 2κ and the period is π/κ. In the case (55b) the quasidistribution $\phi_{1/2}(\varphi,t)$ is rotating with the angular velocity κ and with respect to the formula (65) it consists generally of two different parts, thus the circular frequency is κ and the period is $2\pi/\kappa$.

For the coherent state $|\xi\rangle$ it holds particularly that

$$f_{mn}(0)=\frac{1}{\pi}\frac{\xi^{*m}}{m!}\frac{\xi^n}{n!}\exp(-|\xi|^2) \quad (74)$$

and the corresponding quasidistribution $\phi_l(\varphi,0)$ reads

$$\phi_l(\varphi,0)=\frac{1}{2\pi^{1/2}}\exp(-|\xi|^2)\frac{1}{\Gamma(l+\frac{1}{2})}[|\xi|\cos(\varphi-\psi)]^{2l} \ ,$$
$$\xi=|\xi|\exp(i\psi) \ . \quad (75)$$

The time development of (75) with respect to the two nonlinear oscillator models is governed by the formula

$$\phi_l(\varphi,t)=\frac{1}{2\pi^{1/2}}\exp(-|\xi|^2)\frac{1}{\Gamma(l+\frac{1}{2})}\times\{|\xi|\cos[\varphi-\psi+\kappa(2l-1)t]\}^{2l} \ , \quad (76a)$$

$$\phi_l(\varphi,t)=\frac{1}{2\pi^{1/2}}\exp(-|\xi|^2)\frac{1}{\Gamma(l+\frac{1}{2})}\times\{|\xi|\cos[\varphi-\psi+\kappa(2l)t]\}^{2l} \ . \quad (76b)$$

Analyzing these relations, we can find that although the period in the model (55a) does amount to one-half the period in the model (55b), the physical state at the time $t=\frac{1}{2}(\pi/2\kappa)$ in the case (55a) corresponds to that at the time $t=\frac{1}{4}(\pi/\kappa)$ in the case (55b), i.e., at equal times, with regard to the squeezing properties.

The considerations in this section confirm the picture that the quantum coherence is sensitive to dissipation since it consists in the harmony of the orbits $|\alpha|^2=0,\frac{1}{2},1,\frac{3}{2},2,\ldots$. The dissipation does not only mean that the system as described by $\phi_l(\varphi,t)$ descends to the lower values of half-integer intensities l, but also that the quasidistribution $\phi_l(\varphi,t)$ may rotate with velocities which do not preserve the harmony of motion.

V. CONCLUSION

Using the master-equation approach, we determined the quasidistribution related to the antinormal ordering of field operators for the third-order nonlinear dissipative oscillator being initially in an arbitrary state. With the aid of the standard techniques the photon number distribution, its factorial moments, and squeezing characteristics were established. The photon statistics proved to be independent of the nonlinearity, so that they coincide with those for the linear dissipative oscillator. The traditional and principal squeeze variances were expressed in terms of the lowest antinormally ordered moments. The investigation of new quasidistributions connected with the phase of the complex field amplitude conveyed more information on some important properties of the considered radiation, especially the recurrences of the initial state. For the initial coherent state results consistent with those of Daniel and Milburn were obtained.

[1]G. Dattoli and A. Torre, Phys. Rev. A **37**, 1571 (1988).

[2]P. D. Drummond and D. F. Walls, J. Phys. A **13**, 725 (1980).

[3]G. J. Milburn, Phys. Rev. A **33**, 674 (1986).

[4]F. Haake, H. Risken, C. Savage, and D. F. Walls, Phys. Rev. A **34**, 3969 (1986).

[5]G. J. Milburn and G. A. Holmes, Phys. Rev. Lett. **56**, 2237 (1986).

[6]V. Peřinová and A. Lukš, J. Mod. Opt. **35**, 1513 (1988).

[7]D. J. Daniel and G. J. Milburn, Phys. Rev. A **39**, 4628 (1989).

[8]V. Bužek, Phys. Rev. A **39**, 5432 (1989).

[9]J. Peřina, *Quantum Statistics of Linear and Nonlinear Optical Phenomena* (Reidel, Dordrecht, 1984).

[10]A. Lukš and V. Peřinová, Czech. J. Phys. B **37**, 1224 (1987).

[11]D. F. Walls, Nature **306**, 141 (1983).

[12]A. Lukš, V. Peřinová, and J. Peřina, Opt. Commun. **67**, 149 (1988).

[13]A. Lukš, V. Peřinová, and Z. Hradil, Acta Phys. Pol. A **74**, 713 (1988).