Dark-pulse solitons in nonlinear-optical fibers

S. A. Gredeskul and Yu. S. Kivshar*

Institute for Low Temperature Physics and Engineering, UkrSSR Academy of Sciences, 47 Lenin Avenue, Kharkov 310164, Union of Soviet Socialist Republics

M. V. Yanovskaya

Kharkov State University, 4 Dzerjinskaya Square, Kharkov 310022, Union of Soviet Socialist Republics

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We study analytically the generation and propagation of dark-pulse solitons in nonlinear-optical fibers in the normal dispersion regime. We demonstrate that dark-pulse solitons may be created as pairs by an arbitrary dark pulse with equal boundary conditions without a threshold. We also predict soliton generation by a step of an input pulse phase and obtain the parameters of the generated solitons. The case of soliton generation by a random input pulse is described, and probability of the generation for a Gaussian random pulse is calculated. We also consider the case when dark pulses are generated on a background of finite extent. This situation is related to experiments by Krökel *et al.* [Phys. Rev. Lett. **60**, 29 (1988)] and Weiner *et al.* [Phys. Rev. Lett. **61**, 2445 (1988)], who produced dark pulses (e.g., ~1 psec) on the long bright pulse (e.g., ~10^2 psec). We demonstrate that in this case dark pulses are not, strictly speaking, solitons, and in terms of the inverse scattering transform they correspond not to bound states (discrete spectrum) but to quasistationary ones lying in a continuous spectrum. We calculate the parameters of the pulses produced and study their dynamics. It is demonstrated that propagation of these dark pulses is similar to soliton motion. We also study the influence of small perturbations on dynamics of dark solitons, e.g., dispersive broadening of a background and dissipative losses.

I. INTRODUCTION

Although solitons arise in many areas of physics (e.g., solid-state physics, plasmas, etc.), the single-mode optical fiber has been an especially convenient object for their study. As was first shown by Hasegawa and Tappert,[1,2] the nonlinear refractive index in glass optical fibers may compensate for group-velocity dispersion (GVD) and may lead to propagation of solitary waves without distortion.

The propagation of short optical pulses in single-mode optical fibers is described by the well-known nonlinear Schrödinger (NLS) equation.[1,2] In an appropriate system of normalized coordinates this equation is

$$i\frac{\partial u}{\partial z}+\sigma\frac{\partial^2 u}{\partial t^2}+2|u|^2u=0\ , \tag{1.1}$$

where u is the (complex) amplitude envelope of the pulse, x is the distance along the fiber, and the time variable t is a retarded time measured in a frame of reference moving along the fiber at the group velocity. The normalizing length z_0 is defined by

$$z_0=4\pi(cT_0)^2/|D(\lambda_0)|\lambda_0\ . \tag{1.2}$$

In Eq. (1.2) the parameter T_0 is a width for the input pulse, $D(\lambda)=\lambda^2 d^2n/d\lambda^2$ is the GVD in dimensionless units, n is the refractive index of the core material, c is the velocity of light, and λ_0 is the vacuum wavelength (see, e.g., Ref. 3).

The solutions of this equation divide into two different regimes depending on the sign of σ ($\sigma=\pm1$), i.e., the relative sign of the fiber's GVD and the nonlinear Kerr coefficient. In silica glass optical fiber GVD is zero at a wavelength of about 1.3 μm, positive at larger wavelengths, and negative at shorter ones (see, e.g., Ref. 4). Optical communication systems generally operate in one of two ways, known as coherent and incoherent. Incoherent systems use pulses of light and detect the total energy of the pulse at the receiver. In this case the appropriate boundary condition in Eq. (1.1) is $|u|\to0$ at $t\to\pm\infty$. For the negative GVD ($\sigma=+1$), Zakharov and Shabat[5] showed that Eq. (1.1) with the zero "boundary" condition possesses bright soliton solutions. Since then, soliton propagation of bright optical pulses has been verified in a number of elegant experiments performed in the negative GVD region of the spectrum (see, e.g., a pioneer work by Mollenauer, Stolen, and Gordon[6]). Most recently, transmission of 55-psec optical pulses through 6000 km of fiber was achieved by use of a combination of nonlinear soliton propagation and Raman amplification.[7] For the positive GVD ($\sigma=-1$) there are no bright solitons; instead the pulses undergo enhanced broadening[8] and chirping.[9,10]

Coherent optical systems use a modulated continuous beam and detect the modulation by mixing with a local oscillator at the receiver. In this case the boundary condition to Eq. (1.1) becomes $|u|\to$const, at $t\to\pm\infty$. At $\sigma=+1$ (the negative GVD) a monochromatic plane wave, $|u|=$const, is unstable to the formation of sidebands (it is the so-called Benjamin-Feir instability) and, as a result, the solutions with $|u|\to$const at $t\to\pm\infty$ are unstable too. At $\sigma=-1$ (the positive GVD) the solution $|u|=$const is stable, and, therefore, Eq. (1.1) may have soliton solutions as localized nonlinear excitations of a cw background. Indeed, the NLS equation with the positive GVD is exactly integrable[11] and admits the so-called dark-soliton solutions, consisting of a rapid dip in the intensity of a cw background. The general form of the dark soliton is

$$u(z,t)=u_0\frac{(\tilde{\lambda}+i\nu)^2+\exp Z e^{2iu_0^2 z}}{(1+\exp Z)}\ , \tag{1.3}$$

$$Z=2\nu u_0(t+2\tilde{\lambda}u_0 z),\quad \nu=(1-\tilde{\lambda}^2)^{1/2}\ , \tag{1.4}$$

which corresponds to the boundary conditions $|u|\to u_0$ at $t\to\pm\infty$. The soliton [Eqs. (1.3) and (1.4)] has the only parameter ν which characterizes the soliton intensity.

Recently, Krökel *et al.*[12] have observed experimentally the formation of dark-pulse solitons on a broad bright pulse with a rapid intensity dip stimulated by a driving pulse. Because the sign of the self-phase modulation for the dark pulse is reversed, it becomes possible to balance GVD to allow the dark-pulse propagation along the bright pulse without a distortion. Krökel *et al.*[12] observed two 0.6-psec dark pulses generated by a single 0.3-psec input dark pulse which was produced on the much longer duration (100 psec) bright pulse. Other researchers have also reported observations of dark solitons.[13,14] In particular, in the recent paper by Weiner *et al.*[14] the experimental observation of the fundamental dark soliton in a 1.4-m optical fiber was presented. These experiments utilize a specially shaped antisymmetric input pulse which closely corresponds to the form of the fundamental dark soliton, i.e., the quiescent (hyperbolic-tangent) dark pulse with zero intensity at its center [i.e., $\tilde{\lambda}=0$ in Eqs. (1.3) and (1.4)].

The paper aims to consider generation and dynamics of dark solitons in relation to the above-mentioned real experimental studies of dark pulses in nonlinear single-mode optical fibers. First of all, in Sec. II of the paper we consider the generation of the dark solitons. This problem is very important for the explanation of some experimental results by Krökel *et al.*[12] and Weiner *et al.*,[14] and also for the potential use of dark solitons in optical communication systems. As is well known, the process of the generation of bright solitons described by the NLS equation is threshold. Namely, the bright solitons are created from a localized pulse if the area under its envelope is more than the threshold value $\pi/2$ (see, e.g., Ref. 15):

$$\int_{-\infty}^{\infty}|u(z=0,t)|dt\geq\frac{\pi}{2}\ . \tag{1.5}$$

In Sec. II A of the paper we demonstrate that, unlike the case of bright solitons, the creation of dark solitons takes place without a threshold. We calculate the parameters of the solitons generated by an arbitrary small dark pulse, and also consider more general case when the input pulse is a random Gaussian one.

It is interesting to note that in the case of the positive GVD the new method of soliton generation may be used. Namely, the dark solitons may be generated by a background phase modulation. We consider the cases of a phase step or two phase steps and calculate parameters of created dark solitons (Sec. II C).

In Sec. III we study the dark-pulse dynamics for the cases experimentally investigated by Krökel *et al.*[12] and Weiner *et al.*[14] We consider dark pulses on a background of finite extent, and demonstrate that in this case dark pulses are not, strictly speaking, solitons. In the terms of the inverse scattering transform these dark pulses correspond not to the bound states of the discrete spectrum, but to the so-called quasistationary ones. Using a simple δ-function input pulse on a large background, we calculate the parameters of such dark pulses and also demonstrate that propagation of these soliton-like pulses is similar to the soliton motion with a slowly decreasing amplitude.

In Sec. IV we consider another very important problem related to dark solitons, i.e., the decay of a random optical pulse in a nonlinear optical fiber in the positive GVD region. We demonstrate that an input random pulse of large duration will decay mainly into dark solitons. The connection between the above-mentioned problem and the well-known results of the theory of wave scattering by a disordered system of impurities is discussed. Namely, we demonstrate that for the optical system under consideration one may introduce a so-called "localization length," i.e., the time scale τ_0, so that for $T\gg\tau_0$, T being the duration of the random pulse, properties of the random pulse will be described by localized states only, i.e., by dark solitons.

In Sec. V we briefly discuss the influence of real perturbations on the soliton dynamics. First of all, we study the influence of dispersive broadening of the finite background and demonstrate that the dark pulse adiabatically maintains its soliton characteristics as the background pulse evolves, i.e., if $u_0\sim z^{-1/2}$ as for linear wave packets then, according to Eqs. (1.3) and (1.4), the duration of the dark soliton is proportional to $\tau_s=(2\nu u_0)^{-1}\sim z^{1/2}$ and its intensity $\max(u_0^2-|u|^2)\equiv I_s\sim u_0^2\nu^2$, so that $I_s\tau_s^2=\text{const}$. We also discuss the influence of small dissipative losses on the soliton motion.

In conclusion (Sec. VI) we summarize our results and discuss possible applications of dark-pulse solitons to optics communication systems.

II. GENERATION OF DARK SOLITONS

A. An arbitrary small dark pulse

1. General approach

Let us consider the NLS equation (1.1) at $\sigma=-1$ with the boundary conditions

$$u(z,t)\to u_0e^{i\alpha}=\text{const}\quad\text{at } t\to\pm\infty\ . \tag{2.1}$$

For the symmetric boundary conditions (2.1) the generation of dark solitons by a small intensive hole produced by a driving pulse at the edge of a fiber (similar to the experiments by Krökel *et al.*[12]) may be described by the initial condition at $z=0$,

$$u(0,t)=u_0e^{i\alpha}-u_1(t)\ . \tag{2.2}$$

Here $u_1(t)$ is a localized function, i.e., $|u_1|\to 0$ at $t\to\pm\infty$, where u_0 and an initial phase shift α are real constant parameters. In experiments by Krökel *et al.*,[12] the short dark pulse $u_1(t)$ was produced on the much longer duration bright pulse, so that, strictly speaking, the condition $|u|\to$const at $t\to\pm\infty$ was not held. As will be demonstrated in Sec. III, to study a soliton generation we may consider more simple input pulses because finite duration of a cw background will change mainly the subsequent evolution of dark pulses.

According to the inverse scattering transform for the NLS equation (1.1), to find which type of initial function generates solitons one has to investigate the eigenvalue Zakharov-Shabat (ZS) problem,[11]

$$\frac{\partial}{\partial t}\Psi_1 = i\lambda\Psi_1 - iu(0,t)\Psi_2 , \tag{2.3a}$$

$$\frac{\partial}{\partial t}\Psi_2 = -i\lambda\Psi_2 + iu^*(0,t)\Psi_1 , \tag{2.3b}$$

where the asterisk denotes complex conjugation. As was shown by Zakharov and Shabat,[11] each real discrete eigenvalue $|\lambda|<u_0$, $\lambda^2=u_0^2-w^2$, corresponds to a dark soliton with the amplitude w moving with the velocity 2λ [see Eqs. (1.3) and (1.4), where $\nu=w/u_0$, $\tilde{\lambda}=\lambda/u_0$].

First of all, we study the soliton creation by an arbitrary small driving pulse, $|u_1|\ll u_0$. In this case the condition for the soliton generation and parameters of generated solitons may be obtained in a quite general form.

After the substitution $\Psi_2\to\Psi_2 e^{-i\alpha}$ we obtain the same eigenproblem (2.3) but with an initial potential

$$u(0,t)=u_0-e^{-i\alpha}u_1(t) .$$

It is also convenient to rewrite the eigenvalue problem for the linear combinations of the functions, $\Phi_-=\Psi_1-\Psi_2$, and $\Phi_+=\Psi_1+\Psi_2$,

$$-i\frac{\partial}{\partial t}\Phi_- = (\lambda-u_0)\Phi_+ + a(t)\Phi_+ - ib(t)\Phi_- , \tag{2.4a}$$

$$-i\frac{\partial}{\partial t}\Phi_+ = (\lambda+u_0)\Phi_- - a(t)\Phi_- + ib(t)\Phi_+ , \tag{2.4b}$$

where we use the notation

$$a(t)=\mathrm{Re}[u_1(t)e^{-i\alpha}],\quad b(t)=\mathrm{Im}[u_1(t)e^{-i\alpha}] .$$

For the small-intensity input pulse $u_1(t)$ it is natural to present the eigenvalues of the discrete spectrum as $\lambda_{1,2}=\pm u_0(1-\delta)$. Here δ is a positive parameter because for $\delta<0$ eigenvalues of the discrete spectrum are absent. Solutions of the eigenproblem (2.4) at $u_1=0$ and $\lambda_1=u_0(1-\delta)$ may be presented as follows ($\delta>0$):

$$(\Phi_\pm)_0 \sim \exp[\pm|t|u_0\sqrt{\delta(2-\delta)}] \approx \exp(\pm|t|u_0\sqrt{2\delta}) . \tag{2.5}$$

For $u_1\neq 0$ and the conditions

$$u_0\delta \ll \max|u_1| \tag{2.6}$$

and

$$\tau u_0\delta \ll 1 , \tag{2.7}$$

τ being the characteristic duration of the driving pulse $u_1(t)$, we may obtain an exact result for δ by means of the perturbative approach.

According to the condition (2.6), from Eq. (2.4a) we have

$$\Phi_-(t_*)-\Phi_-(-t_*)-i\Phi_+(0)\int_{-\infty}^{\infty}a(t)dt - \int_{-t_*}^{t_*}b(t)\Phi_-(t)dt=0 , \tag{2.8}$$

where t_* is an arbitrary large value [$t_*\gg\tau$ but $t_*\ll(u_0^2\delta)^{-1/2}$]. From Eq. (2.4b) it follows that for $t\sim t_*$ $u_1=0$ and, as a result, in the lowest approximation we have $\Phi_-=-(i/2u_0)(\partial/\partial t)\Phi_+$. Using the asymptotic solutions (2.5), from Eq. (2.8) we obtain in the lowest order in δ the equation

$$i\sqrt{2\delta}\Phi_+(0)-i\Phi_+(0)\int_{-\infty}^{\infty}a(t)dt[1+O(\sqrt{\delta})]=0 , \tag{2.9}$$

i.e., $\delta=\frac{1}{2}[\int_{-\infty}^{\infty}a(t)dt]^2$. The same result may be obtained for another discrete eigenvalue $\lambda_2=-u_0(1-\delta)$.

Therefore a small arbitrary pulse $u_1(t)$ always generates two eigenvalues of the discrete spectrum,

$$\lambda_{1,2}=\pm\lambda_0=\pm u_0(1-\tfrac{1}{2}\Delta^2) , \tag{2.10}$$

at the condition [see Eq. (2.9)]

$$\Delta\equiv\mathrm{Re}\left[e^{-i\alpha}\int_{-\infty}^{\infty}u_1(t)dt\right]>0 . \tag{2.11}$$

As follows from the inverse scattering transform,[11] the eigenvalues (2.10) correspond to a pair of dark solitons with the equal amplitudes $u_0\Delta$ and opposite velocities $\pm 2\lambda_0$ [see Eqs. (1.3) and (1.4)]. Therefore, for $\Delta>0$, the dark-pulse solitons may be created without a threshold, i.e., by an infinitely small driving pulse (see Fig. 1). This analytical result explains some experimental observations by Krökel *et al.*[12] who, in particular, did not notice any threshold power for dark-soliton generation. Our analytical results are also related to results of numerical simulations of the dark-soliton generation for a special dark pulse by Blow and Doran.[16]

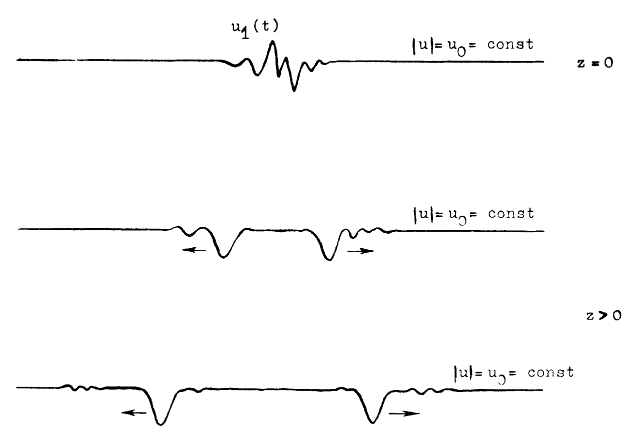

FIG. 1. Decay of an arbitrary small pulse into two dark solitons. The condition (2.11) is valid.

It is interesting to note that our results (2.10) and (2.11) for the eigenproblem (2.3) have an analog with the famous Peierls problem in quantum mechanics: A one-dimensional well always contains a discrete level of energy.[17]

2. *Small random pulses*

It is interesting to consider the generation of dark solitons by a random driving pulse which may be described by the random function,

$$\beta(t)=\mathrm{Re}[u_1(t)e^{-i\alpha}] \ , \tag{2.12}$$

so that $\langle\beta(t)\rangle=0$. The angle brackets mean the averaging over all realizations of $\beta(t)$. According to Eqs. (2.10) and (2.11), in the case of small-intensity pulses the soliton generation is defined by the sign of the random value

$$\xi\equiv\int_{-\infty}^{\infty}\beta(t)dt \ .$$

The probability density $P(\xi)$ of the random values ξ may be easily calculated in the case of the Gaussian function $\beta(t)$, i.e., $\langle\beta(t)\beta(t')\rangle=B(t,t')$, $B(0,0)=\beta_0^2$ and $B(t,t')$ is a Gaussian binary correlator.

For the Gaussian random function $\beta(t)$ the probability density $P(\xi)$ may be easily connected with the characteristic functional $F(k)$,

$$P(\xi)=\left\langle\delta\left[\xi-\int_{-\infty}^{\infty}\beta(t)dt\right]\right\rangle =\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\, e^{ik\xi}F(k) \ , \tag{2.13}$$

where (see, e.g., Ref. 18)

$$F(k)=\left\langle\exp\left[-ik\int_{-\infty}^{\infty}\beta(t)dt\right]\right\rangle =\exp\left[-\frac{k^2}{2}\int_{-\infty}^{\infty}\int B(t,t')dt\,dt'\right] \ .$$

After simple calculations we obtain the result

$$P(\xi)=\frac{1}{\sqrt{2\pi B}}\exp(-\xi^2/2B) \ , \tag{2.14}$$

where

$$B\equiv\int_{-\infty}^{\infty}\int B(t,t')dt\,dt'\equiv\tfrac{1}{2}\beta_0^2t_c^2 \ .$$

The probability density (2.14) describes all statistical characteristics of the soliton generation by the small-intensity random Gaussian pulse. For example, the probability P_n to find the value ξ in the region $\beta_0/n<\xi<\beta_0 n$ with an arbitrary integer n is the following:

$$P_n=\int_{\beta_0/n}^{\beta_0 n}P(\xi)d\xi=\tfrac{1}{2}\left[\mathrm{erf}\left[\frac{n}{t_c}\right]-\mathrm{erf}\left[\frac{1}{nt_c}\right]\right] \ , \tag{2.15}$$

where

$$\mathrm{erf}x=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \ .$$

According to Eqs. (2.10) and (2.11), the probability (2.15) has the sense of the probability to generate two dark solitons with equal amplitudes w lying in the interval $u_0\beta_0/n<w<u_0\beta_0 n$. The total probability of the soliton generation by a small random driving pulse may be found as

$$p=\lim_{n\to\infty}P_n=\tfrac{1}{2} \ .$$

In the case of nonsmall values $\beta(t)$ this probability will be more than $\frac{1}{2}$ because eigenvalues may also appear for negative ξ.

3. *Phase-modulated pulses*

A simple, but rather important case of the soliton generation is the phase-modulated input pulse,

$$u(0,t)=\begin{cases}u_0e^{i\alpha}, & |t|>\tau \ ,\\ u_0e^{i\beta(t)}, & |t|<\tau \ .\end{cases} \tag{2.16}$$

If the initial pulse is not very large, we may consider the driving pulse using the results (2.10) and (2.11). After simple transformations, one can obtain, from Eq. (2.11),

$$\Delta=2u_0\int_{-\tau}^{\tau}dt\,\sin^2\{\tfrac{1}{2}[\beta(t)-\alpha]\} \ . \tag{2.17}$$

It is important to note that $\Delta\geq 0$ for any modulation process. Therefore, according to the condition (2.11), this type of input pulse will always produce two dark solitons with equal amplitudes. For example, in the case $\beta(t)=\beta=\mathrm{const}$, we have a positive value

$$\Delta=4u_0\tau\sin^2[(\beta-\alpha)/2]$$

for any difference $\beta-\alpha$, so that the soliton amplitudes are equal to $4u_0^2\tau\sin^2[(\beta-\alpha)/2]$. This result is valid provided $u_0\tau\ll 1$.

At last, let us consider a small fluctuation of the input phase, i.e., $\beta(t)=\alpha+\alpha_1(t)$ for $|t|<\tau$, where the following condition for smallness of the deviation must hold: $|u_0\int_{-\infty}^{\infty}\alpha_1(t)dt|\ll 1$. Using the general formula (2.17), it is easy to obtain at the above condition $\Delta=(u_0/2)\int_{-\infty}^{\infty}\alpha_1^2(t)dt$. The same result may be obtained in the framework of the direct consideration used in Sec. II A 1.

B. Boxlike dark pulse

1. *General case*

Another important case which may be treated analytically is a boxlike dark pulse. The case is mathematically interesting and has an exact solution. Let us consider the case of an even-symmetry boxlike pulse, when the boundary condition at $z=0$ has the equal asymptotic behavior at $t\to\pm\infty$ (see Fig. 2),

$$u(0,t)=\begin{cases}u_0e^{i\alpha}, & |t|>\tau \ ,\\ u_0e^{i\alpha}-u_1(t)=\tilde{u}(t), & |t|<\tau \ ,\end{cases} \tag{2.18}$$

where u_1 is a complex constant, $|u_1|\leq u_0$.

The solution of the eigenproblem (2.3) with the potential (2.18) may be obtained in a closed form. In particular, the discrete spectrum is defined by the transcendent

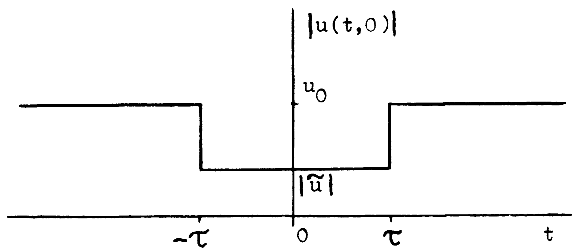

FIG. 2. A boxlike dark input pulse.

equation for real values of the spectral parameter λ ($|\tilde{u}|^2<\lambda^2<u_0^2$):

$$(\lambda^2-u_0\mathrm{Re}\tilde{u})\tan[2\tau(\lambda^2-|\tilde{u}|^2)^{1/2}]=[(\lambda^2-|\tilde{u}|^2)(u_0^2-\lambda^2)]^{1/2}\ . \tag{2.19}$$

In the limit $\tilde{u}=0$ (zero intensity inside the region $|t|<\tau$) we have from Eq. (2.19) the eigenvalue problem obtained by Zakharov and Shabat[11]

$$u_0\cos(2\lambda\tau)=\pm\lambda\ . \tag{2.20}$$

For small u_1 Eq. (2.19) has two real solutions,

$$\lambda_{1,2}=\pm u_0[1-2\tau^2(\mathrm{Re}u_1e^{-i\alpha})^2]\ , \tag{2.21}$$

if $\mathrm{Re}(u_1e^{-i\alpha})>0$. The result (2.21) corresponds to our previous results (2.10) and (2.11). In a more general case we have a number of dark-soliton pairs with the amplitudes $(u_0^2-\lambda_1^2)^{1/2}$, $(u_0^2-\lambda_2^2)^{1/2},\dots$, and velocities $\pm2\lambda_1,\pm2\lambda_2,\dots$, where $\lambda_1,\lambda_2,\dots$ are real solutions of Eq. (2.19). In particular, for $|\tilde{u}|\ll u_0$ and $u_0\tau\gg1$ the number N of the dark-soliton pairs may be estimated as follows: $N\sim2u_0\tau/\pi$.

2. The δ-function limit

The most interesting and important limit case of the boxlike input pulse considered above is the δ-function pulse which may be described by only one parameter related to its intensity. To obtain the δ-function limit one needs to consider an effective boxlike pulse and in the limit when the total square of this pulse is fixed, its duration tends to zero and the intensity tends to infinity, one has to obtain a number of results corresponding to the δ-function approximation, and also the exact formula for the form of the δ-like potential. Such a definition is not unique for the Zakharov-Shabat spectral problem (2.3) and the amplitude of the δ function is a functional of its square (see, e.g., the similar situation in Refs. 19 and 20). In the paper we will use the approximation using a boxlike pulse.

To this end, let us consider the effective boxlike input potential shown in Fig. 3. We must choose $u=-u_1<0$ for $|t|<\tau$ because below we will use the limit $u_1\to\infty$ (cf. Figs. 2 and 3). To obtain the exact limit case, we will use the relation between eigenfunctions

$$\Psi(t;\lambda)=\begin{bmatrix}\Psi_1\\ \Psi_2\end{bmatrix}$$

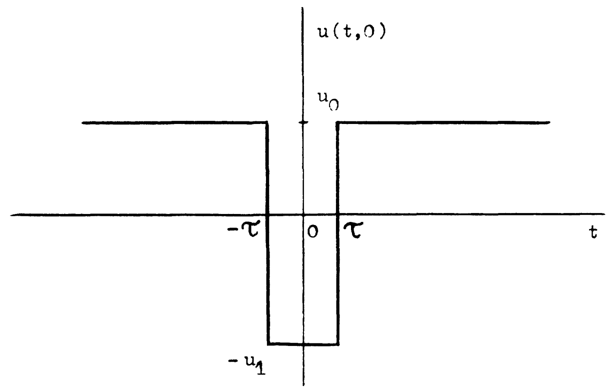

FIG. 3. An effective dark pulse $u(0,t)$ for exact construction of the δ-function approximation.

calculating on two edges of the effective pulse, i.e., at $t=\pm\tau$. The relation may be presented in the form

$$\Psi(-\tau;\lambda)=T(\tau,u_1)\Psi(\tau;\lambda)\ , \tag{2.22}$$

where $T(\tau,u_1)$ has the sense of the local transfer matrix (from $t=-\tau$ to $t=\tau$). Using the standard and straightforward calculations of the Jost functions in the ZS eigenproblem with the input potential shown in Fig. 3, we may calculate the exact form of the needed transfer matrix $T(\tau,u_1)$. Therefore the limit $u_1\to\infty$, $\tau\to0$ but

$$\tau u_1=\gamma=\mathrm{const}>0 \tag{2.23}$$

must lead to the exact definition of the δ-like potential. We do not present the total transfer matrix $T(\tau,u_1)$ because it is described by a rather cumbersome formula, but in the limit case we have the following result:

$$T_\delta=\lim_{\substack{\tau\to0\\ u_1\to\infty\\ \tau u_1=\gamma}}T(\tau,u_1)=\begin{bmatrix}\cosh(2\gamma) & -i\sinh(2\gamma)\\ i\sinh(2\gamma) & \cosh(2\gamma)\end{bmatrix}\ . \tag{2.24}$$

In the same limit the condition $\Psi(-0;\lambda)=T_\delta\Psi(+0;\lambda)$ is the matching condition for the δ-function potential

$$u(0,t)=u_0-\epsilon\delta(t) \tag{2.25}$$

with an unknown ϵ. The direct comparison of the matching conditions for Eq. (2.25) with the result (2.24) leads to the exact definition of the δ-function input dark pulse,

$$u(0,t)=u_0-2\delta(t)\tanh\gamma\ , \tag{2.26}$$

where γ is defined by Eq. (2.23). As a result, any dark pulse with a small duration and the square 2γ may be described as a δ-pulse driving pulse with the intensity $2\tanh\gamma$.

The solution of the ZS direct scattering problem (2.3) with the initial potential (2.26) demonstrates that the potential corresponds to the two eigenvalues of the discrete spectrum,

$$\lambda_{1,2}=\pm\frac{u_0}{\cosh(2\gamma)} \tag{2.27}$$

so that two dark solitons will be created. These solitons have the equal amplitudes $w=(u_0^2-\lambda^2)^{1/2}=u_0|\tanh(2\gamma)|$ and opposite velocities $\pm 2u_0\text{sech}(2\gamma)$.

To conclude the section, it is interesting to note that our result (2.27) gives a rather good approximation for any driving pulse. For example, in the limit $\gamma \ll 1$ from the formula (2.27) it follows

$$\lambda_{1,2}=\pm u_0(1-2\gamma^2) ,$$

that is the same as in Eq. (2.10) because $2\gamma=\int_{-\tau}^{\tau}u_1(t)dt=\Delta$ is the square of the driving pulse.

C. Phase steps

The more general case of an odd-symmetry boxlike pulse is the pulse with unequal phases at $t=\pm\infty$. In particular, experiments by Weiner *et al.*[14] utilize a specially shaped antisymmetric input pulse which closely corresponds to the form of the fundamental dark soliton. The pulse has a phase difference at its edges which equals π. We will consider the simplest example of such a pulse, i.e., a phase step on a cw background, when

$$u(0,t)=\begin{cases}u_0e^{i\alpha}, & t<0 ,\\ u_0e^{i\beta}, & t>0 .\end{cases} \tag{2.28}$$

The eigenfunctions of the ZS spectral problem (2.3) may be presented as the following:

$$\Psi_-(t;\lambda)=\begin{bmatrix}u_0e^{i\alpha}\\ \lambda+i(u_0^2-\lambda^2)^{1/2}\end{bmatrix}e^{t(u_0^2-\lambda^2)^{1/2}}, \quad t<0 ,$$

$$\Psi_+(t;\lambda)=C_1\begin{bmatrix}u_0e^{i\beta}\\ \lambda-i(u_0^2-\lambda^2)^{1/2}\end{bmatrix}e^{-t(u_0^2-\lambda^2)^{1/2}}, \quad t>0 .$$

Using the condition $\Psi_-(-0;\lambda)=\Psi_+(+0,\lambda)$ we may find the equation for the discrete spectrum and, as a result, its single real solution

$$\lambda=-u_0\cos\varphi, \quad \varphi=\tfrac{1}{2}(\beta-\alpha) . \tag{2.29}$$

The eigenvalue (2.29) corresponds to a dark soliton with the intensity $w=u_0|\sin\varphi|$ and the velocity $-2u_0\cos\varphi$. In the case $\beta=\alpha+\pi$ we have a so-called "black" soliton, i.e., the fundamental (quiescent) pulse with zero intensity at its center. In another case this soliton will be "gray" so that it has the lower-contrast intensities (see Fig. 4).

To obtain a number of dark solitons by the same way, one need to choose a cw background with a variable phase, $u(t)=u_0e^{i\beta(t)}$, where, for example, $\beta(t)=\alpha+\sum_j\beta_j\Theta(t-t_j)$, $t_1<t_2<\cdots<t_N$. In this case the input background will evolve into N dark solitons, if periods between the steps $|t_{j+1}-t_j|$ will be more than some limit value. To obtain the condition, let us consider two phase steps on a cw background as an input pulse.

Let us introduce new notation $\lambda=-u_0\cos\varphi$ then $(u_0^2-\lambda^2)^{1/2}=u_0\sin\varphi$. For the two steps, when $\beta(t)=\alpha+\beta_1\Theta(t-t_1)+\beta_2\Theta(t-t_2)$, a simple consideration of the

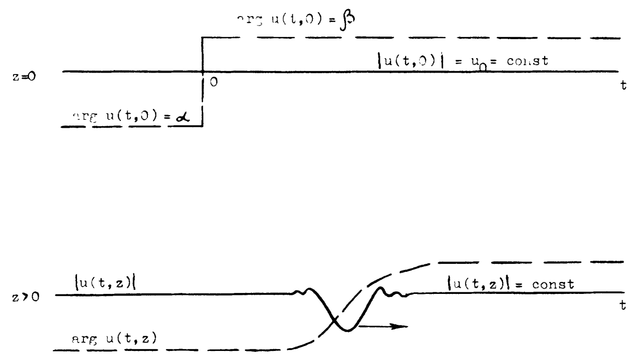

FIG. 4. Decay of a phase step into a dark soliton, the case $\beta-\alpha>\pi$.

direct scattering problem (2.3) yields the equation for eigenvalues, i.e., the equation for φ,

$$(e^{2i\varphi-i\beta_1}-1)(e^{2i\varphi-i\beta_2}-1)=(e^{-i\beta_2}-1)(e^{-i\beta_1}-1)e^{-2u_0\tau\sin\varphi+2i\varphi} , \tag{2.30}$$

where $\tau\equiv|t_2-t_1|$. It is interesting to note that the result for a step obtained above follows from (2.24) at $\beta_2=0$, $\varphi=\beta/2$. In the case $u_0\tau\sin\varphi\gg 1$ Eq. (2.30) has two independent solutions which describe two independent dark solitons. So, the condition $u_0|t_{j+1}-t_j|\sin(\beta_j/2)\gg 1$ is the condition when from the stepwise phase modulation with N steps the same quantity of dark solitons will arise.

For special cases $\beta_2=\pm\beta_1$ the general equation (2.30) has simple representations. Let us consider both these cases separately.

Case $\beta_2=\beta_1=\beta$. Equation (2.30) may be transformed into the following equation:

$$\frac{\sin^2(\varphi-\beta/2)}{\sin^2(\beta/2)}=e^{-i2u_0\tau\sin\varphi} , \tag{2.31}$$

which has one or two solutions. For $\tau\to\infty$ we have two solutions

$$\varphi_{1,2}=\beta/2\pm\sin(\beta/2)e^{-u_0\tau\sin(\beta/2)} ,$$

and for $\tau\to 0$ we have only one solution,

$$\varphi_1=\beta-2u_0\tau\sin^2(\beta/2), \quad 0<\beta<\pi ,$$

or

$$\varphi_1=\beta-\pi+2u_0\tau\sin^2(\beta/2), \quad \pi<\beta<2\pi .$$

Direct analysis of Eq. (2.31) yields the threshold value at which the second solution will arise,

$$\tau_{\text{thr}}=u_0^{-1}|\text{cotan}(\beta/2)| . \tag{2.32}$$

Therefore, for $\tau<\tau_{\text{thr}}$ two equal phase steps generate only one dark soliton corresponding to the value of the spectral parameter $\lambda_1=-u_0\cos\varphi_1$. But for $\tau>\tau_{\text{thr}}$ two dark solitons will arise, and in the case $\tau\gg\tau_{\text{thr}}$ they will be independent, $\lambda_1=\lambda_2=-u_0\cos(\beta/2)$. Above the threshold $\tau=\tau_{\text{thr}}$, this new solution may be presented as follows

($\tau > \tau_{thr}$):

$$\varphi_2 \simeq 2u_0(\tau - \tau_{thr})\sin^2(\beta/2), \quad 0<\beta<\pi ,$$

or

$$\varphi_2 \simeq \pi - 2u_0(\tau - \tau_{thr})\sin^2(\beta/2), \quad \pi<\beta<2\pi .$$

The amplitudes of created dark solitons are related to values of φ as follows: $w = u_0|\sin\varphi|$, so that the second dark soliton appears at infinitely small amplitude.

Case $\beta_2 = -\beta_1 = \beta$. Equation (2.30) takes the form

$$\frac{\cos(2\varphi)-\cos\beta}{1-\cos\beta} = e^{-2u_0\tau\sin\varphi} \tag{2.33}$$

and always has two solutions. In particular, for $\tau\to\infty$ we obtain $\varphi_1=\beta/2$, $\varphi_2=\pi-\beta/2$ that yields $\lambda_{1,2} = -u_0\cos(\beta/2)$, but for $\tau\to 0$ we may obtain the result [(2.10) and (2.19)] directly from Eq. (2.33).

Therefore the quantity and velocities of dark solitons produced by a step phase modulation is related to the signs of the steps and the distance between them. In the case when all steps are positive, the condition $|t_{j+1}-t_j| \gg u_0^{-1}|\cot\text{an}(\beta_j/2)|$ guarantees that the quantity of dark solitons will be equal to that of steps. This result may be useful for production of dark solitons.

D. Arbitrary random pulse on a cw background

In Sec. II A 2 we have considered the dark-soliton creation by a small random pulse. The main condition used in that section was a smallness of the binary correlator β_0^2 of the random pulse. As a result, the small random pulse generates *only two* symmetric dark solitons with the probability density (2.14). In the case of an arbitrary random pulse on a cw background $|u|=u_0$ we have to study the more general problem and, in particular, the number of dark solitons is changing. In according with the inverse scattering transform (IST) (see Sec. II A), the number of the solitons N is exactly equal to the quantity of eigenvalues lying on the real λ axis in the region $|\lambda|<u_0$. Therefore, to find N, we have to calculate the number of eigenvalues of the discrete spectrum stipulated by the random pulse.

To describe the situation analytically, let us consider the ZS eigenproblem (2.3) with the following "potential" [random input pulse $v(t)$ with the duration τ on the cw background u_0]:

$$u(t)=\begin{cases} u_0, & t<0, \quad t>\tau , \\ u_0+v(t), & 0<t<\tau , \end{cases} \tag{2.34}$$

where, for simplicity, we use the real functions u_0 and $v(t)$. Here $v(t)$ is a random Gaussian function describing a white noise,

$$\langle v(t)v(0)\rangle = 2D\delta(t) . \tag{2.35}$$

Considering the system (2.3) with the potential [(2.34) and (2.35)] as a function of the spatial variable t yields the problem which is similar to the well-known problem considered in the theory of one-dimensional disordered systems (see, e.g., Ref. 21, Chap. 2, Sec. 8). The main problem in our case related to the dark-soliton creation is determination of the number N of discrete spectrum eigenvalues lying in the region $|\lambda|\le u_0$. The similar problem was discussed in Ref. 21 (see also references therein). We present the corresponding calculations related to Eqs. (2.3) and (2.34) in Appendix A. Due to the technique described we may obtain the value N in a general form (see Appendix A). Simple analysis yields the following asymptotics:

$$N \sim u_0\tau(D/u_0)^{1/3} \quad \text{for } D \ll u_0 , \tag{2.36}$$

$$N \sim D\tau \sim u_0\tau \quad \text{for } D\sim u_0 , \tag{2.37}$$

and

$$N\sim u_0\tau \quad \text{for } D\gg u_0 . \tag{2.38}$$

The results have a simple physical sense. First of all, the number of eigenvalues lying in the region $|\lambda|\le u_0$ is exactly proportional to the duration τ of the random pulse $v(t)$ in all cases; the result is evident. For small intensities of the random pulse, when $D\ll u_0$ [see Eq. (2.36)], the number of dark solitons is proportional to dimensionless parameter $(D/u_0)^{1/3}$, but for $D\sim u_0$, when random fluctuations are of the order of the intensity of the cw background, the influence of the latter decreases. As a result, in the limit $D\gg u_0$, i.e., for strong fluctuations, the result does not depend on D at all [see Eq. (2.38)].

III. DYNAMICS OF DARK PULSES ON A BACKGROUND OF FINITE EXTENT

A. The shape of an input pulse

As was noted in the Introduction, Krökel *et al.*[12] observed two 0.6-psec dark pulses generated by a single 0.3-psec input dark pulse (driving pulse) which was produced on a much longer duration (100-psec) bright pulse in the positive GVD region. To describe the experimental situation, we consider dark-pulse generation and dynamics on a background of finite extent. For simplicity, we consider an even-symmetry pulse (see Fig. 5) which may be presented as a large bright pulse with smoothed edges and a driving even-symmetry pulse stipulated by an external driving force (see Ref. 12) in the moment $t=0$.

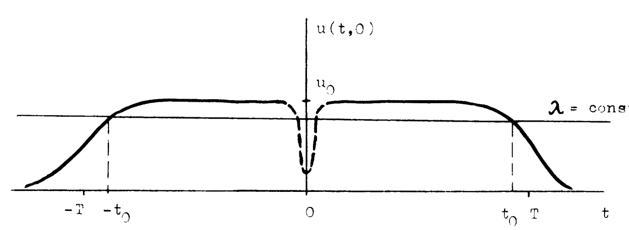

FIG. 5. Dark input pulse on a background of finite extent. The large background pulse (solid line) is described as a bright pulse in the semiclassical (WKB-type) limit, the driving pulse (dashed line) is considered in the δ-function approximation.

According to results of Sec. II B 2, we will use for this driving pulse the δ-function approximation. So, a realistic input pulse which we will consider may be presented as a simple superposition of two pulses,

$$u(0,t)=u(t)-2\delta(t)\tanh\gamma \ , \tag{3.1}$$

where $u(t)$ is a large bright pulse which we will describe in the WKB-type approximation, and the last term ($\sim\tanh\gamma$) in the right-hand side (rhs) of Eq. (3.1) is the δ-function pulse which represents an arbitrary driving pulse with the square 2γ. We suppose that $u(t)$ is a real function, and it is constant u_0 in the region $|t|\lesssim T$ (see Fig. 5), where T is a large parameter. So, in the case $T\to\infty$ we have to obtain results for a cw background.

B. Semiclassical limit

To describe the dynamics of the long bright pulse, the ZS scattering problem (2.3) must be solved in the so-called semiclassical limit using WKB-type expansions. A similar approach but for the effect of a smooth phase modulation on the soliton generation was used in the paper by Lewis[22] for another version of the NLS equation (in the case of an anormal dispersion).

The semiclassical limit of the ZS eigenproblem equations (2.3) is defined to be the situation where the Jost functions $\Psi_1(t,\lambda)$ and $\Psi_2(t,\lambda)$ oscillate far more rapidly than the envelope of the potential function $u(t)$ [see Eq. (3.1)]. In this case WKB-type solutions can be constructed for both $t<0$ and $t>0$ regions separately, which are local plane waves with a wavelength determined by the local value of the potential. According to Refs. 17 and 22, it is convenient to formalize the stated approximation by introducing an ordering parameter, ϵ, supposed small with respect to all other quantities. Therefore we rescale such that $\partial_t\to\epsilon\partial_t$, reversible at the end of the calculations. Hence Eq. (2.3) is rewritten in the following form:

$$\begin{aligned}\epsilon\frac{\partial}{\partial t}\Psi_1&=i\lambda\Psi_1-iu(t)\Psi_2 \ ,\\ \epsilon\frac{\partial}{\partial t}\Psi_2&=-i\lambda\Psi_2+iu(t)\Psi_1 \ ,\end{aligned} \tag{3.2}$$

[where, for a simplicity, we suppose that $u(t)$ in Eq. (3.1) is a real function], which becomes (after eliminating Ψ_2)

$$\epsilon^2\frac{\partial^2\Psi_1}{\partial t^2}-\epsilon^2\left[\frac{u_t}{u}\right]\frac{\partial}{\partial t}\Psi_1+\left[-i\epsilon\lambda\frac{u_t}{u}+\lambda^2+|u|^2\right]\Psi_1=0 \ . \tag{3.3}$$

So, at any order in ϵ we can determine $\Psi_1(t,\lambda)$ up to a pair of constants A_0 and B_0, and, as a result, we can also derive $\Psi_2(t,\lambda)$ using the first of Eqs. (3.2) in the same order in ϵ. The same procedure must be fulfilled for two regions $t<0$ and $t>0$ separately, so that two sets of constants arise, $A_0^{(1)},B_0^{(1)},\ldots$, and $A_0^{(2)},B_0^{(2)},\ldots$. These constant values are then given by the boundary conditions

$$\Psi_1(t,\lambda)\to e^{i\lambda t/\epsilon}, \quad \Psi_2(t,\lambda)\to 0 \quad \text{at } t\to-\infty \ , \tag{3.4}$$

and the matching condition at the point $t=0$, where an additional δ-function pulse is installed; see Fig. 5. The scattering data may be obtained using the other boundary conditions,

$$\begin{aligned}&\Psi_1(t,\lambda)\to a(\lambda)e^{i\lambda t/\epsilon} \ ,\\ &\Psi_2(t,\lambda)\to b(\lambda,t)e^{-i\lambda t/\epsilon} \quad \text{at } t\to+\infty \ .\end{aligned} \tag{3.5}$$

In the final formulas we must put $\epsilon=1$ to obtain the results in the previous scale. The described procedure is the same as used in the semiclassical limit of quantum mechanics (see more details in Ref. 17).

Omitting a number of cumbersome (but rather standard from the viewpoint of the well-known WKB technique in the quantum-mechanics approximation) formulas we present the results for the eigenfunctions of the ZS scattering problem with the potential shown in Fig. 5 (solid line). For $\lambda>u(t)$,

$$\Psi_1(t,\lambda)=\frac{1}{\{2(\lambda^2-u^2)^{1/2}[\lambda-(\lambda^2-u^2)^{1/2}]\}^{1/2}}\left[\begin{matrix}u\\ \lambda-(\lambda^2-u^2)^{1/2}\end{matrix}\right]\exp\left[i\int^t dt'(\lambda^2-u^2)^{1/2}\right] \ , \tag{3.6a}$$

$$\Psi_2(t,\lambda)=\frac{1}{\{-2(\lambda^2-u^2)^{1/2}[\lambda-(\lambda^2-u^2)^{1/2}]\}^{1/2}}\left[\begin{matrix}u\\ \lambda+(\lambda^2-u^2)^{1/2}\end{matrix}\right]\exp\left[-i\int^t dt'(\lambda^2-u^2)^{1/2}\right] \ . \tag{3.6b}$$

For $\lambda<u(t)$,

$$\Psi_1(t,\lambda)=\frac{1}{\{-2i(u^2-\lambda^2)^{1/2}[\lambda+i(u^2-\lambda^2)^{1/2}]\}^{1/2}}\left[\begin{matrix}u\\ \lambda+i(u^2-\lambda^2)^{1/2}\end{matrix}\right]\exp\left[\int^t dt'(u^2-\lambda^2)^{1/2}\right] \ . \tag{3.7a}$$

$$\Psi_2(t,\lambda)=\frac{1}{\{2i(u^2-\lambda^2)^{1/2}[\lambda-i(u^2-\lambda^2)^{1/2}]\}^{1/2}}\left[\begin{matrix}u\\ \lambda-i(u^2-\lambda^2)^{1/2}\end{matrix}\right]\exp\left[-\int^t dt'(u^2-\lambda^2)^{1/2}\right] \ . \tag{3.7b}$$

Therefore, in the limit $t\to\pm\infty$ from Eqs. (3.6) and (3.7) we have

$$\Psi_1(t,\lambda)\xrightarrow[t\to\pm\infty]{}\left[\begin{matrix}1\\0\end{matrix}\right]e^{i\lambda t\mp i\Phi_\pm} \ , \tag{3.8}$$

where

$$\Phi_\pm \equiv \lim_{t\to\pm\infty}\left[\mp\int_{\pm t_0}^{t} dt'[(\lambda^2-u^2)^{1/2}-\lambda]-|\lambda|t_0\right],$$

t_0 being a turning point (see Fig. 5).

Using the matching conditions at the point $t=t_0$, $t=-t_0$, and $t=0$ (for the additional δ-function pulse), we obtain the result for $a(\lambda)$

$$a(\lambda)=-\frac{\exp[i(\Phi_+ +\Phi_-)]}{2\xi^2(u_0^2-\lambda^2)^{1/2}}\left\{\left[i\cosh(2\gamma)\left[\frac{z^2-1}{z}\right]+2\sinh(2\gamma)\right]+i\sinh(2\gamma)\left[\frac{z^2+1}{z}\right]\xi + i(\xi^2/4)\left[\cosh(2\gamma)\left[\frac{z^2-1}{z}\right]+2i\sinh(2\gamma)\right]\right\}, \quad (3.9)$$

where

$$\xi\equiv\exp\left[-2\int_0^T[u_0^2(t)-\lambda^2]^{1/2}dt\right] \quad (3.10)$$

and

$$z\equiv(\lambda/u_0)+iu_0^{-1}(u_0^2-\lambda^2)^{1/2}. \quad (3.11)$$

In the formula (3.10) we put approximately $t_0\approx T$.

The results (3.9)–(3.11) are obtained in the WKB-type approximation when the value ξ is rather small, i.e., $T\gg 1$.

C. Quasistationary states and the inverse scattering transform

According to the IST (Refs. 23 and 24), to find which type of initial function generates solitons, one has to find *real* eigenvalues of the ZS scattering problem (2.3) which are solutions of the equation $a(\lambda)=0$. It is easy to prove that real solutions of the above equation are absent, so that solitons are absent, too. That is the well-known result in the inverse problem for the NLS Eq. (1.1) at $\sigma=-1$ and localized initial pulses (see, e.g., Ref. 24). But due to the δ-function hole in the bright pulse, our problem has a number of features. Let us investigate the equation $a(\lambda)=0$ with more accuracy. In the linear approximation in ξ (the strong WKB limit and very large T) the equation $a(\lambda)=0$ may be rewritten as follows:

$$i\left[\frac{z^2-1}{z}\right]+2\tanh(2\gamma)+i\xi\tanh(2\gamma)\left[\frac{z^2+1}{z}\right]=0. \quad (3.12)$$

In the zeroth approximation in ξ we obtain the following results:

$$\lambda=\pm\lambda_0=\pm u_0\mathrm{sech}(2\gamma), \quad (3.13)$$

$$z_0=i\tanh(2\gamma)\pm\mathrm{sech}(2\gamma). \quad (3.14)$$

The values of the spectral parameter λ the same as in the problem considered in Sec. II B 2 [cf. Eqs. (3.13) and (2.29)], i.e., the δ-function pulse on a cw background, because at $\xi=0$ the ends of the background tend to infinity. Therefore the semiclassical parameter ξ characterizes a deviation of zeros of $a(\lambda)$ from the real axis λ due to the background of finite extent. Taking into account the last term in Eq. (3.12) yields small addenda to (3.13) and (3.14),

$$\delta\lambda=-i\xi u_0^{-1}(u_0^2-\lambda_0^2),\quad \delta z=-z_0\xi\tanh(2\gamma),$$

i.e., for zeros of $a(\lambda)$ we have the result

$$\lambda=\pm u_0\mathrm{sech}(2\gamma)-iu_0\xi\tanh(2\gamma). \quad (3.15)$$

The small additional term in Eq. (3.15) shifts the zeros into the lower half of the complex plane λ, which is why these zeros do not exactly match the eigenvalues and do not correspond to exact solitons. But, on the other hand, these values in the limit $\xi\to 0$ go to the real axis and strongly influence the continuous spectrum of the spectral problem in the vicinities of the points $\lambda=\pm\lambda_0$. Therefore we may denote that situation as appearance of quasistationary states in the inverse scattering problem.

The Jost coefficient $a(\lambda)$ in the vicinities of the points $\lambda=\pm\lambda_0$ may be presented as follows:

$$|a(\lambda)|^{-2}\approx\frac{\delta^2}{(\lambda\pm\lambda_0)^2+\delta^2}, \quad (3.16)$$

where

$$\delta\equiv(\xi/u_0)(u_0^2-\lambda_0^2)=\xi u_0\tanh(2\gamma) \quad (3.17)$$

and has two maxima (see Fig. 6 for $\lambda>0$). In the limit $\delta\ll 1$ ($\xi\ll 1$) the function (3.16) corresponds to two δ-functions with intensities $\pi\delta$. In the points $\lambda=\pm\lambda_0$ the relation $|a(\pm\lambda_0)|=1$ is valid.

The contribution of the continuous spectrum N_d related to the imaginary part of the zeros into the total pulse

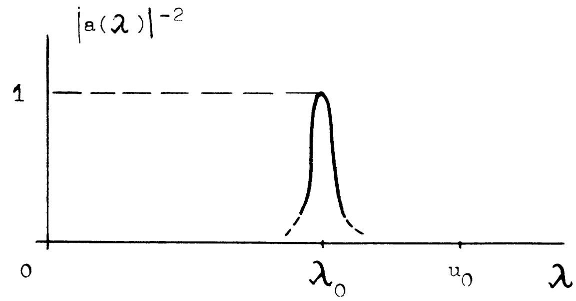

FIG. 6. The function $|a(\lambda)|^{-2}$ in the vicinity of the point $\lambda=\lambda_0$; the part for $\lambda<0$ is symmetric.

intensity,

$$N=(1/\pi)\int d\lambda \ln|a(\lambda)|^2 ,$$

is also a small value,

$$N_d=(1/\pi)\int d\lambda \ln\left[\frac{\delta^2}{(\lambda-\lambda_0)^2+\delta^2}\right]\sim\delta . \tag{3.18}$$

It means that at the points $\lambda=\pm\lambda_0$ in the limit $\delta\rightarrow 0$ the exact eigenvalues of the discrete spectrum will appear, and the input pulse will generate two dark solitons. But for $\delta\neq 0$ these dark pulses are not exact dark solitons, and for very large time they will decay into wave trains according to the formula of IST (Ref. 24),

$$|u(z,t)|^2\approx\frac{1}{4\pi z}\ln|a(\lambda)|^2,\quad \lambda=-t/4z . \tag{3.19}$$

The exact investigation of the asymptotic $|u(z,t)|$ for large z in the case of two sharp peaks of $|a(\lambda)|^{-2}$ is a very difficult problem. The corresponding calculations must demonstrate a relation between nonsoliton and soliton wave packets when the parameter ξ tends to zero.

For $\delta\neq 0$ the generated dark pulses evolve slowly due to a broadening of the background. Numerical study of the problem was given by Tomlinson *et al.*[25] An approximate analytical description of the same effect is presented below in Sec. V A.

IV. RANDOM PULSE

A very interesting situation arises when the input pulse $u(0,t)$ in an optical fiber is a stochastic pulse *without a background*. In this case we will consider a model input pulse when the "initial" potential $u(0,t)$ may be presented as a piecewise function,

$$u(0,t)=\begin{cases}0, & t<0,\quad t>T ,\\ u(t), & 0\leq t\leq T ,\end{cases} \tag{4.1}$$

where $u(t)$ is, for simplicity, a real random function with a zero mean value and the correlator,

$$B(t)=\langle u(t+t')u(t')\rangle . \tag{4.2}$$

As a result, we have the ZS eigenproblem with a random input pulse for zero "boundary" conditions at $t\rightarrow\pm\infty$.

Spectral properties of the ZS scattering problem (2.3) with a Markov random potential [(4.1) and (4.2)] defined for $|t|<\infty$ and the properties of the corresponding scattering data for the same pulse are similar to those of the (linear) Schrödinger equation used in quantum mechanics.[17] The latter equation is used in the theory of one-dimensional disordered solids, and the general results and technique for the investigations are described in Ref. 21. To this end, we do not present in the section detailed calculations for the ZS scattering problem, and will formulate only our main results related to the optical problem under consideration. The major part of the results is based on the effective Fokker-Planck (FP) equation; its derivation is presented in Appendix B.

The spectrum of the scattering problem (2.3) with the Markov random potential [(4.1) and (4.2)] is a pure point, and all eigenstates are exponentially localized with the localization time $\tau_0(\lambda)$. That is reflected in the behavior of scattering characteristics. Let us introduce the transmitted, t_1, and reflected, r, amplitudes by the relations [cf. Eq. (B1)]

$$\Psi(t<0)=\begin{bmatrix}0\\ t_1 e^{-i\lambda t}\end{bmatrix},\quad \Psi(t>0)=\begin{bmatrix}re^{i\lambda t}\\ e^{-i\lambda t}\end{bmatrix} .$$

Then the transmission coefficient $|t_1|^2$ of a very large pulse $[T\gg\tau_0(\lambda)]$ is exponentially small in T/τ_0 with a probability exponentially close to unity:

$$|t_1|^2=1-|r|^2\sim\exp[-T/\tau_0(\lambda)] . \tag{4.3}$$

Let u_0^2 and τ_c be characteristic scales of the correlation function $B(t)$. In respect to slowly varying (with time scales which much more than τ_c) functions, the function $u(t)$ is a δ-correlated one,

$$B(t)=2D\delta(t),\quad D\sim u_0^2\tau_c . \tag{4.4}$$

More exactly, in the case $u_0\tau_c\ll 1$ we may always consider $u(t)$ as a Gaussian white noise in the spectral region $|\lambda|\tau_c\ll 1$. If, additionally, the condition $|\lambda|\gg D$ is valid (the region of large "energy" in the white-noise approximation), then the localization time $\tau_0(\lambda)$ does not depend on the spectral parameter and may be presented as follows (see Appendix B):

$$\tau_0(\lambda)=\frac{1}{2D} . \tag{4.5}$$

Taking into account the above-mentioned results (based on Appendix B) we will demonstrate that the spectrum of the ZS scattering problem (2.3) with the potential (4.1), (4.2), and (4.4) under the conditions $u_0\tau_c\ll 1$ and $DT\gg 1$ has quasistationary states in the region

$$D\ll|\lambda|\ll\tau_c^{-1} , \tag{4.6}$$

and find their characteristics. (It is important to note that in the case of the linear Schrödinger equation the similar states were found in Ref. 26.)

The transmission coefficient $|t_1|^2$ for the segment [0,T] may be presented as follows:

$$|t_1|^2=\frac{|t_1^{(1)}t_1^{(2)}|^2}{|1-r^{(1)}r^{(2)}|^2} ,$$

where $t_1^{(1,2)}$ and $r^{(1,2)}$ are corresponding amplitudes of half segments $[0,T/2]$ and $[T/2,T]$, respectively, for wave propagation to the left and to the right. At complex values λ_n of the spectral parameter which are solutions of the equation

$$1-r^{(1)}(\lambda)r^{(2)}(\lambda)=0 ,$$

the transmission and reflection coefficients tend to infinity. These states have no incident waves, and have the sense of decomposing states which outside of the segment [0,T] are described only as leaving waves. If the values $\mathrm{Im}\lambda_n$ are small in comparison with $\mathrm{Re}\lambda_n$, then the states are quasistationary ones.

For the problem under consideration, in the case

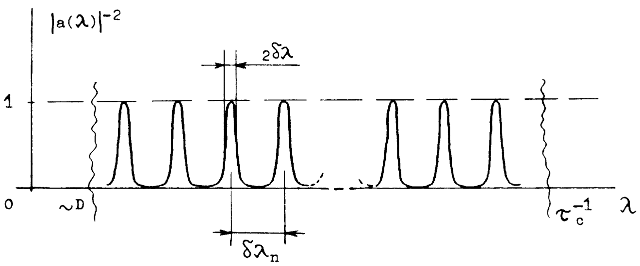

FIG. 7. The even function $|a(\lambda)|^{-2}$ for a random input pulse without a background. Peaks correspond to quasistationary states which in the limit $T\to\infty$ yields exactly eigenvalues of the discrete spectrum. The axis $\lambda>0$ is presented only.

$T\gg\tau_0(\lambda)$ the product $r^{(1)}r^{(2)}$ due to the estimation (4.3) is exponentially close to unity. As a result, the complex values of the spectral parameter corresponding to the quasistationary states are near the real axis λ, but at

$$\arg[r^{(1)}(\lambda_n)r^{(2)}(\lambda_n)]=2\pi n \; . \tag{4.7}$$

A more detailed analysis, which is based on the study of the phase ϕ of the reflection amplitude [see Eq. (B6)] as a function of λ, allows us to obtain the following results:

$$\mathrm{Re}\lambda_n \simeq \frac{n\pi}{T} \; , \tag{4.8}$$

$$\mathrm{Im}\lambda_n \sim \exp(-DT/2) \; . \tag{4.9}$$

As a result, the function $|a(\lambda)|^{-2}=|t_1(\lambda)|^2$ which describes the spectral density of the pulse intensity [see, e.g., Eq. (3.18)] has in the region (4.6) a set of equidistant,

$$\delta\lambda_n \equiv \mathrm{Re}(\lambda_n-\lambda_{n-1})\cong\pi/T \; ,$$

and exponentially narrow,

$$\delta\lambda\sim(2T)^{-1}\exp(-DT/2) \; ,$$

peaks (see Fig. 7). Those peaks in the limit $T\gg\tau_0$ are similar to those for dark pulses. In the case $T\to\infty$ all the states will be localized, so the random pulse decays into dark solitons only. That situation is similar to the case considered in Secs. II D and III: the role of the cw background is played by the parameter $(D/\tau_c)^{1/2}$; the parameter T is the duration of the effective background.

To conclude the section, we note that the considered quasistationary states must play an important role in the study of optical pulses in the normal dispersion region. Characteristics of such pulses with randomly varying parameters must demonstrate dependences related to properties of dark solitons. We are sure that the same situation was observed in Ref. 27 where a decay of a large bright pulse with a randomly changing envelope demonstrated in the normal GVD region creation of dark pulses which had properties of dark solitons.

V. DARK SOLITONS UNDER PERTURBATIONS

In the previous sections we studied the soliton dynamics and soliton creation in the framework of the exactly integrable NLS equation (1.1). But in real optical fibers there is a number of additional effects which, as a rule, lead to new terms in the NLS equation. If amplitudes of the terms may be considered as small parameters (this is usually valid for optical fibers), their influence on soliton dynamics may be considered in the framework of perturbative approaches. The most effective and elaborated method to study soliton dynamics under perturbations is the perturbation theory for solitons based on the IST.[28] Due to a general outline of the method we may obtain equations for evolution of soliton parameters and a radiation generated by solitons in the case of the NLS equation with the positive GVD when the solitons are dark ones. But the most sufficient assumption of the theory is unchanging of boundary conditions, that is not valid for real systems. Indeed, according to optical experiments,[12–14] dark-pulse solitons were observed as holes on a background of finite extent. That is why the sufficient effect in the dark-soliton dynamics is the influence of dispersive broadening of the finite-extent background on the parameters of dark solitons.

A. Broadening of a finite-extent background

In experiments by Krökel *et al.*[12] and Weiner *et al.*[14] dark pulses were produced on a background of finite extent; the latter had the form of a large bright pulse. Therefore the influence of the broadening background on dynamics of dark solitons is the very important effect in real experiments. Some results related to the problem were obtained by numerical calculations in Ref. 25. In this section we consider the effects analytically.

As was demonstrated by the inverse scattering transform,[24] a bright pulse (nonlinear wave packet) in the region of the positive GVD will broaden in accordance with the asymptotic formula [cf. (3.19)]

$$|u(z,t)|^2=\frac{1}{4\pi z}\ln|a(-t/4z)|^2, \quad z,t\gg1 \; , \tag{5.1}$$

where $a(\lambda)$ is the Jost coefficient for the *total* potential used in the inverse scattering transformations (e.g., the pulse shown in Fig. 5). The result (5.1) is the "nonlinear generalization" of the well-known formula for a broadening linear wave packet under a dispersion. Indeed, let us consider the case of a small-amplitude input pulse $u(0,t)$ in the ZS eigenproblem (2.3). In the case $\max|u|\ll1$, solutions of the ZS eigenproblem may be found by means of perturbation theory. Simple calculations yield the general perturbative expansions in $u(0,t)$,

$$\Psi_1(t,\lambda)=C_1e^{i\lambda t}-iC_3e^{i\lambda t}\int_{-\infty}^{t}dt'u(0,t')e^{-2i\lambda t'} \; , \tag{5.2a}$$

$$\Psi_2(t,\lambda)=C_2e^{-i\lambda t}+iC_4e^{-i\lambda t}\int_{-\infty}^{t}dt'u^*(0,t')e^{2i\lambda t'} \; . \tag{5.2b}$$

To define the constants we use asymptotic relations at $t\to\pm\infty$ [see, e.g., Ref. 24]. After simple and straightforward calculations, we obtain the following result for the Jost coefficient $b(\lambda)$ to first order in $u(0,t)$:

$$|b(\lambda)|=\left|\int_{-\infty}^{\infty}dt'u(0,t')e^{-2i\lambda t'}\right| \; . \tag{5.3}$$

Using the relation of the IST, $|a|^2=1+|b|^2$, we obtain from Eq. (5.3) the asymptotic representation for $|u(z,t)|$,

$$|u(z,t)|^2=\frac{1}{4\pi z}\left|\int_{-\infty}^{\infty}dt'u(0,t')e^{-itt'/2z}\right|^2 . \qquad (5.4)$$

It is easy to verify that (5.4) is exactly the result of the linear theory. Indeed, let us find the asymptotic expansion of the general solution

$$u(z,t)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}U(k)e^{ikt+ik^2z} \qquad (5.5)$$

of the linearized NLS equation $iu_z-u_{tt}=0$ corresponding to the positive GVD, here $U(k)$ being the Fourier transformation of the input pulse $u(0,t)$. In the limit $t,z\gg 1$ and t/z fixed, we may obtain from Eq. (5.5) the well-known result,

$$u(z,t)\approx\frac{1}{2\sqrt{\pi z}}U\left[\frac{t}{2z}\right]e^{it^2/4z}e^{i\pi/4} . \qquad (5.6)$$

Taking into account that

$$U(k)=\int_{-\infty}^{\infty}dt'u(0,t')e^{-ikt'} ,$$

we obtain directly the result (5.4).

So, in the general case of an arbitrary bright (nonsoliton) pulse $u(z,t)$ the following estimation is valid:

$$u(t,z)\approx\frac{\text{const}}{\sqrt{z}},\quad \text{const}\sim 1 \qquad (5.7)$$

for large z. As a result, in the problem of dark-pulse soliton generation we may simply take $u_0\sim z^{-1/2}$ in the general formulas obtained for the case of a cw background. Our estimations will be valid for a very large (but finite) background.

Therefore, in accordance with Eqs. (1.3) and (1.4), the duration of the dark soliton is proportional to

$$\tau_s\sim(u_0\nu)^{-1}\sim z^{1/2} , \qquad (5.8)$$

and its intensity defined as

$$I_s\equiv\max(u_0^2-|u|^2) \qquad (5.9)$$

is proportional to

$$I_s\sim u_0^2\nu^2\sim z^{-1} . \qquad (5.10)$$

In the result, the value $I_s\tau_s^2$ is not dependent on z. The latter was obtained as an approximate numerical result in Ref. 25 for the case of odd and even dark pulses produced on a background of finite extent. Our results (5.8) and (5.9) estimate analytically the changing of the soliton parameters along a fiber for the broadening background.

It is interesting to compare our analytical results and numerical data obtained in Ref. 25. Let us choose the dependences of the soliton parameters in the form

$$\tau_s=c_1\sqrt{z},\quad I_s=c_2/z , \qquad (5.11)$$

where c_1 and c_2 are unknown constants. Using the numerical data for $z=15$ (Fig. 7 in Ref. 25), we have calculated the following values: $c_1\approx 0.79$ and $c_2\approx 4.5$. The values allow us to present analytical dependences (5.11) on the figure (Fig. 8) together with numerical points (circles on Fig. 8). As a result, there is a rather good agreement between analytical formulas (5.11) and numerical data. Thus, in the presence of dispersive broadening of the finite-extent background dark pulses adiabatically maintain their main soliton characteristics as the background intensity decreases along a fiber.

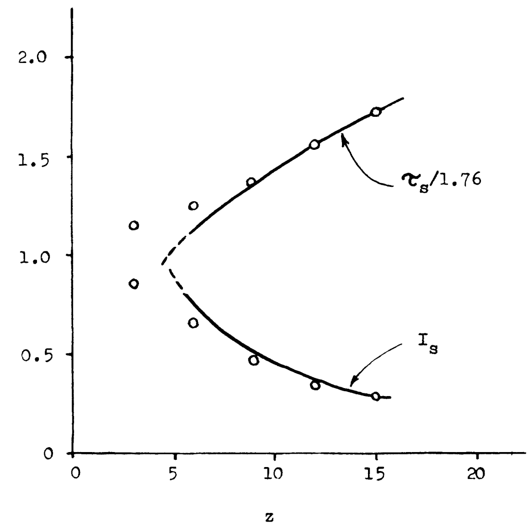

FIG. 8. The comparison of analytical [formulas (5.11) with $c_1=0.79$ and $c_2=4.5$] (solid lines) and numerical (circles from Ref. 25) results for the dark-soliton duration τ_s and its intensity I_s vs the normalized distance along a fiber. The constants c_1 and c_2 are calculated at $z=15$.

B. Dissipative losses

In real experiments there are additional dissipative losses acting on pulses in fibers. As is well known, due to the dissipative losses the amplitude of the bright soliton decreases as $\sim\exp(-2\gamma z)$ along a fiber, γ being a dissipative coefficient (see, e.g., Ref. 28). In the case of a dark soliton along with a changing of soliton parameters there is a dissipation-induced decay of a background. That is why the general solution of the problem with the evolving background of finite extent is difficult. We briefly discuss the problem assuming the changing background in the presence of the dissipation (without a discussion of its origin).

In the case of a perturbation the NLS equation may be presented in the form [cf. (1.1)]

$$i\frac{\partial u}{\partial z}-\frac{\partial^2 u}{\partial t^2}+2|u|^2u=\epsilon R(u) , \qquad (5.12)$$

$\epsilon R(u)$ being a perturbation with a small ϵ. Using simple transformations we may obtain the following integral relation:

$$\frac{\partial}{\partial z}\int_{-\infty}^{\infty}dt|u|^2 =-i\epsilon\int_{-\infty}^{\infty}dt\,u^*R(u)+i\epsilon^*\int_{-\infty}^{\infty}dt\,uR^*(U) . \qquad (5.13)$$

If we assume that the background u_0 is changed only, then in the case of dissipative losses, $\epsilon R(u)=-\gamma u$, it is easy to obtain directly from Eqs. (1.3), (1.4), and (5.13) the simple equation

$$\frac{\partial \tau_s}{\partial z}=\gamma\tau_s, \quad \tau_s \sim (\nu u_0)^{-1} , \tag{5.14}$$

which describes a dissipation-induced decay of the dark soliton $\tau_s=\tau_s(0)\exp(\gamma z)$. When the soliton intensity I_s decreases as $I(0)\exp(-2\gamma z)$, the soliton accelerates because the velocity $2\lambda=2(u_0^2-\nu^2)^{1/2}$ tends to the limit value. The situation qualitatively differs from the case of a bright optical soliton; in the latter case the soliton velocity is constant.

VI. CONCLUSIONS

In conclusion, we briefly summarize our results. In the framework of the NLS equation that describes the propagation of short optical pulses in single-mode optical fibers in the normal GVD region, we considered the problem of dark-soliton creation by a driving input pulse on a cw background. We demonstrated that dark solitons may be created as pairs by an arbitrary dark pulse without a power threshold (see also Ref. 29). Additionally, the driving pulse may be approximately presented with a good accuracy by the effective δ-function dark pulse on a cw background. As a result of the above conclusion, a long stochastic pulse decays into a set of dark pulses which are similar to dark solitons in the limit of infinite extension of the input random pulse.

Besides, we proposed a new (and rather simple) way to generate dark solitons in optical fibers. Using the inverse scattering transform, we demonstrated that dark solitons will always be produced by changing of a cw background phase only (the so-called phase modulation of the input pulse). In the case of a phase step we obtained both "black" and "gray" dark solitons; the former, i.e., the fundamental dark soliton with the zero intensity in its center, will arise when the step is exactly equal to π. As it seems for us, this way may be more simple than the preparation of a special input pulse as in Ref. 14.

The mentioned experiments[12,14] produced dark solitons from a driving dark pulse on a background of finite extent. We studied an effective input pulse of similar shape and demonstrated that in terms of the inverse scattering transform the generated dark pulses correspond not to the bound states, but to the quasistationary states, and their motion is similar to the motion of the NLS equation solitons with slowly decreasing amplitudes. The pulses adiabatically maintain their soliton characteristics as the background evolves, in particular, if the background undergoes a dispersive broadening, the duration of the dark soliton τ_s is proportional to $z^{1/2}$ along a fiber, so that the value $I_s\tau_s^2$ is approximately equal to a constant, I_s being the dark-soliton intensity.

Thus the above results for dark-soliton generation and dynamics in optical fibers lead to important conclusions. One can easily create dark solitons in optical fibers by a small driving pulse or a phase modulation of a background in the form of a rather long bright pulse, but, on the other hand, small (random or systematic) fluctuations acting on dark pulses will create additional secondary dark solitons with the probability $p\geq\frac{1}{2}$. The latter, probably, will make impossible the effective use of dark solitons in optical communication systems directly. On the other hand, the creation of dark solitons without a threshold stipulates their importance in dynamics of optical pulses in the normal GVD region of single-mode optical fibers: For rather long irregular pulses without a background most of the "dark" intensity (the deviation from a mean intensity of the pulse) is related to dark solitons.

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APPENDIX A: THE NUMBER OF DARK SOLITONS GENERATED BY A RANDOM PULSE

In this appendix we estimate the number of dark solitons produced by a random pulse on a cw background $|u|=u_0$. The value is exactly equal to the number of eigenvalues of the discrete spectrum lying in the region $|\lambda|\leq u_0$ of the ZS eigenproblem (2.3) with the "potential" [cf. (2.34) and (2.35)]:

$$u(t)=\begin{cases} u_0, & t<0, \quad t>\tau , \\ u_0+v(t), & 0<t<\tau , \end{cases} \tag{A1}$$

$$\langle v(t)v(0)\rangle=2D\delta(t) , \tag{A2}$$

where τ is the pulse duration and $v(t)$ is the white noise. In our consideration we use the technique presented in Ref. 21.

For a rather long pulse the number of the eigenvalues is approximately equal to[21]

$$N\cong\tau\mathcal{N}(u_0,-u_0) ,$$

where $\mathcal{N}(u_0,-u_0)$ is the quantity of the eigenvalues in the region for a unit "length." For simplicity, we consider the Gaussian white noise, in the case the latter relation has the form

$$N\cong 2\tau\mathcal{N}(u_0,0) . \tag{A3}$$

For localized eigenvalues of the ZS eigenproblem it is convenient to present the eigenfunctions in the form $\Psi_2=\Psi_1 e^{2i\varphi}$, then the phase φ evolves according to the equation

$$\frac{d\varphi}{dt}=-\lambda+[u_0+v(t)]\cos(2\varphi) .$$

The phase φ has this important property: its derivation in respect to the spectral parameter λ in each point of t has a constant sign.

Following the general theory (Ref. 21, Chap. 2, Secs. 6.1 and 6.2), the number of eigenvalues on the unit length may be rewritten as follows:

$$\mathcal{N}(\lambda,0)=|J(\lambda)| ,$$

where $J(\lambda)$ is the stationary flux of the probability density (PD) $p(\varphi)$. The Fokker-Planck equation for the PD $p(t,\varphi)$ has the form (cf. Ref. 21)

$$\frac{\partial p(t,\varphi)}{\partial t}+\frac{\partial}{\partial\varphi}\{[-\lambda+u_0\cos(2\varphi)]p(t,\varphi)\} =2D\frac{\partial}{\partial\varphi}\left[\cos(2\varphi)\frac{\partial}{\partial\varphi}[p(t,\varphi)\cos(2\varphi)]\right] ,$$

and the flux $J(\lambda)$ is equal to the following:

$$J(\lambda)=[-\lambda+u_0\cos(2\varphi)]p-2D\cos(2\varphi)\frac{\partial}{\partial\varphi}[p\cos(2\varphi)] . \tag{A4}$$

The relation (A4) is the equation for the stationary PD $p(\varphi)$. We have to find its π-periodic solution,

$$p(\varphi)=p(\varphi+\pi) , \tag{A5}$$

and then, using the normalized condition

$$\int_0^{\pi}p(\varphi)d\varphi=1 ,$$

we obtain the flux $J(\lambda)$, and, as a result, the number of eigenvalues $\mathcal{N}(\lambda,0)$.

The general problem for calculation of $\mathcal{N}(\lambda,0)$ in the case of systems which are similar to the ZS eigenproblem (2.3) was solved in Refs. 30–32 (see also Ref. 21, Chap. 2, Sec. 8). In this appendix we calculate the value $\mathcal{N}(u_0,0)$ which describes the total number of the eigenvalues.

If $\lambda=u_0$ in Eq. (A4), then the solution of the corresponding equation, which has the property (A5), may be presented in the form

$$p(\varphi)=-\frac{J(u_0)}{2D}\int_{-\pi/4}^{\varphi}\frac{\exp[R(\varphi)-R(\varphi')]}{\cos^2(2\varphi')}d\varphi' \tag{A6}$$

for $-\pi/4<\varphi<\pi/4$, and a similar formula for $\pi/4<\varphi<3\pi/4$ after replacing $-\pi/4$ by $\pi/4$ in (A6). Here

$$R(\varphi)=-\frac{1}{4d}\left[\tan(2\varphi)-\ln\left|\frac{1+\tan\varphi}{1-\tan\varphi}\right|\right]-\ln|\cos(2\varphi)| ,$$

$$d\equiv D/u_0 .$$

Straightforward but rather cumbersome analysis yields that in the limit $d\ll 1$ the integral in Eq. (A6) is a spike function of φ with the amplitude and width of the order of $d^{1/3}$. The latter lead to the estimation

$$|J(u_0)|\sim u_0(D/u_0)^{1/3} , \tag{A7}$$

i.e.,

$$N\sim u_0\tau(D/u_0)^{1/3},\quad D\ll u_0$$

[see (2.35)]. In the case $D\sim u_0$ the integral in (A6) has no parameters and may be estimated as follows:

$$N\sim D\tau\sim u_0\tau,\quad D\sim u_0 \tag{A8}$$

[see (2.36)]. At last, for $D\gg u_0$ the integral has the amplitude of the order of D/u_0 and its width ~ 1. As a result, we have the formula

$$N\sim u_0\tau,\quad D\gg u_0 ,$$

which coincides with (A8). The latter has the simple physical explanation. For $D\sim u_0$ fluctuations strongly change a gap in the initial spectrum [at $v(t)=0$], and for more large intensities the situation is the same.

APPENDIX B: DERIVATION OF THE FOKKER-PLANCK EQUATION

In this appendix we present an exact derivation of the effective FP equation related to the ZS eigenproblem with the random input pulse (4.1) to (4.3). The calculations described below allow us to obtain the FP equation in the form which is known in the theory of disordered systems (see Ref. 21); that is why we may directly use those results.

To describe statistical properties of the problem under consideration, we seek the solution of Eq. (2.3) in the form

$$\Psi(t,\lambda)=\begin{bmatrix}0\\ t_1e^{-i\lambda t}\end{bmatrix},\quad t<0 , \tag{B1a}$$

$$\Psi(t,\lambda)=\begin{bmatrix}re^{i\lambda t}\\ e^{-i\lambda t}\end{bmatrix},\quad t>T , \tag{B1b}$$

and

$$\Psi(t,\lambda)=\begin{bmatrix}\Psi_1(t,\lambda)\\ \Psi_2(t,\lambda)\end{bmatrix},\quad 0<t<T . \tag{B2}$$

Then, using the invariant imbedding method[18] we may obtain from the set of equations for Ψ_1 and Ψ_2 defined above in (B2) the equation for the ratio

$$r(t)=\Psi_1(t,\lambda)/\Psi_2(t,\lambda) \tag{B3}$$

in the following form (see also Refs. 32 and 33):

$$\frac{1}{r}\frac{dr}{dt}=2i\lambda-iu(t)r-iu(t)r^{-1} , \tag{B4}$$

with the initial condition

$$r(t=0)=0 . \tag{B5}$$

The stochastic Eq. (B4) is the fundamental equation for the study of the stochastic properties of the input pulse.

Let us introduce the notation

$$r(t)=\exp[-w(t)+i\phi(t)] \tag{B6}$$

and then Eq. (B4) takes the form

$$\frac{dw}{dt}=\sinh w(-iue^{i\phi}+iue^{-i\phi}) , \tag{B7a}$$

$$\frac{d\phi}{dt}=2\lambda+\cosh w(-ue^{i\phi}-ue^{-i\phi}) . \tag{B7b}$$

The first term in Eq. (B7b) is a regular addendum to the phase ϕ. Taking into account standard approaches, we may average the equations on the fast phase variable $2\lambda t$. If we substitute

$$\phi=2\lambda t+\theta \; , \tag{B8}$$

θ being a slowly varying value, after averaging we may obtain a single stochastic equation for w,

$$\frac{dw}{dt}=2D\sinh w\cosh w+V(t)\sinh w \; , \tag{B9}$$

where the effective potential $V(t)$ is related to the random pulse $u(t)$ as follows:

$$V(t)=-iu_1(t)e^{i\theta}+iu_1^*(t)e^{-i\theta} \tag{B10}$$

and

$$u(0,t)=u_1(t)e^{2i\lambda t}+u_1^*(t)e^{-2i\lambda t} \; . \tag{B11}$$

As a result,

$$\langle V(t)V(t')\rangle=4D\delta(t-t') \; . \tag{B12}$$

At last, we may directly obtain the FP equation corresponding to Eq. (4.9) with the potential (B12),

$$\tau_0\frac{\partial P(w,t)}{\partial t}=\frac{\partial^2}{\partial w^2}[\sinh^2 wP(w,t)] \; . \tag{B13}$$

Here the time parameter

$$\tau_0=1/2D \tag{B14}$$

has the sense of the localization "length," which does not depend on the spectral parameter λ. The FP equation (B13) with the localization length of another form is well known in the theory of disordered systems,[21] and we may use directly the general results of that theory.

It is important to note once again that the FP equation (B13) is valid for a number of conditions. If we denote $D\approx u_0^2\tau_c$ (the same estimation is always valid), then those conditions may be presented as follows.

(i) $\tau_c \ll D^{-1}$, the condition for parameters of the random pulse: it must be δ correlated.

(ii) $D\ll|\lambda|$, the condition for the averaging of Eq. (B7). As a result, the FP equation [(B13) and (B14)] is valid for [cf. (4.6)]

$$D\ll|\lambda|\ll\tau_c^{-1} \; . \tag{B15}$$

*The author to whom all correspondence should be addressed.

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