PHYSICAL REVIEW A VOLUME 41, NUMBER 7 1 APRIL 1990

Microscopic theory of the continuous measurement of photon number

Nobuyuki Imoto and Masahito Ueda
NTT Basic Research Laboratories, 3-9-11 Midori-cho, Musashino-shi, Tokyo 180, Japan

Tetsuo Ogawa
Department of Applied Physics, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113, Japan
(Received 6 November 1989)

This paper proposes a microscopic model for the continuous measurement of photon numbers, where the continuous measurement is simulated by a sequence of infinitesimal processes. Each process is composed of a unitary evolution for the coupled system, which consists of the field and the measuring apparatus, followed by a projection according to the readout of the measuring apparatus. The proposed model consists of two-level atoms that are coupled one by one with an optical cavity field via the electric-dipole interaction. Each atomic level is then read out after the interaction, causing a nonunitary state reduction of the field. In particular, it is shown that this microscopic model leads to the Srinivas-Davies model that has been postulated for the photodetection process.

Continuous photodetection is an interesting quantum measurement process because the photon number is measured continuously rather than instantaneously owing to the one-by-one conversion of photons into photoelectrons. The quantum theory of the photodetection process, which takes the measurement back action into account, was first treated by Mollow.[1] In his treatment, however, unitary evolution of the field-detector system is assumed until the photoelectron number is measured. The quantum theory of *continuous* photon counting was initiated by Srinivas and Davies,[2] in which they postulated a model for one-count and no-count processes, namely, the Srinivas-Davies model (SD model). A theory for nonunitary evolution of the field under continuous photon counting has been developed by the authors[3,4] based on the SD model. The SD model has been, however, a postulate which has not been explained microscopically. The present paper gives a microscopic foundation of the SD model, using the Jaynes-Cumming Hamiltonian (JC Hamiltonian) for the field and a two-level atom. Zoller, Marte, and Walls[5] also pointed out that the solution of a master equation for photoemission, which is derived from the JC Hamiltonian,[6] can be expressed by one-emission and no-emission processes similar to the SD model. In contrast, we derive the SD model directly from the JC Hamiltonian. A general theory of continuous measurement has been developed by Barchielli, Lanz, and Prosperi[7] and specific models are examined by Caves and Milburn for the position of a free particle.[8] The present paper, on the other hand, treats the continuous measurement of photon numbers.

In the SD model, the one-count process and the no-count process play major roles. (i) The one-count process is described by a superoperator J such that

$$\hat{\rho}(t^+) \propto J\hat{\rho}(t) \equiv \lambda \hat{a}\hat{\rho}(t)\hat{a}^\dagger , \tag{1}$$

where $\hat{\rho}(t)$ and $\hat{\rho}(t^+)$ are the field-density operators before and after the one-count process, λ is a constant that denotes the magnitude of coupling between the field and the detector and it means the reciprocal expectation value of waiting time for a single-photon state, and $\hat{a}$ ($\hat{a}^\dagger$) is the photon annihilation (creation) operator.[9] (ii) The no-count process is described by

$$\hat{\rho}(t+\tau) \propto S_\tau \hat{\rho} t \equiv \exp[(-i\omega - \lambda/2)\hat{n}\tau]\hat{\rho}(t) \times \exp[(i\omega - \lambda/2)\hat{n}\tau] , \tag{2}$$

where ω is the optical frequency of the mode under consideration, $\hat{n}$ ($\equiv \hat{a}^\dagger \hat{a}$) is the photon-number operator, and τ is the time during which no photon is counted from t to $t+\tau$. Equation (2) is derived from Eq. (1) by assuming that the one-count process and the no-count process form an exclusive exhaustive set of events.[2] Equations (1) and (2) are, however, postulates and a microscopic derivation of the above SD model has not been presented to date.

As has been pointed out in Refs. 2 and 10, the SD model cannot be described by von Neumann's projection postulate,[11] according to which an arbitrary density operator $\hat{\rho}$ is projected onto $\hat{P}\hat{\rho}\hat{P}$ with a probability $P(X) \equiv \mathrm{Tr}[\hat{P}\hat{\rho}(t)\hat{P}]$, where $\hat{P} \equiv |X\rangle\langle X|$ is the projection operator, and $|X\rangle$ is an eigenvector of the measured observable $\hat{X}$ with eigenvalue X. Since annihilation operator $\hat{a}$ is not a projection operator, it is obvious that the SD model, expressed by Eq. (1), cannot be described by von Neumann's projection postulate. In other words, von Neumann's postulate is applicable only to a first-kind (nondestructive) measurement while the photodetection process is a second-kind (destructive) measurement. It is therefore necessary to generalize the concept of quantum measurement so that it includes not only measurements described by an instantaneous projection but also includes continuous measurement processes, such as the photodetection process described above.

In this paper, we propose a general microscopic model of continuous measurement and apply this model to the photodetection process. The main idea is that continuous measurement can be accomplished by the simultaneous progress of system-apparatus coupling and readout of the measuring apparatus. This process is simulated by a sequence of infinitesimal processes, each of which represents system-apparatus coupling via unitary interaction followed by the readout of the measuring apparatus.

Let us consider that the system and the apparatus are coupled via an interaction Hamiltonian, $\hat{H}_{\rm int}$. The coupled density operator after the interaction then becomes

$$\hat{\rho}_{s-a}(\Delta t) = \hat{U}\hat{\rho}_s(0) \otimes \hat{\rho}_a(0)\hat{U}^\dagger , \quad (3)$$

where $\hat{\rho}_{s-a}(\Delta t)$ is the coupled density operator after interaction time Δt, $\hat{U}$ is a unitary operator defined as $\hat{U} \equiv \exp(-i\hat{H}_{\rm int}\Delta t/\hbar)$, and $\hat{\rho}_s(0)$ and $\hat{\rho}_a(0)$ are the initial density operators for the system and apparatus, respectively. When the apparatus is measured by a process described by a projection operator in a Hilbert space, the state of the system after the measurement of the apparatus is obtained, using the probability-operator measures for the coupled systems,[12] as

$$\hat{\rho}_s(\Delta t) = \frac{{}_a\langle X|\hat{\rho}_{s-a}(\Delta t)|X\rangle_a}{{\rm Tr}_s[{}_a\langle X|\hat{\rho}_{s-a}(\Delta t)|X\rangle_a]} = \frac{{\rm Tr}_a[\hat{\rho}_{s-a}(\Delta t)\hat{\rho}_a^{(\rm read)}]}{{\rm Tr}_{s-a}[\hat{\rho}_{s-a}(\Delta t)\hat{\rho}_a^{(\rm read)}]} , \quad (4)$$

where $\hat{\rho}_s(\Delta t)$ is the system density operator just after the measurement of the apparatus, $|X\rangle_a$ is an eigenvector of the apparatus observable $\hat{X}_a$ with an eigenvalue of X, and $\hat{\rho}_a^{(\rm read)} \equiv |X\rangle_{a\,a}\langle X|$. Symbols ${\rm Tr}_s$, ${\rm Tr}_a$, and ${\rm Tr}_{s-a}$ denote traces over the system, the apparatus, and the system and apparatus, respectively. After the apparatus observable is measured, the state of the apparatus is reset to its initial state in order to prepare for the next measurement. If we write the measurement process expressed by Eq. (4) as $\hat{\rho}_s(\Delta t) = M[\hat{\rho}_{s-a}(\Delta t)]$, then the total process within a time interval of Δt is characterized by the state change for the system as

$$\hat{\rho}_s(0) \rightarrow \hat{\rho}_s(\Delta t;X) = M[\hat{U}\hat{\rho}_s(0) \otimes \hat{\rho}_a(0)\hat{U}^\dagger] , \quad (5)$$

with measurement readout X.

Repeating the above process yields a sequence of readout values $X_1, X_2, \ldots$ for every time interval Δt. The time evolution of the system density operator after $t = N\Delta t$ is obtained by making N successive operations of the above process as

$$\hat{\rho}_s(N\Delta t;X_1,X_2,\ldots,X_N) = M_N\{\cdots M_2[\hat{U}M_1[\hat{U}\hat{\rho}_s(0) \otimes \hat{\rho}_a(0)\hat{U}^\dagger] \otimes \hat{\rho}_a(0)\hat{U}^\dagger] \otimes \cdots\} , \quad (6)$$

where M_n depends on readout X_n. If we make $\Delta t \rightarrow 0$ (and $N \rightarrow \infty$ with $N\Delta t = t$), the readout becomes a function of time as $X(t)$. The nonunitary state evolution of the system depends on all readouts $X(\tau)$ $(0 \le \tau \le t)$ and the initial state $\hat{\rho}_s(0)$. Therefore, $\hat{\rho}_s(t)$ is expressed in a functional form as $\hat{\rho}_s(t) = f[X(\tau)(0 \le \tau \le t), \hat{\rho}_s(0)]$. In order to obtain an explicit expression of $f[X(\tau)(0 \le \tau \le t), \hat{\rho}_s(0)]$, a more specific model for the system and apparatus is needed. It should be noted that in making $\Delta t \rightarrow 0$, the magnitude of the coupling between the system and the apparatus should be changed appropriately, as will be discussed in a later example [see Eq. (16)].

Now, let us apply the above theory to the photodetection process schematically shown in Fig. 1(a). The photodetector is coupled to the optical field in a closed cavity so weakly that the detector does not absorb more than one photon in an infinitesimal time interval. The output of the detector is a time sequence of photoelectric current pulses, each of which represents the detection of one photon. Figure 1(b) shows the physical model we propose for the photodetection scheme in Fig. 1(a). Here, two-level atoms that are prepared initially in the ground state are driven into the cavity one by one. Each atom passes through the cavity within the same time duration Δt so that it interacts with the cavity field via the electric-dipole interaction during Δt. This interaction causes a unitary evolution of the atom-field system. After passing through the field, the level of each atom is measured. This causes a state change in the optical field. An atom in the upper level corresponds to the one-count process, and an atom in the ground state corresponds to the no-count process.

The electric-dipole interaction can be described[13] by the JC interaction Hamiltonian:

$$\hat{H}_{\rm int} = \hbar g(\hat{a}\hat{\sigma}^\dagger + \hat{a}^\dagger\hat{\sigma}) , \quad (7)$$

where $\hat{\sigma}$ is the level-lowering operator for the two-level atom defined by $\hat{\sigma} \equiv |g\rangle_{a\,a}\langle e|$, where $|g\rangle_a$ and $|e\rangle_a$ are state vectors for the ground and excited states, respectively. For each time duration $[t_0, t_0+\Delta t]$, the initial state is expressed as

$$\hat{\rho}(t_0) = \hat{\rho}_f(t_0) \otimes |g\rangle_{a\,a}\langle g| , \quad (8)$$

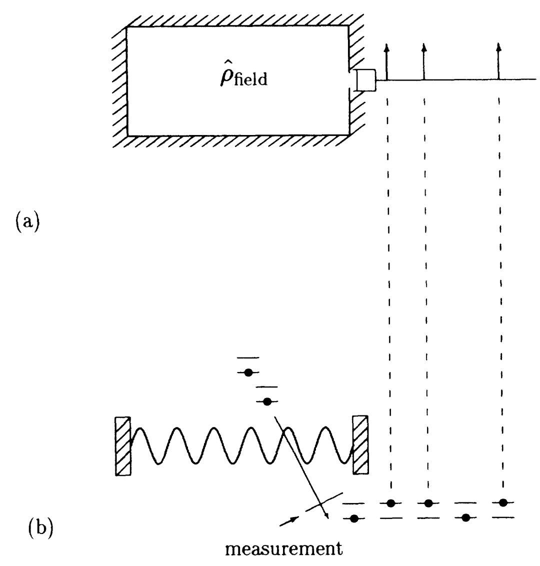

FIG. 1. Photon-counting system for an optical field in a closed cavity. (a) The real system being considered. The density operator of the field gradually changes into the vacuum state as the photodetection process proceeds. (b) Corresponding physical model. Two-level atoms in the ground state are successively driven into the cavity one by one. The atoms are excited with a small probability, and the atomic level is measured. Atoms in the excited level correspond to one-count processes, and atoms in the ground level correspond to no-count processes.

where $\hat{\rho}(t_0)$ is the initial state of the atom-field system, $\hat{\rho}_f(t_0)$ is the initial state of the field, and $|g\rangle_{a\,a}\langle g|$ is the ground state of the atom. The evolution of the density operator for the total system can be written, in the interaction picture, as

$$\hat{\rho}(t)=\hat{\rho}(t_0)+\sum_{m=0}^{\infty}\left[\frac{1}{i\hbar}\right]^m\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\cdots\int_{t_0}^{t_{m-1}}dt_m[\hat{H}_{\rm int}(t_1),[\hat{H}_{\rm int}(t_2),\ldots,[\hat{H}_{\rm int}(t_m),\hat{\rho}(t_0)]\ldots]]\,. \tag{9}$$

Using Eqs. (7), (8), and $t\equiv t_0+\Delta t$, $\hat{\rho}(t_0+\Delta t)$ is expressed as[13]

$$\begin{aligned}\hat{\rho}(t_0+\Delta t)=\hat{\rho}(t_0)&+ig\Delta t[\hat{\rho}_f(t_0)\hat{a}^\dagger\otimes|g\rangle_{a\,a}\langle e|-\hat{a}\hat{\rho}_f(t_0)\otimes|e\rangle_{a\,a}\langle g|]\\&+\frac{(ig\Delta t)^2}{2}\{[\hat{\rho}_f(t_0)\hat{n}+\hat{n}\hat{\rho}_f(t_0)]\otimes|g\rangle_{a\,a}\langle g|-2\hat{a}\hat{\rho}_f(t_0)\hat{a}^\dagger\otimes|e\rangle_{a\,a}\langle e|\}\,,\end{aligned} \tag{10}$$

where the two-level atom is assumed to be resonant with the field frequency. Here, we consider up to the second order of the perturbation because the second-order term is the lowest order that describes the real transition. Higher-order terms can be neglected for the present case because each atom is assumed to pass through the cavity in an infinitesimal time Δt.

When the atomic level is measured, the associated state reduction of the field is calculated using Eq. (4) for the one-count process and the no-count process. (i) For the one-count process, $\hat{\rho}_a^{(\rm read)}$ is equal to $|e\rangle_{a\,a}\langle e|$. Substituting this $\hat{\rho}_a^{(\rm read)}$ and Eq. (10) into Eq. (4), we obtain

$$J[\hat{\rho}_f(t_0)]\Delta t=[g^2\Delta t\hat{a}\hat{\rho}_f(t_0)\hat{a}^\dagger]\Delta t\,. \tag{11}$$

(ii) For the no-count process, $\hat{\rho}_a^{(\rm read)}(t)$ is equal to $|g\rangle_{a\,a}\langle g|$. Using Eqs. (4) and (10), we obtain

$$S_{\Delta t}[\hat{\rho}_f(t_0)]=\exp(-i\omega\hat{n}\Delta t)\left[\hat{\rho}_f(t_0)-\frac{(g\Delta t)^2}{2}[\hat{\rho}_f(t_0)\hat{n}+\hat{n}\hat{\rho}_f(t_0)]\right]\exp(i\omega\hat{n}\Delta t)\,, \tag{12}$$

where the picture is changed into the Schrödinger picture from the interaction picture. Since Δt is very small, we can write

$$\hat{\rho}_f(t_0)-\frac{(g\Delta t)^2}{2}[\hat{\rho}_f(t_0)\hat{n}+\hat{n}\hat{\rho}_f(t_0)]=\exp\left[-\frac{(g\Delta t)^2}{2}\hat{n}\Delta t\right]\hat{\rho}_f(t_0)\exp\left[\frac{(g\Delta t)^2}{2}\hat{n}\Delta t\right]. \tag{13}$$

Using this, Eq. (12) can be written as

$$S_{\Delta t}[\hat{\rho}_f(t_0)]=\exp\left[\left(-\omega-\frac{g^2\Delta t}{2}\right)\hat{n}\Delta t\right]\hat{\rho}_f(t_0)\exp\left[\left(i\omega-\frac{g^2\Delta t}{2}\right)\hat{n}\Delta t\right]. \tag{14}$$

If we set $\tau=N\Delta t$ (N is an integer), the N successive operations of Eq. (14) yield

$$S_\tau[\hat{\rho}_f(t_0)]=\exp\left[\left(-i\omega-\frac{g^2\Delta t}{2}\right)\hat{n}\tau\right]\hat{\rho}_f(t_0)\exp\left[\left(i\omega-\frac{g^2\Delta t}{2}\right)\hat{n}\tau\right]. \tag{15}$$

Equations (11) and (15) describe the one-count and no-count processes of the present model, respectively. Both processes describe the nonunitary evolution for the field owing to the continuous measurement. The factor $g^2\Delta t$ in Eqs. (11) and (14) represents the photon-counting rate, which is the reciprocal expectation value of waiting time for a single-photon state. We set this factor as

$$\lambda\equiv g^2\Delta t\,. \tag{16}$$

Hence, Eqs. (11) and (14) become Eqs. (1) and (2), respectively, that is, the SD model.

The present argument indicates that the SD model, which has been postulated, turns out to be equivalent to a sequence of infinitesimal processes, each of which is composed of a unitary evolution via the electric-dipole interaction and the projection onto the atomic level. Thus, continuous-state reduction by one-by-one photon counting is describable by successive-state reductions with the readout of the coupled measuring apparatus.

The present theory is not limited to destructive continuous measurements but can also be applied to nondestructive continuous measurement. If the unitary coupling between the system and apparatus is associated with an interaction which satisfies the quantum nondemolition (QND) conditions,[14] then the process becomes a continuous QND measurement. This subject will be reported elsewhere.

The authors would like to thank Dr. Y. Yamamoto of the NTT Basic Research Laboratories for useful discussions.

[1]B. R. Mollow, Phys. Rev. **168**, 1896 (1968).

[2]M. D. Srinivas and E. B. Davies, Opt. Acta **28**, 981 (1981); **29**, 235 (1982).

[3]M. Ueda, Quantum Optics **1** (2), 131 (1989); M. Ueda, this issue, Phys. Rev. A **41**, 3875 (1990).

[4]M. Ueda, N. Imoto, and T. Ogawa, this issue, Phys. Rev. A **41**, 3891 (1990).

[5]P. Zoller, M. Marte, and D. F. Walls, Phys. Rev. A **35**, 198 (1987).

[6]B. R. Mollow, Phys. Rev. A **12**, 1919 (1975).

[7]A. Barchielli, L. Lanz, and G. M. Prosperi, Nuovo Cimento B **72**, 79 (1982).

[8]C. M. Caves and G. J. Milburn, Phys. Rev. A **36**, 5543 (1987).

[9]A caret is used to denote a Hilbert space operator throughout this paper. For a superoperator such as J and S, however, a caret is not used.

[10]C. A. Holmes, G. J. Milburn, and D. F. Walls, Phys. Rev. A **39**, 2493 (1989).

[11]J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton Univ. Press, Princeton, NJ, 1955).

[12]For example, C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976), pp. 83–85.

[13]For example, M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, MA, 1974), pp. 230–233.

[14]N. Imoto, H. A. Haus, and Y. Yamamoto, Phys. Rev. A **32**, 2287 (1985).