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Quantum theory for continuous photodetection processes

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We develop a quantum theory for continuous photodetection processes that describes nonunitary time development of the field under continuous measurement of photon number. Exact expressions are obtained for time evolutions of the photon-field density operator, average and variance of the photon number, and the Fano factor. These are applied to typical quantum states, i.e., number, coherent, thermal, and squeezed states. The continuous photodetection process is made up of two elementary processes in terms of the referring measurement process, that is, one-count and no-count processes. Just after the one-count process in which a photodetector registers one photoelectron, the average photon number $\langle n(t) \rangle$ of the remaining field is shown to increase for super-Poissonian states (e.g., thermal state) and decrease for sub-Poissonian states (e.g., number state); for the Poissonian state (e.g., coherent light), $\langle n(t) \rangle$ does not change. During the no-count process in which the photodetector registers no photoelectrons, on the other hand, $\langle n(t) \rangle$ decreases in time for all states except the number state. The physical origins for these results are clarified from the viewpoint of nonunitary state reduction by continuous measurement of photon number. Furthermore, we introduce a nonreferring measurement process in which the detector registers photocounts, but we discard all readout information. We discuss the difference in the way the photon field evolves in this process compared to the referring measurement process.

I. INTRODUCTION

According to von Neumann's quantum theory of measurement, a quantum photodetection process is categorized into two stages.[1] In the first stage, the photon field and the detector couple with each other via a unitary interaction, establishing a quantum correlation between them. This process is reversible because the interaction is unitary. In the second stage, the number of photoelectric conversions is read out instantaneously, producing a new quantum state of the photon field via nonunitary state reduction. Thus, the measurement process is irreversible only at the second stage.

An actual photodetection process, however, differs from the above picture because the number of photoelectrons is measured not at a single time but one by one. Information concerning registration of a photocount is read out in *real time* throughout the measurement period. The state reduction of the photon field therefore occurs at every moment when the detector is active, and the photon field thus evolves nonunitarily. There are a number of articles[2–5] which correctly incorporate the effect of state reduction by continuous measurement of photon number based on or similar to the Srinivas-Davies model.[6] Ueda further developed a general theory for the nonunitary time evolution of the photon field by continuous measurement of photon number.[7]

In the present paper, we develop several general formulas to describe a nonunitary evolution of the field under continuous photodetection process, and examine the time evolution in detail for typical quantum states: number, coherent, thermal, and squeezed states. We show that the manner of the state reduction depends strongly on both the initial photon statistics and the readout information concerning registrations of photocounts (photoelectron statistics). In particular, it is found that for a super-Poissonian state, the average photon number immediately after one photoelectron was registered (one-count process) *increases*, whereas while no photoelectrons are being registered (no-count process) the average photon number *decreases* in time except for the number state. The physical origins for such counterintuitive results are clarified from the viewpoint of nonunitary state reduction by continuous measurement of photon number.

The process we have described so far may be called a *referring measurement process* (RMP) because we refer to all information concerning registrations of photocounts throughout a measurement period. We can, however, discard this available information. That is, we certainly know that the detector is active, but we do not read out the results of measurement. Such a process may be called a *nonreferring measurement process* (NMP) because we do not refer to the results of measurement. We investigate the difference in the way the photon field evolves in this process compared to the referring measurement process.

This paper is organized as follows. Section II describes a nonunitary time evolution of the photon field in the RMP and develops several formulas for the time evolution of the photon-number moments. Section III applies these formulas to four typical quantum states. Here, the paradoxial time evolutions of the average photon number and variance are schematically illustrated. Section IV describes the state evolutions of typical quantum states in the NMP and compares them with those in the RMP. Section V discusses the physical origins for some counterintuitive results obtained so far from the viewpoint of nonunitary state reduction by continuous measurement of photon number. Some complicated algebra is relegated to the appendices to avoid digressing from the main subjects.

II. STATE EVOLUTION IN A REFERRING MEASUREMENT PROCESS

A. General description

We adopt a model of a continuous photodetection process proposed by Srinivas and Davies[6] because it abstracts the essential features of a quantum photodetection process without assuming any microscopic structure of the photodetector. Before going into the detailed analysis, let us outline our scheme. We consider the time evolution of a photon field in a closed optical cavity as shown in Fig. 1. The photodetector begins to count photons at $t=0$ when a small window on the cavity well is opened. We assume an ideal cavity such that the photons dissipate only through the window. Let us consider a regular point process[8] where the probability of more than one photon being registered during an infinitesimal time interval is negligible. Then the one-count and no-count processes form an exclusive exhaustive set of events in any infinitesimal time interval. We refer to the process in which one photoelectron is registered as a *one-count process*, and to the process in which no photoelectrons are registered as a *no-count process*.

In general, the density operator of the photon field is changed discontinuously by the one-count process because one photon is instantaneously absorbed from the field by the detector. It is also changed by the no-count process because it gives us information that no photon is detected. This change is, however, continuous in time.[7] Therefore an initial photon field changes continuously during the first no-count process, then discontinuously at the first one-count process, and again continuously up to the second one-count process, etc. After a sufficiently long time, the total number of registered photoelectrons is identified with the total number of photons in the initial state. Here we assume unit detection efficiency for simplicity. The photon field after the measurement process is completed is the vacuum state because all photons have been absorbed. If we stop the measurement process by closing the window at a finite time t_{final} ($<\infty$), then the photon field in the cavity is not the vacuum state but is an intermediate state which is reducing towards the vacuum state. Thus an actual photodetection process affords a good example of a quantum measurement process in which the effects of both continuous measurement and state reduction must be considered in real time. Henceforth, we shall refer to such a process as a *referring measurement process* (RMP).

We shall also investigate another process where we do not read out any information concerning the results measurement, although we know that the detector is active. In this case, the density operator of the photon field changes nonunitarily, too, but in a completely different way from that in the RMP, as will be discussed in Sec. IV. We refer to such a process as a *nonreferring measurement process* (NMP).

In general, a quantum measurement process plays two distinct roles with respect to the past and future of the observed system.[7,9] With respect to the past, repeated measurements of the same quantum state verify the probability distribution of photoelectrons that was predicted from the previous measurement. With respect to the future, a single measurement produces a new quantum state via nonunitary state reduction caused by the measurement back action on the photon field, as schematically shown in Fig. 2. The density operator of the photon field at a particular time determines the probability that a photoelectron is registered at the same time (measurement action); whether a photoelectron is registered or not determines the photon density operator at an infinitesimally later time (measurement back action). Thus the density operator of the photon field evolves nonunitarily due to continuous measurement of photon number and its back action on the photon field. In the rest of Sec. II we shall discuss how the photon field

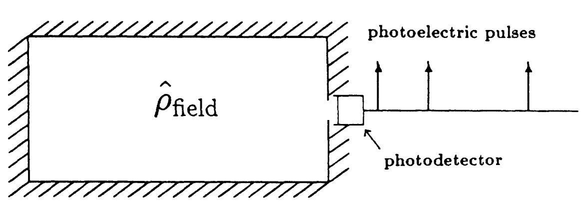

FIG. 1. Photon counting system for an optical field in a closed cavity. The density operator of the photon field gradually changes into the vacuum state, producing photoelectric pulses one by one.

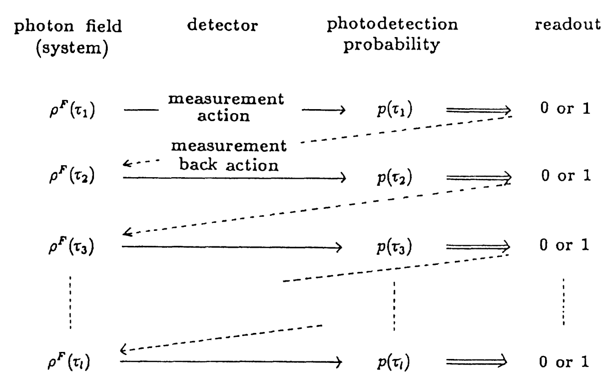

FIG. 2. Measurement action and back action in a continuous photodetection context.

evolves in time according to the results of continuous measurement of photon number.

B. One-count process

The one-count process is described by a superoperator J as[6]

$$J\rho(t)=\lambda a\rho(t)a^{\dagger} , \tag{2.1}$$

where $\rho(t)$ is the density operator just before the one-count process, λ is a parameter which represents the probability of one photoelectron being registered per unit time when the initial field is in a single-photon state, and a ($a^{\dagger}$) is the photon annihilation (creation) operator. The operator J expresses the measurement action of annihilating one photon from the photon field. Although Eq. (2.1) is a postulate as described in Ref. 3, it can be justified if we use a physical model of photon counting.[10]

An operator which describes a quantum photodetection process should give both the probability for the result of measurement (photoelectron statistics) and the quantum state immediately after the measurement (photon statistics). The probability, $P(J)dt$, that a one-count process occurs in the interval from t to $t+dt$ is given by

$$P(J)dt=\mathrm{Tr}[J\rho(t)]dt=\lambda\langle n(t)\rangle dt , \tag{2.2}$$

where

$$\langle n(t)\rangle \equiv \mathrm{Tr}[\rho(t)a^{\dagger}a] \tag{2.3}$$

is the average photon number just before the one-count process. The density operator of the post-measurement state is related to that of the pre-measurement state by

$$\rho(t^{+})=\frac{J\rho(t)}{\mathrm{Tr}[J\rho(t)]}=\frac{a\rho(t)a^{\dagger}}{\langle n(t)\rangle} , \tag{2.4}$$

where the symbol t^{+} denotes a time infinitesimally later than t (just after the one-count process). Then the average photon number immediately after the one-count process defined by $\langle n(t^{+})\rangle \equiv \mathrm{Tr}[\rho(t^{+})a^{\dagger}a]$ is given by

$$\langle n(t^{+})\rangle=\langle n(t)\rangle-1+\frac{\langle[\Delta n(t)]^2\rangle}{\langle n(t)\rangle} , \tag{2.5}$$

where $\Delta n(t)\equiv n(t)-\langle n(t)\rangle$. This result expresses the average photon number of the post-measurement state in terms of the pre-measurement photon statistics. We find that the difference between the average photon numbers before and after the one-count process is not exactly equal to 1, but it has an additional term depending on the variance of the photon number before the measurement. This term is nothing but the Fano factor,

$$F(t)\equiv\frac{\langle[\Delta n(t)]^2\rangle}{\langle n(t)\rangle} , \tag{2.6}$$

which takes a value greater than unity for super-Poissonian states, less than unity for sub-Poissonian states, or equal to unity for Poissonian states. Thus we find that the average photon number immediately after the one-count process increases, decreases, or remains unchanged according to whether the pre-measurement photon statistics are super-Poissonian, sub-Poissonian, or Poissonian, respectively. Results for several initial photon statistics are summarized in Table I.

TABLE I. One-count process.

Average photon number	Initial state
$\langle n(t^{+})\rangle=\langle n(t)\rangle-1$	number
$\langle n(t)\rangle-1<\langle n(t^{+})\rangle<\langle n(t)\rangle$	sub-Poissonian
$\langle n(t^{+})\rangle=\langle n(t)\rangle$	Poissonian
$\langle n(t)\rangle<\langle n(t^{+})\rangle<2\langle n(t)\rangle$	sub-thermal
$\langle n(t^{+})\rangle=2\langle n(t)\rangle$	thermal
$\langle n(t^{+})\rangle>2\langle n(t)\rangle$	super-thermal

The variance immediately after the one-count process, which is defined by

$$\langle[\Delta n(t^{+})]^2\rangle=\mathrm{Tr}[\rho(t^{+})(a^{\dagger}a)^2]-\{\mathrm{Tr}[\rho(t^{+})a^{\dagger}a]\}^2 ,$$

is represented by the photon statistics before the one-count process as

$$\langle[\Delta n(t^{+})]^2\rangle=\langle[\Delta n(t)]^2\rangle+\left[\frac{\langle[\Delta n(t)]^3\rangle}{\langle n(t)\rangle}-\left[\frac{\langle[\Delta n(t)]^2\rangle}{\langle n(t)\rangle}\right]^2\right] , \tag{2.7}$$

where $\langle[\Delta n(t)]^3\rangle$ is the third cumulant of photon number at time t. This equation shows that the sign of the change in photon number variance depends on the second and third cumulants of the original photon statistics. Appendix A shows that the kth moment of the post-measurement state is expressed in terms of the moments of the pre-measurement state up to $(k+1)$th as

$$\langle n(t^{+})^k\rangle=\frac{1}{\langle n(t)\rangle}\sum_{m=0}^{k}\begin{bmatrix}k\\ m\end{bmatrix}(-1)^{k-m}\langle n(t)^{m+1}\rangle , \quad k=1,2,3,\ldots . \tag{2.8}$$

From this result we find that the one-count process, in general, changes not only the average photon number but also changes the whole statistics of the original photon field. This reflects the fact that the photon counting is a second-kind unsharp measurement of photon number. A unique exception is the coherent state; the coherent state suffers no change by the one-count process as will be shown in Sec. V. With respect to only the kind of statistics, the number state is another exception; it remains a number state, although its eigenvalue decreases by 1.

C. No-count process

Next, let us examine the no-count process. Associated with the no-count process is a superoperator S_τ such that the probability, $P(S_\tau)$, of no count being registered in the interval from t to $t+\tau$ is

$$P(S_\tau)=\mathrm{Tr}[S_\tau\rho(t)] , \tag{2.9}$$

where τ is an arbitrary time interval. It has been postulated that[6]

$$S_\tau \rho(t) = \exp\left[-\left(i\omega + \frac{\lambda}{2}\right)a^\dagger a \tau\right] \times \rho(t) \exp\left[\left(i\omega - \frac{\lambda}{2}\right)a^\dagger a\tau\right] . \quad (2.10)$$

Note that when the coupling between the detector and the field is zero, that is when $\lambda=0$, the field evolves according only to the free Hamiltonian $\hbar\omega a^\dagger a$, as it should. The density operator immediately after the no-count process is given by

$$\rho(t+\tau) = \frac{S_\tau \rho(t)}{\mathrm{Tr}[S_\tau \rho(t)]} = \frac{\exp[-(i\omega+\lambda/2)a^\dagger a\tau]\rho(t)\exp[(i\omega-\lambda/2)a^\dagger a\tau]}{\mathrm{Tr}[\rho(t)\exp(-\lambda a^\dagger a\tau)]} . \quad (2.11)$$

This evolution is also nonunitary. The state evolution in the one-count process is discontinuous in general. The state reduction in the no-count process, in contrast, proceeds continuously because the nonunitary part $\exp[-(\lambda/2)a^\dagger a\tau]$ vanishes as τ approaches zero. The no-count process does not extract any photons from the initial photon field and, therefore, $\rho_{mn}(t+\tau)$ depends only on $\rho_{mn}(t)$. Nevertheless, the average photon number decreases for all but one quantum state because the readout information that no count has been registered in the interval from t to $t+\tau$ requires us to modify the knowledge about the original photon-density operator according to Eq. (2.11). Thus the statistical properties of the observed photon field change in time during the no-count process. A unique exception is the number state; it suffers no change in the no-count process, as shown later. With respect to the kind of statistics alone, the coherent state is another exception; it remains the coherent state, although its amplitude attenuates exponentially in time.

The time evolution of the average photon number in the no-count process,

$$\langle n(t+\tau)\rangle = \mathrm{Tr}[\rho(t+\tau)a^\dagger a] ,$$

is given by

$$\langle n(t+\tau)\rangle = -\frac{1}{\tau}\frac{\partial}{\partial\lambda}\ln \mathrm{Tr}[\rho(t)\exp(-\lambda a^\dagger a\tau)] . \quad (2.12)$$

Expanding the right-hand side (rhs) of Eq. (2.12) in powers of small $\lambda\tau$, and taking the limit $\lambda\tau\to 0$ leads to a differential equation:

$$\frac{d}{dt}\langle n(t)\rangle = -\lambda\langle[\Delta n(t)]^2\rangle . \quad (2.13)$$

Thus we find that the average photon number decreases in time at a rate of the photon number variance multiplied by the reciprocal expectation value of waiting times. Therefore, it does not change for number state ($\langle[\Delta n(t)]^2\rangle=0$), but decays for all other states. Equation (2.13) is integrated to give another expression of $\langle n(t+\tau)\rangle$:

$$\langle n(t+\tau)\rangle = \langle n(t)\rangle - \lambda\int_t^{t+\tau}\langle[\Delta n(t')]^2\rangle dt' . \quad (2.14)$$

Results for several initial photon statistics are summarized in Table II. The time evolution of the photon-number variance depends only on the third cumulant as

$$\frac{d}{dt}\langle[\Delta n(t)]^2\rangle = -\lambda\langle[\Delta n(t)]^3\rangle , \quad (2.15)$$

or

$$\langle[\Delta n(t+\tau)]^2\rangle = \langle[\Delta n(t)]^2\rangle - \lambda\int_t^{t+\tau}\langle[\Delta n(t')]^3\rangle dt' . \quad (2.16)$$

Therefore, the Fano factor evolves in the no-count process as

$$F(t+\tau) = F(t) - \lambda\int_t^{t+\tau}\left[\frac{\langle[\Delta n(t')]^3\rangle}{\langle n(t')\rangle} - \left(\frac{\langle[\Delta n(t')]^2\rangle}{\langle n(t')\rangle}\right)^2\right]dt' . \quad (2.17)$$

In a similar way, it can be shown that the kth moment obeys the following differential equation:

$$\frac{d}{dt}\langle n(t)^k\rangle = -\lambda[\langle n(t)^{k+1}\rangle - \langle n(t)\rangle\langle n(t)^k\rangle] , \quad k=1,2,3,\ldots . \quad (2.18)$$

This equation shows that the rate of change of the kth moment is expressed in terms of the pre-measurement moments up to the $(k+1)$th. Equation (2.15) implies that the photon statistics change in time during the no-count process, in spite of the fact that no photons are detected in the process.

D. Quantum photodetection process of forward recurrence times

Now that we have studied the one-count process and the no-count process, we are in a position to describe an actual time evolution of the photon field where we read out all information concerning registrations of photocounts in *real time* throughout the measurement period. We refer to such a process as the quantum photodetection process of forward recurrence times (QPF). Suppose that the measurement process started as $t=0$ and ended

TABLE II. No-count process.

Average photon number	Initial state
$\langle n(t+\tau)\rangle = \langle n(t)\rangle$	number
$\langle n(t+\tau)\rangle = \langle n(t)\rangle e^{-\lambda\tau}$	coherent
$\langle n(t+\tau)\rangle = \dfrac{\langle n(t)\rangle e^{-\lambda\tau}}{1+\langle n(t)\rangle(1-e^{-\lambda\tau})}$	thermal

at $t=T$, and that m photons were registered at times τ_j $(j=1,2,\dots,m)$ with no further photons registered in the measurement period. Then, the density operator of the photon field, $\rho_m^{\rm QPF}(\tau_1,\tau_2,\dots,\tau_m;0,T)$, immediately after the measurement process is given by[7]

$$\rho_m^{\rm QPF}(\tau_1,\tau_2,\dots,\tau_m;0,T) = \frac{S_{T-\tau_m}JS_{\tau_m-\tau_{m-1}}J\cdots JS_{\tau_1}\rho(0)}{{\rm Tr}[S_{T-\tau_m}JS_{\tau_m-\tau_{m-1}}J\cdots JS_{\tau_1}\rho(0)]}, \tag{2.19}$$

where $\rho(0)$ is the initial density operator of the photon field. Here the denominator is sometimes called the probability distribution of forward recurrence times (PDF).[7,11] It is denoted as $P_m^{(\rm forward)}(\tau_1,\tau_2,\dots,\tau_m;0,T)$ and gives the probability per (unit time)^m that one count is registered at m times with no further counts registered at other times:

$$P_m^{(\rm forward)}(\tau_1,\tau_2,\dots,\tau_m;0,T) = {\rm Tr}[S_{T-\tau_m}JS_{\tau_m-\tau_{m-1}}J\cdots JS_{\tau_1}\rho(0)] . \tag{2.20}$$

It has been shown that[7]

$$S_{T-\tau_m}JS_{\tau_m-\tau_{m-1}}J\cdots S_{\tau_1}\rho(0)=\lambda^m \exp\left(-\lambda\sum_{j=1}^{m}\tau_j\right)\exp\left[-\left(i\omega+\frac{\lambda}{2}\right)a^\dagger aT\right]a^m\rho(0)(a^\dagger)^m\exp\left[\left(i\omega-\frac{\lambda}{2}\right)a^\dagger aT\right] . \tag{2.21}$$

Substituting Eq. (2.21) into Eq. (2.19) yields[7]

$$\rho_m^{\rm QPF}(\tau_1,\tau_2,\dots,\tau_m;0,T)=\frac{\exp[-(i\omega+\lambda/2)a^\dagger aT]a^m\rho(0)(a^\dagger)^m\exp[(i\omega-\lambda/2)a^\dagger aT]}{{\rm Tr}[\rho(0)(a^\dagger)^m\exp(-\lambda a^\dagger aT)a^m]} . \tag{2.22}$$

This equation gives the nonunitary time evolution of the photon field in the QPF. Note that the rhs of this equation no longer depends on the times at which photoelectrons were registered. This is because the denominator and the numerator in Eq. (2.19) share the same factor $\exp(-\lambda\sum_{j=1}^m\tau_j)$ and are therefore canceled out in forming the ratio. This is characteristic of closed-system photocounting and does not hold for open-system photocounting.[12] Henceforth we will denote the quantity in the left-hand side of Eq. (2.22) simply as $\rho_m^{\rm QPF}(T)$. On the other hand, the PDF retains the lost information concerning the times of registration. In fact, substituting Eq. (2.21) into Eq. (2.20) yields[7,13]

$$P_m^{(\rm forward)}(\tau_1,\tau_2,\dots,\tau_m;0,T) = \lambda^m\exp\left(-\lambda\sum_{j=1}^m\tau_j\right) \times{\rm Tr}[\rho(0)(a^\dagger)^m\exp(-\lambda a^\dagger aT)a^m] . \tag{2.23}$$

To examine how the original photon statistics develop in time, let us calculate the photon-number moments, $\langle n(t^+)^k\rangle$, immediately after the QPF. They are given by

$$\langle n(t^+)^k\rangle={\rm Tr}[\rho_m^{\rm QPF}(t)(a^\dagger a)^k] . \tag{2.24}$$

Using the operator identity which expands the moments in terms of the normally ordered ones[14]

$$(a^\dagger a)^k=\sum_{n=1}^{k}:(a^\dagger a)^n:\sum_{m=1}^{n}\frac{(-1)^{n+m}m^k}{m!(n-m)!} , \tag{2.25}$$

we obtain a useful formula,

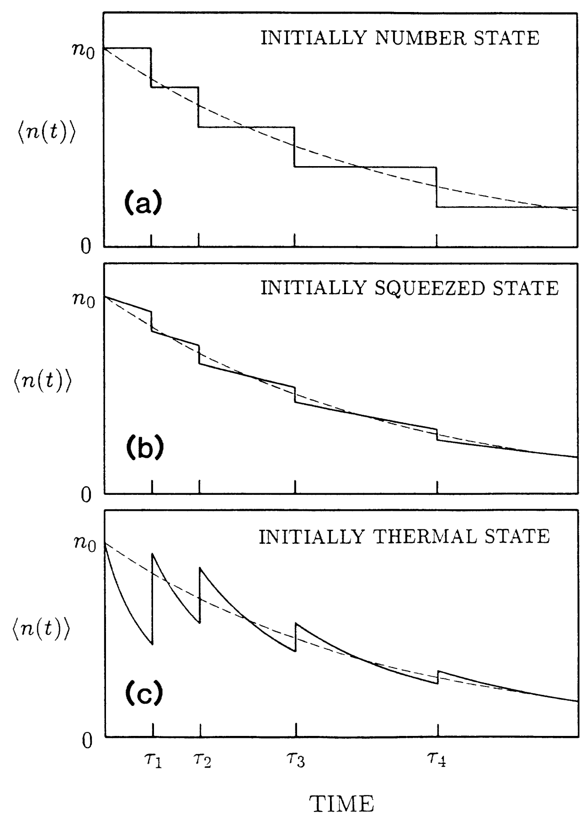

FIG. 3. Time evolution of the average photon number $\langle n(t)\rangle$ for initially (a) number, (b) squeezed, and (c) thermal states. Their initial average photon numbers and λ are chosen to be the same as one another. The dashed curves show the average photon-number evolution for an initially coherent state with the same initial average photon number. One-count processes are assumed to occur at $\tau_1,\tau_2,\dots$.

$$\langle n(t)^k \rangle = \sum_{j=1}^{k} \frac{1}{\lambda^j} \frac{P_{m+j}^{(\text{forward})}(\tau_1, \ldots, \tau_m, t, \ldots, t; 0, t)}{P_m^{(\text{forward})}(\tau_1, \tau_2, \ldots, \tau_m; 0, t)} \sum_{i=1}^{j} \frac{(-1)^{j+i} i^k}{i!(j-i)!} , \tag{2.26}$$

where $t > \tau_m$. In particular, the average and variance of the photon number are given by

$$\langle n(t) \rangle = \frac{1}{\lambda} \frac{P_{m+1}^{(\text{forward})}(\tau_1, \ldots, \tau_m, t; 0, t)}{P_m^{(\text{forward})}(\tau_1, \tau_2, \ldots, \tau_m; 0, t)} , \tag{2.27}$$

$$\langle [\Delta n(t)]^2 \rangle = \frac{1}{\lambda^2} \frac{P_{m+2}^{(\text{forward})}(\tau_1, \ldots, \tau_m, t, t; 0, t)}{P_m^{(\text{forward})}(\tau_1, \tau_2, \ldots, \tau_m; 0, t)} + \langle n(t) \rangle - \langle n(t) \rangle^2 , \tag{2.28}$$

where $t > \tau_m$. These formulas are very useful when we follow the time evolution of the photon field in the QPF. Figure 3 illustrates the time evolution of the average photon number in the QPF for initially (a) a number state, (b) a squeezed state, and (c) a thermal state. The times $\tau_1, \tau_2, \ldots$ indicate the times when photoelectrons were registered. The dashed curves correspond to an initially coherent state. Figure 3(a) indicates that when the initial state is a number state, $\langle n(t) \rangle$ does not change under the no-count process, and decreases by 1 for every one-count process. Figure 3(b) indicates that when the initial state is a sub-Poissonian squeezed state, $\langle n(t) \rangle$ is not constant: under the no-count process, $\langle n(t) \rangle$ decays more slowly than the dashed curve, which corresponds to the coherent state, and for the one-count process $\langle n(t) \rangle$ decreases by less than 1. Figure 3(c) indicates that when the initial state is a thermal state, $\langle n(t) \rangle$ decays faster than the dashed curve under the no-count process, and *increases* for the one-count process in contrast to the other states. The detailed calculation will be given in Sec. III.

E. Quantum photodetection process for the number of counts

In an actual photocounting experiment, we sometimes read out partial information concerning registrations of photocounts. One example is the quantum photodetection process for the number of counts (QPN) in which we measure only the number of counts registered in a measurement period. Since the information concerning the times of photodetection is discarded, the superoperator, $N_T(m)$, describing the QPN is given by

$$N_T(m) = \int_0^T d\tau_m \int_0^{\tau_m} d\tau_{m-1} \cdots \int_0^{\tau_2} d\tau_1 S_{T-\tau_m} J S_{\tau_m - \tau_{m-1}} \cdots J S_{\tau_1} . \tag{2.29}$$

Clearly, the probability, $P(m;0,T)$, of m counts being registered in an interval $[0,T)$ is given by

$$P(m;0,T) = \text{Tr}[N_T(m)\rho(0)] . \tag{2.30}$$

Substituting Eq. (2.29) into Eq. (2.30) and using Eq. (2.20), we have

$$P(m;0,T) = \int_0^T d\tau_m \int_0^{\tau_m} d\tau_{m-1} \cdots \int_0^{\tau_2} d\tau_1 P_m^{(\text{forward})}(\tau_1, \tau_2, \ldots, \tau_m; 0, T) . \tag{2.31}$$

The density operator, $\rho_m^{\text{QPN}}(T)$, immediately after the QPN is therefore given by

$$\rho_m^{\text{QPN}}(T) = \frac{N_T(m)\rho(0)}{\text{Tr}[N_T(m)\rho(0)]} . \tag{2.32}$$

Substituting Eqs. (2.21) and (2.29) into Eq. (2.32), we obtain

$$\rho_m^{\text{QPN}}(T) = \frac{\exp[-(i\omega + \lambda/2)a^\dagger a T] a^m \rho(0) (a^\dagger)^m \exp[(i\omega - \lambda/2)a^\dagger a T]}{\text{Tr}[\rho(0)(a^\dagger)^m \exp(-\lambda a^\dagger a T) a^m]} , \tag{2.33}$$

which is identical to Eq. (2.22). That is, with respect to the post-measurement state, the QPF and QPN gives the same result.[7] This statement holds for any initial quantum state as long as the closed-system photodetection process is considered.

III. APPLICATIONS TO TYPICAL QUANTUM STATES

Using the general equations obtained in Sec. II, we examine the nonunitary state evolution for four typical initial states, i.e., number state, coherent state, squeezed state, and thermal state. The evolution can be traced exactly and is graphically visualized using the average photon number $\langle n(t) \rangle$, the photon number variance $\langle [\Delta n(t)]^2 \rangle$, and the Fano factor $F(t) \equiv \langle [\Delta n(t)]^2 \rangle / \langle n(t) \rangle$.

A. Initially number state

We choose the initial condition as the number state $\rho(0) = |n_0\rangle\langle n_0|$. Equation (2.22) gives the density operator immediately after the QPF as

$$\rho_m^{\text{QPF}}(T) = |n_0 - m\rangle\langle n_0 - m| . \tag{3.1}$$

The photon field remains the same number state under the no-count process, but its eigenvalue decreases by 1 for the one-count process. Hence the result (3.1). Thus the time evolution of the average photon number is given by

$$\langle n(t)\rangle = n_0 - k \quad \text{for } \tau_k < t \leq \tau_{k+1} \ . \tag{3.2}$$

The photon number variance remains zero throughout the measurement process:

$$\langle [\Delta n(t)]^2 \rangle = 0 \quad \forall t \ . \tag{3.3}$$

The Fano factor of the number state remains, therefore, zero. It should be noted that these results stem from the fact that we read out all available information about registrations of photocounts; if we discard some of the information, the number state will no longer remain a number state. This observation is crucially important from a measurement-theoretical point of view, as will be discussed in Sec. IV, in connection with the nonreferring measurement process.

Next, let us examine the nonunitary time development of the number state in the QPN. Since the QPF and QPN give the same post-measurement states, we have

$$\rho_m^{\mathrm{QPN}}(T) = |n_0 - m\rangle\langle n_0 - m| \ . \tag{3.4}$$

With respect to the measured photoelectron statistics, however, the QPF and QPN give different results. The probability distribution of forward recurrence times, which corresponds to the QPF, is obtained from Eq. (2.23) as

$$P_m^{(\mathrm{forward})}(\tau_1,\tau_2,\ldots,\tau_m;0,T) = \lambda^m \exp\left[-\lambda \sum_{j=1}^{m} \tau_j\right] \frac{n_0!}{(n_0-m)!} (e^{-\lambda T})^{n_0-m} \ . \tag{3.5}$$

This probability distribution contains both the term of information concerning times of photocounts, $\exp(-\lambda \sum_{j=1}^{m} \tau_j)$, and the term concerning times of no counts, $(e^{-\lambda T})^{n_0-m}$. On the other hand, the probability distribution for the number of counts, which corresponds to the QPN, is obtained by substituting Eq. (3.4) into (2.30):

$$P(m;0,T) = \begin{bmatrix} n_0 \\ m \end{bmatrix} (1-e^{-\lambda T})^m (e^{-\lambda T})^{n_0-m} \ . \tag{3.6}$$

This probability distribution no longer contains the information concerning the times of photocounts because we discarded it. These results will be compared to those obtained in the nonreferring measurement process (see Sec. IV).

B. Initially coherent state

The density operator of the coherent state is $\rho(0) = |\alpha_0\rangle\langle\alpha_0|$, where $|\alpha_0\rangle$ is a coherent state vector with complex amplitude α_0. This can be expressed in terms of the number state as[15]

$$|\alpha_0\rangle = \exp\left[-\frac{|\alpha_0|^2}{2}\right] \sum_{n=0}^{\infty} \frac{\alpha_0^n}{(n!)^{1/2}} |n\rangle \ . \tag{3.7}$$

The nonunitary time evolution of the initially coherent state in the QPF is obtained with the aid of Eq. (2.22) as[13]

$$\rho_m^{\mathrm{QPF}}(T) = \left|\alpha_0 \exp\left[-\left[i\omega + \frac{\lambda}{2}\right]T\right]\right\rangle \times \left\langle \alpha_0 \exp\left[-\left[i\omega + \frac{\lambda}{2}\right]T\right]\right| \ . \tag{3.8}$$

In contrast to the number state, the coherent state does not change in the one-count process because the coherent state is an eigenstate of the annihilation operator. Therefore when operated on by the superoperator J, the coherent state does not change but simply produces a constant factor $|\alpha_0|^2$, which is canceled out by the denominator and numerator in Eq. (2.4). However, the amplitude of the coherent state decreases exponentially under the no-count process for the reason given in Sec. II C. Consequently, both the average photon number and the photon number variance decay exponentially at the same rate, i.e.,

$$\langle n(t)\rangle = \langle [\Delta n(t)]^2\rangle = |\alpha_0|^2 \exp(-\lambda t) \ . \tag{3.9}$$

Hence, the Fano factor remains unity at all times: $F(t)=1$. An initially coherent state remains Poissonian throughout the referring measurement process.

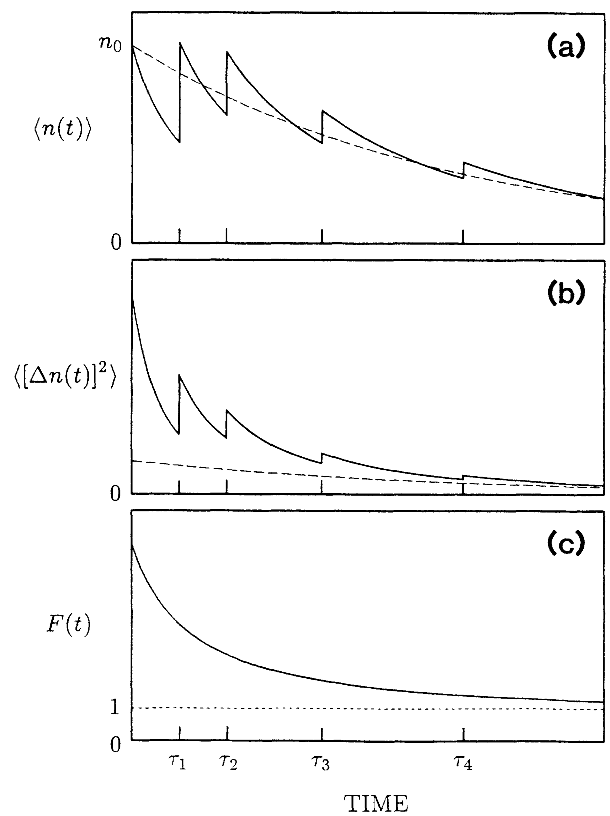

FIG. 4. Time evolution of the average photon number $\langle n(t)\rangle$, photon-number variance $\langle[\Delta n(t)]^2\rangle$, and the Fano factor $F(t)$ for an initially thermal state. The dashed curves show the corresponding evolution for an initially coherent state with the same initial average photon number. One-count processes are assumed to occur at $\tau_1, \tau_2, \ldots$.

C. Initially thermal state

The density operator of the thermal state is given by

$$\rho(0)=\frac{1}{1+n_0}\sum_{n=0}^{\infty}\left[\frac{n_0}{1+n_0}\right]^n|n\rangle\langle n| , \quad (3.10)$$

where n_0 is the average photon number of the initial state. Substituting this into Eq. (2.22) yields the nonunitary time evolution in the QPF of the initially thermal state as

$$\rho_m^{\mathrm{QPF}}(T)=\left[\frac{1+n_0(1-e^{-\lambda T})}{1+n_0}\right]^{m+1}\sum_{n=0}^{\infty}\begin{bmatrix}n+m\\ m\end{bmatrix}\left[\frac{n_0e^{-\lambda T}}{1+n_0}\right]^n|n\rangle\langle n| . \quad (3.11)$$

To calculate the time evolution of the photon-number moments, let us evaluate the PDF. From Eqs. (2.23) and (3.10), we obtain

$$P_m^{(\mathrm{forward})}(\tau_1,\tau_2,\ldots,\tau_m;0,T)=m!\exp\left[-\lambda\sum_{j=1}^{m}\tau_j\right]\frac{(\lambda n_0)^m}{[1+n_0(1-e^{-\lambda T})]^{m+1}} . \quad (3.12)$$

Substituting this equation into Eq. (2.27) yields

$$\langle n(t)\rangle=\frac{(k+1)n_0e^{-\lambda t}}{1+n_0(1-e^{-\lambda t})}\quad \text{for } \tau_k<t\le\tau_{k+1},\quad k=0,1,2,\ldots . \quad (3.13)$$

From this equation we find that every time a photon is detected, the average number of photons remaining in the photon field *increases* by the ratio

$$\frac{\langle n(\tau_k^+)\rangle}{\langle n(\tau_k)\rangle}=\frac{k+1}{k},\quad k=1,2,\ldots , \quad (3.14)$$

while under the no-count process the average photon number *decreases* in time by the ratio

$$\frac{\langle n(t)\rangle}{\langle n(\tau_k^+)\rangle}=\exp[-\lambda(t-\tau_k)]\frac{1+n_0(1-e^{-\lambda\tau_k})}{1+n_0(1-e^{-\lambda t})} ,\quad \text{for } \tau_k<t\le\tau_{k+1} . \quad (3.15)$$

This equation shows that the average photon number decreases faster than for the initially coherent state. The time evolution of the photon number variance can be obtained using Eq. (2.28) as

$$\langle[\Delta n(t)]^2\rangle=\frac{(k+1)n_0(n_0+1)e^{-\lambda t}}{[1+n_0(1-e^{-\lambda t})]^2} . \quad (3.16)$$

Therefore, the Fano factor becomes

$$F(t)=\frac{1+n_0}{1+n_0(1-e^{-\lambda t})} , \quad (3.17)$$

which is always larger than unity, but approaches unity as the measurement time progresses. Thus, we find that an initially thermal state approaches the Poisson statistics as a result of the referring measurement. Here we note that although $\langle n(t)\rangle$ and $\langle[\Delta n(t)]^2\rangle$ change discontinuously in the one-count process, the Fano factor has no discontinuities even for the one-count process. This is a special feature of the thermal state whose photocount distribution obeys a power law. Figures 4(a), 4(b), and 4(c) illustrate the time evolution of the average photon number, photon number variance, and the Fano factor, respectively, where the dotted curves correspond to an initially coherent state with the same initial average photon number.

The probability distribution for the number of counts is obtained by substituting Eq. (3.12) into Eq. (2.31):

$$P(m;0,T)=\frac{[n_0(1-e^{-\lambda T})]^m}{[1+n_0(1-e^{-\lambda T})]^{m+1}} . \quad (3.18)$$

This result shows that the Bose-Einstein character of photoelectron statistics for the initially thermal state holds in the QPN, even though the photon statistics develop into different statistics as in Eq. (3.11).

D. Initially squeezed state

A squeezed state of light, $|\alpha,r\rangle$, can be generated from a coherent state $|\alpha\rangle$ via a unitary transformation:[16]

$$|\alpha,r\rangle\equiv\exp\left[\frac{r}{2}[a^2-(a^\dagger)^2]\right]|\alpha\rangle=\frac{\exp[-|\alpha|^2/2+(\alpha^2/2)\tanh r]}{(\cosh r)^{1/2}}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\left[\frac{\tanh r}{2}\right]^{n/2}H_n\left[\frac{\alpha}{(\sinh 2r)^{1/2}}\right]|n\rangle , \quad (3.19)$$

where r is a squeezing parameter and $H_n(z)$ is the nth Hermite polynomial defined as

$$H_n(z)=\sum_{m=0}^{[n/2]}\frac{(-1)^m n!}{m!(n-2m)!}(2z)^{n-2m} . \quad (3.20)$$

Using Eq. (3.19), the density operator of the squeezed state becomes

$$\rho(0)=|\alpha,r\rangle\langle\alpha,r|=\frac{\exp\{-|\alpha|^2+[\alpha^2+(\alpha^*)^2](\tanh r)/2\}}{(\cosh r)^{1/2}}\sum_{n,m=0}^{\infty}\frac{1}{(m!n!)^{1/2}}\left[\frac{\tanh r}{2}\right]^{(m+n)/2}H_m\left[\frac{\alpha}{(\sinh 2r)^{1/2}}\right]\times H_n^*\left[\frac{\alpha}{(\sinh 2r)^{1/2}}\right]|m\rangle\langle n| . \quad (3.21)$$

We assume that α is real because the phase of the coherent state does not affect the following discussion. Then Eq. (3.21) becomes

$$\rho(0)=\frac{\exp[\alpha^2(\tanh r-1)]}{(\cosh r)^{1/2}}\sum_{n,m=0}^{\infty}\frac{1}{(m!n!)^{1/2}}\left[\frac{\tanh r}{2}\right]^{(m+n)/2}H_m\left[\frac{\alpha}{(\sinh 2r)^{1/2}}\right]H_n^*\left[\frac{\alpha}{(\sinh 2r)^{1/2}}\right]|m\rangle\langle n| . \quad (3.22)$$

Substituting this into Eq. (2.22) gives (see Appendix B)

$$\rho_m^{\text{QPF}}(\tau)=\frac{1}{2^m N}\sum_{n,k=0}^{\infty}\frac{1}{(k!n!)^{1/2}}\left[\frac{\tanh r}{2}\right]^{(k+n)/2}\exp(-i\Omega k\tau+i\Omega^* n\tau)H_{k+m}\left[\frac{\alpha}{(\sinh 2r)^{1/2}}\right]H_{n+m}^*\left[\frac{\alpha}{(\sinh 2r)^{1/2}}\right]|k\rangle\langle n| \quad (3.23)$$

where

$$N\equiv\frac{d^m}{dz^m}\left[\exp\left[\frac{2z}{1+z}\frac{\alpha^2}{\sinh 2r}\right](1-z^2)^{-1/2}\right]_{z=e^{-\lambda\tau}\tanh r} \quad (3.24)$$

and $\Omega=\omega-i\lambda/2$. As before, we use formulas (2.27) and (2.28) to obtain the average photon number and variance, where the PDF is given in Appendix B as

$$P_m^{(\text{forward})}(\tau_1,\tau_2,\ldots,\tau_m;0,T)=N\frac{(\lambda\tanh r)^m}{\cosh r}\exp\left[-\lambda\sum_{j=1}^{m}\tau_j+\alpha^2(\tanh r-1)\right] . \quad (3.25)$$

Figures 5(a), 5(b), and 5(c) illustrate the time evolution of the average photon number, the photon number variance, and the Fano factor of an initially squeezed state, respectively, where the dotted curves show the corresponding time development of an initially coherent state. We choose parameters α and r such that the squeezed state shows sub-Poissonian photon statics, and the complex amplitude of the coherent state is chosen such that the initial average photon number is equal to that of the squeezed state. Note here that the average photon number and the photon number variance of the squeezed state $|\alpha,r\rangle$ are evaluated as $\alpha^2e^{-2r}+\sinh^2 r$ and $\alpha^2e^{-4r}+2\sinh^2 r\cosh^2 r$, respectively. In contrast to the thermal state, the average photon number decreases by less than 1 for the one-count process, which is a common feature of the sub-Poissonian state [see Eq. (2.5)]. On the other hand, under the no-count process it decreases at a smaller rate than that of the thermal and coherent states. The time development of the Fano factor shows that the photon statistics approach Poissonian as time progresses. In Fig. 5(c), we observe that the time development of the Fano factor has discontinuities at times when photocounts are registered. This presents a sharp contrast to the cases of number, coherent, and thermal states. The probability distribution for the number of counts is obtained if we substitute Eq. (3.25) into Eq. (2.31):

$$P(m;0,T)=N\frac{[(1-e^{-\lambda T})\tanh r]^m}{m!\cosh r}\exp[\alpha^2(\tanh r-1)] . \quad (3.26)$$

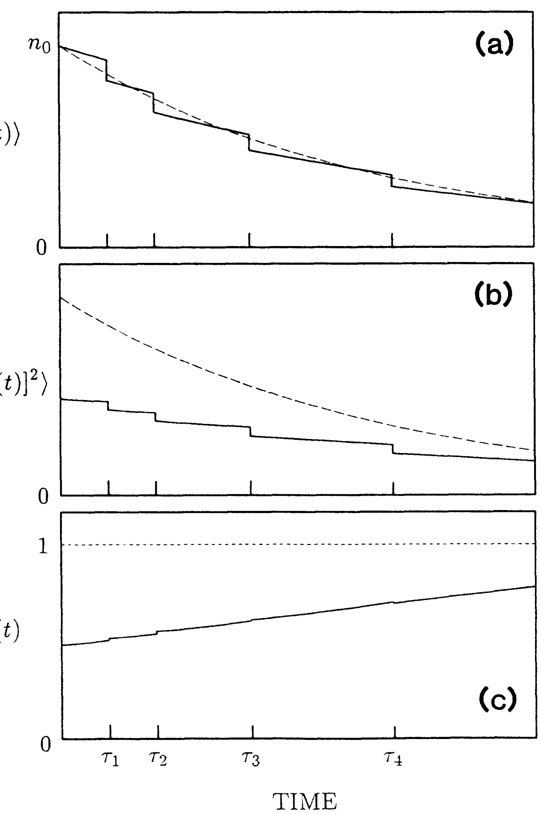

FIG. 5. Time evolution of the average photon number $\langle n(t)\rangle$, photon-number variance $\langle[\Delta n(t)]^2\rangle$, and the Fano factor $F(t)$ for an initally squeezed state. The dashed curves show the corresponding evolution for an initially coherent state with the same initial average photon number. One-count processes are assumed to occur at $\tau_1,\tau_2,\ldots$.

IV. STATE EVOLUTION IN NONREFERRING MEASUREMENT PROCESS

Thus far we have examined nonunitary time evolution of the photon field in the RMP. A characteristic feature of this process lies in the fact that we read out all or at least partial information concerning registrations of photocounts in real time throughout the measured period. Such a process consists of the one-count and no-count processes. On the other hand, it is completely at our choice whether we actually read out available information or discard some of the information. What happens to the photon field if we discard all information about photocounts except for knowledge that the detector is active? Such a process is called the nonreferring measurement process (NMP) with emphasis on the fact that we do not observe the photoelectric pulses although we certainly know that the detector is active. Let us introduce a superoperator T_τ such that it describes the time development of the density operator in the NMP of duration τ. Since we do not refer to the result of measurements, T_τ must satisfy the following identity:

$$\mathrm{Tr}[T_\tau \rho(t)]=1 . \tag{4.1}$$

Since the one-count and no-count processes form a complete set of events in an infinitesimal time interval dt, we have the equation

$$T_{dt}\rho(t)=J\rho(t)dt+S_{dt}\rho(t) . \tag{4.2}$$

This equation, combined with Eqs. (2.1) and (2.10), leads to a differential equation for the density operator of the photon field in the NMP:

$$\frac{d\rho(t)}{dt}=\lambda a\rho(t)a^\dagger-\left[i\omega+\frac{\lambda}{2}\right]a^\dagger a\rho(t)+\left[i\omega-\frac{\lambda}{2}\right]\rho(t)a^\dagger a . \tag{4.3}$$

This operator differential equation can be integrated exactly (see Appendix C) to give

$$\rho(t+\tau)=\frac{T_\tau\rho(t)}{\mathrm{Tr}[T_\tau\rho(t)]}=T_\tau\rho(t)=\sum_{k=0}^{\infty}\frac{(1-e^{-\lambda\tau})^k}{k!}\exp\left[-\left[i\omega+\frac{\lambda}{2}\right]a^\dagger a\tau\right]a^k\rho(t)(a^\dagger)^k\exp\left[\left[i\omega-\frac{\lambda}{2}\right]a^\dagger a\tau\right] , \tag{4.4}$$

where Eq. (4.1) has been used. This result has been obtained in Ref. 7 using a different method.

From this equation we can exactly evaluate the photon-number moments $\langle n(t+\tau)^k\rangle$ after the nonreferring measurement process (see Appendix D). The average photon number is given as

$$\langle n(t+\tau)\rangle=e^{-\lambda\tau}\langle n(t)\rangle , \tag{4.5}$$

that is, the average photon number decreases exponentially in time. This is because we discard all readout information so that the photodetector comes to play the simple role of a linear absorber with absorption coefficient λ. The photon-number variance is given by

$$\langle[\Delta n(t+\tau)]^2\rangle=e^{-2\lambda\tau}\langle[\Delta n(t)]^2\rangle+e^{-\lambda\tau}(1-e^{-\lambda\tau})\langle n(t)\rangle . \tag{4.6}$$

Hence we obtain the Fano factor

$$F(t+\tau)=e^{-\lambda\tau}F(t)+1-e^{-\lambda\tau} . \tag{4.7}$$

Equation (4.7) shows that the statistics of the premeasurement photon field, which is represented by $F(t)$, lose their feature as time proceeds. No matter what the initial statistics are, they approach the Poissonian. Figures 6(a) and 6(b) compare the time evolutions of the Fano factors for the NMP and RMP.

To elucidate the meaning of the nonreferring measurement, let us consider the initially number state ($\rho(0)=|n_0\rangle\langle n_0|$). The density operator after the NMP of duration τ is given by

$$\rho(\tau)=\sum_{m=0}^{n_0}\begin{bmatrix}n_0\\m\end{bmatrix}(1-e^{-\lambda\tau})^m(e^{-\lambda\tau})^{n_0-m}|n_0-m\rangle\times\langle n_0-m| . \tag{4.8}$$

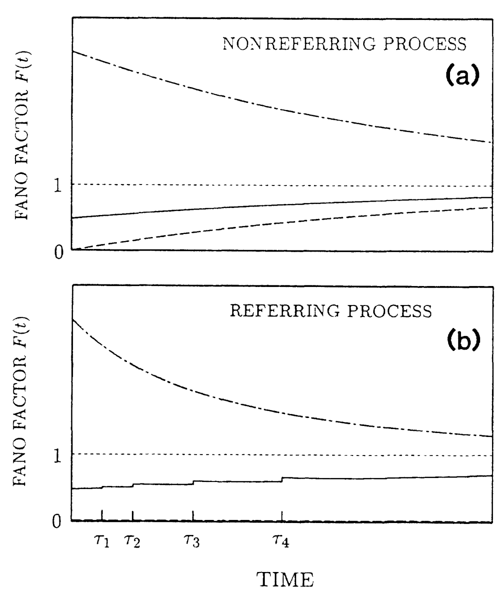

FIG. 6. Time evolution of the Fano factor $F(t)$ in the (a) nonreferring measurement process and (b) referring measurement process. Here the dot-dashed, dotted, solid, and dashed curves correspond to initially thermal, coherent, squeezed, and number states, respectively. One-count processes are assumed to occur at $\tau_1,\tau_2,\ldots$.

This can be understood as follows: our knowledge that the detector is active leads us to a conclusion that some of the initial n_0 photons can be detected by a photodetector with probability $p=1-e^{-\lambda\tau}$. The coefficient in the summand of Eq. (4.8),

$$\begin{pmatrix} n_0 \\ m \end{pmatrix} p^m(1-p)^{n_0-m} , \tag{4.9}$$

gives the probability of m out of n_0 photons being detected with probability p. However, since we do not know the number of photons that are actually detected, the density operator after this measurement process falls into a mixture of all possible numbers. Consequently, the Fano factor increases toward unity as time progresses.

$$F(\tau)=1-\exp(-\lambda\tau) . \tag{4.10}$$

In contrast, for the QPF and QPN, we do know the number of detected photons. Therefore the post-measurement density operator does not fall into a mixture but remains a pure state as in Eq. (3.1). Similar arguments hold true for other states. A unique exception is the coherent state. An initially coherent state does not fall into a mixture state but remains a coherent state (pure state). In fact, substituting $\rho(0)=|\alpha_0\rangle\langle\alpha_0|$ into Eq. (4.4), we obtain

$$\rho(\tau)=\left|\alpha_0\exp\left[-\left(i\omega+\frac{\lambda}{2}\right)\tau\right]\right\rangle \times\left\langle\alpha_0\exp\left[-\left(i\omega+\frac{\lambda}{2}\right)\tau\right]\right| . \tag{4.11}$$

This result is the same as $\rho_m^{\rm QPF}(\tau)$ in Eq. (3.8). Only for an initially coherent state is there no difference between the time evolutions of the density operator in the RMP and the NMP. For both measurement schemes the Fano factor is always unity, independent of time.

V. DISCUSSION

Most results of this paper are intuitively intelligible. For example, it is obvious that the average photon number tends to decrease by one-by-one photon counting. For a number state, the photon number does not change when no photons are detected, and it decreases by exactly 1 when one photon is detected. For the NMP, the average photon number decreases monotonically due to the absorption by the photodetector. Some results, however, seem paradoxial. For example, the average photon number can decrease even when no photons are detected, and increases in some cases even when one photon is detected. The density operator of the photon field in the NRP develops differently from that in the RMP, although the physical situation of the whole system does not appear to be different.

The apparent paradoxes are resolved if we take into account the effects of continuous measurement and its backaction on the photon field. First of all, the initial state is *given a priori* in the present analysis. This means that we *know* the initial photon statistics. For simplicity, let us assume that the initial state is $\frac{1}{2}(|0\rangle\langle 0|+|100\rangle\langle 100|)$. Then the initial average photon number is 50. When one photon is detected for this state, the possibility that the initial state was $|0\rangle\langle 0|$ suddenly vanishes. Thus we can conclude that the initial state was $|100\rangle\langle 100|$. Since one photon has been extracted by the detector just now, we conclude that the present state is $|99\rangle\langle 99|$ with the average photon number *increased* by 49. The fact that the field state is modified by the measurement can be regarded as a measurement back action. Thus, the average photon number has changed from 50 to 99 by extracting one photon from the photon field. In this way the measurement back action consists of two elements. One is that the probability of the state $|0\rangle$ vanishes and hence the probability of the state $|100\rangle$ increases. The other is that the state $|100\rangle$ transforms into $|99\rangle$ by extraction of one photon.

If no photons are detected in the RMP, then we must modify our knowledge of the initial photon statistics so that the probability of $|0\rangle\langle 0|$ is increased and that of $|100\rangle\langle 100|$ is decreased to be in accordance with the fact that we have not detected any photons for a long time. Such a modification results in a decrease in the average photon number, even though no photon has actually been detected.

Generalizing the above discussion, the density operator is modified every moment according to the results of continuous measurements. It is natural to assume that the probability of one photon being detected is proportional to the average photon number in the cavity. This is expressed by Eq. (2.2). Therefore, if the statistics have a long tail for large photon numbers like the Bose-Einstein statistics, the probability that the photon number was large increases as a result of the one-count process. This results in the increase of the average photon number. In fact, writing Eq. (2.4) in matrix elements, we have

$$\rho_{mn}(t^+)=\frac{\sqrt{(m+1)(n+1)}}{\langle n(t)\rangle}\rho_{m+1,n+1}(t) . \tag{5.1}$$

This equation shows that for the one-count process, each matrix element is shifted by one and is multiplied by a factor of $\sqrt{(m+1)(n+1)}/\langle n(t)\rangle$. This factor works as an enhancement factor for large m,n such that $(m+1)(n+1)>\langle n(t)\rangle^2$ and as a reduction factor for small m,n such that $(m+1)(n+1)<\langle n(t)\rangle^2$. If the photon-number distribution is localized around some number, the photon number is roughly determined from the beginning. Thus, we conclude that the average photon number for fields such as a number state decreases approximately by 1 in the one-count process.

A coherent state is a special example in which the increasing and decreasing contributions to the average photon number are balanced. Thus there is no discontinuity in the average photon number, even when the detector counts a photon, as is indicated in Fig. 3 (dashed curves). In fact, the matrix elements for a coherent state can be written as

$$\rho_{mn}=\frac{\alpha^m(\alpha^*)^n}{(m!n!)^{1/2}}\exp(-|\alpha|^2) , \tag{5.2}$$

which exactly satisfies Eq. (5.1) if we set $\rho_{mn}(t^+)=\rho_{mn}(t)$. Therefore, even off-diagonal elements suffer nothing from the one-count process.

All these discussions are based on the fact that the density operator is redetermined according to our knowledge about the results of measurements. The density operator (and, hence, the probability distribution for the number of photons) is a quantity that we determine according to our knowledge. The average photon number is not an actually observed value but is the photon number we expect to obtain by repeated measurements. Such an expectation value can increase when one photon is detected immediately after the observation process began and can decrease when no photons are detected. Therefore, the density operator not only expresses an objective physical state of a system but also reflects our knowledge about the system. This is further understood by considering the NMP. The density operator under the NMP expresses the knowledge about photon statistics when we know the initial state and know that photon counting is being performed, but we do not know the results of measurements. Therefore, the density operator for the NRP develops differently from that for the RMP.

It should be noted that the above discussion can be understood within the context of classical probability theory. In fact, we have used only the diagonal elements of the density operator in the number state basis. Therefore, the apparent paradox mentioned above is not restricted to quantum-mechanical situations but may also be found in some classical situations. However, the complete analysis in Sec. II is done including off-diagonal matrix elements of the density operator. Therefore, the analysis in this paper is thus fully quantum mechanical.

Finally, we wish to discuss the conservation law of the photon number in the context of continuous measurement. Suppose that the measurement process began at $t=0$ and m photons have been detected by the time $t=T$. Then the density operator immediately after $t=T$ is given by Eq. (2.33) and the average photon number of the remaining field is given as

$$\langle n(T)\rangle_m=\mathrm{Tr}[\rho_m^{\mathrm{QPN}}(T)a^\dagger a]=\exp(-\lambda T)\frac{\mathrm{Tr}[\rho(0)(a^\dagger)^{m+1}\exp(-\lambda a^\dagger aT)a^{m+1}]}{\mathrm{Tr}[\rho(0)(a^\dagger)^m\exp(-\lambda a^\dagger aT)a^m]}\ , \tag{5.3}$$

where the subscript m in $\langle n(T)\rangle_m$ indicates the number of detected photons up to time T. It is noted that $\langle n(T)\rangle_m$ is the average photon number but that m is a result of a single measurement. Since the conservation law in quantum mechanics holds true in the ensemble-average context, the sum $m+\langle n(T)\rangle_m$ does not have to be equal to the initial average photon number n_0. The conserved quantity is the ensemble average of the summation with respect to m. It can be shown the way the ensemble-averaged result certainly equals n_0:

$$\sum_{m=0}^{\infty} P(m;0,T)[m+\langle n(T)\rangle_m]=n_0\ . \tag{5.4}$$

VI. CONCLUSIONS

We have developed general formulas for continuous photodetection processes. These formulas describe nonunitary time evolution of the photon field under continuous measurement of photon number. We have shown the way the state reduces towards the vacuum state depends strongly on the initial photon statistics. In particular, we have found that the average photon number after the one-count process increases when the premeasurement state is super-Poissonian. We identify the physical origins of this effect as the vanishing probability of the vacuum state and the associated renormalization of the density operator in the one-count process. We have introduced the nonreferring measurement process, and by comparing it with the referring measurement process we have discussed the effect of discarding observable information of the system on its state reduction. We have derived general formulas which show the way the initially pure state, in general, collapses into a mixed state.

In the quantum theory of measurement process, the nonunitary process due to the measurement is usually considered to be an instantaneous process. The evolution of the state is obtained using the operation-valued measure.[17] The present analysis enables us to trace the time evolution of the photon field in real time as the state reduces towards the vacuum state. In this sense, the present analysis extends the conventional quantum theory of measurement. However, the present analysis is not a kind of measurement theory which "explains" why the wave function collapses; it is a continuation of infinitesimal nonunitary processes (one-count and no-count-processes) which are postulated.

ACKNOWLEDGMENTS

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APPENDIX A: PROOF OF EQ. (2.8)

The kth moment of the post-measurement state $\langle n(t^+)^k\rangle$ is given by

$$\langle n(t^+)^k\rangle=\mathrm{Tr}[\rho(t^+)(a^\dagger a)^k]\ . \tag{A1}$$

Substituting Eq. (2.4) into Eq. (A1) yields

$$\langle n(t^+)^k\rangle=\frac{1}{\langle n(t)\rangle}\mathrm{Tr}[\rho(t)a^\dagger(a^\dagger a)^k a]\ . \tag{A2}$$

It can be shown by mathematical induction that

$$(a^\dagger a)^k a = \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix} (-1)^{k-m} a (a^\dagger a)^m . \quad \text{(A3)}$$

Substituting this into Eq. (A2) yields

$$\langle n(t^+)^k \rangle = \frac{1}{\langle n(t) \rangle} \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix} (-1)^{k-m} \langle n(t)^{m+1} \rangle , \quad \text{(A4)}$$

where

$$\langle n(t)^{m+1} \rangle = \mathrm{Tr}[\rho(t)(a^\dagger a)^{m+1}] . \quad \text{(A5)}$$

APPENDIX B: PROOF OF EQS. (3.23) AND (3.25)

Substituting (3.21) into the numerator of the rhs of (2.22) yields

$$\exp(-i\Omega a^\dagger a\tau) a^m \rho(0) (a^\dagger)^m \exp(i\Omega^* a^\dagger a \tau)$$
$$= r \frac{\exp[|\alpha|^2(\tanh r - 1)]}{\cosh r} \sum_{k,n=m}^{\infty} \left[\frac{\tanh r}{2} \right]^{(k+n)/2} \frac{1}{(k!n!)^{1/2}} H_k(y) H_n(y)$$
$$\times \exp[-i\Omega(k-m)\tau + i\Omega^*(n-m)\tau] \left[\frac{k!n!}{(k-m)!(n-m)!} \right]^{1/2} |k-m\rangle\langle n-m| , \quad \text{(B1)}$$

where $\Omega \equiv \omega - i\lambda/2$ and $y \equiv \alpha/\sqrt{\sinh 2r}$. Taking the trace of this equation, we have

$$\mathrm{Tr}[\exp(-i\Omega a^\dagger a\tau) a^m \rho(0) (a^\dagger)^m \exp(i\Omega^* a^\dagger a \tau)] = \frac{\exp[|\alpha|^2(\tanh r - 1)]}{\cosh r} \exp(m\lambda\tau) z^m \frac{d^m}{dz^m} \sum_{k=0}^{\infty} \frac{z^k}{2^k k!} H_k^2(y) , \quad \text{(B2)}$$

with $z \equiv e^{-\lambda\tau}\tanh r$. With the aid of the identity[18]

$$\sum_{k=0}^{\infty} \frac{z^k}{2^k k!} H_k^2(y) = \exp\left[\frac{2x}{1+z} y^2 \right] \Big/ (1-z^2)^{1/2} , \quad \text{(B3)}$$

we finally obtain

$$\mathrm{Tr}[\exp(-i\Omega a^\dagger a\tau) a^m \rho(0) (a^\dagger)^m \exp(i\Omega^* a^\dagger a \tau)] = \frac{[\exp|\alpha|^2(\tanh r - 1)]}{\cosh r} (\tanh r)^m \frac{d^m}{dz^m} \{ \exp[(2z/1+z)y^2]/(1-z^2)^{1/2} \} . \quad \text{(B4)}$$

Substituting Eqs. (B1) and (B4) into the rhs of Eq. (2.22), we obtain Eq. (3.23). Substituting Eqs. (B1) and (B4) into Eq. (2.23), we obtain Eq. (3.25).

APPENDIX C: STATE EVOLUTION IN NONREFERRING MEASUREMENT PROCESS

In this appendix, we derive Eq. (4.4), which describes the field-density-operator evolution under a nonreferring measurement process, using a different method from a previous work.[7] In a nonreferring measurement process, whether a one-count process or a no-count process occurred is not referred, although one or the other certainly occurred within the interval from t to $t+dt$. Therefore, $\rho(t+dt)$ in the nonreferring process is equal to the statistical summation of $\rho(t+dt)$'s that evolved from $\rho(t)$ by the one-count process and no-count process as

$$\rho(t+dt) = P(J)dt \frac{J\rho(t)}{\mathrm{Tr}[J\rho(t)]} + P(S_{dt}) \frac{S_{dt}\rho(t)}{\mathrm{Tr}[S_{dt}\rho(t)]} , \quad \text{(C1)}$$

where $\rho(t)$ is the density operator which has evolved from the initial state $\rho(0)$ under a nonreferring process. Using the formulas for $P(J)$, $J\rho$, $P(S_{dt})$, and $S_{dt}\rho$ in the text, Eq. (C1) is rewritten as

$$\rho(t+dt) = \rho(t) + \lambda dt\, a\rho(t)a^\dagger - \left[i\omega + \frac{\lambda}{2} \right] dt\, a^\dagger a \rho(t)$$
$$+ \left[i\omega - \frac{\lambda}{2} \right] dt\, \rho(t) a^\dagger a . \quad \text{(C2)}$$

This leads to a differential equation

$$\frac{d\rho}{dt} = \lambda a \rho a^\dagger - \left[i\omega + \frac{\lambda}{2} \right] a^\dagger a \rho + \left[i\omega - \frac{\lambda}{2} \right] \rho a^\dagger a . \quad \text{(C3)}$$

Next, we solve this differential equation. By analogy with the usual interaction picture, we write ρ as

$$\rho(t) = \exp\left[-\left[i\omega + \frac{\lambda}{2} \right] a^\dagger a t \right] \tilde{\rho}(t)$$
$$\times \exp\left[\left[i\omega - \frac{\lambda}{2} \right] a^\dagger a t \right] . \quad \text{(C4)}$$

Substituting this into Eq. (C3), we obtain

$$\frac{d\tilde{\rho}}{dt}=\lambda\exp(-\lambda t)a\tilde{\rho}a^{\dagger}\ . \tag{C5}$$

Here, the well-known formulas

$$\exp(xa^{\dagger}a)a\exp(-xa^{\dagger}a)=a\exp(-x)$$

and

$$\exp(xa^{\dagger}a)a^{\dagger}\exp(-xa^{\dagger}a)=a\exp(x)$$

are used. Equation (C5) means that $\tilde{\rho}$ is expressed by $a\tilde{\rho}a^{\dagger}$ of an infinitesimal time before. This implies that the solution can be written as

$$\tilde{\rho}(t)=\sum_{k=0}^{\infty}C_k(t)a^k\tilde{\rho}(0)(a^{\dagger})^k\ . \tag{C6}$$

Substituting this into Eq. (C5), we obtain a differential-difference equation for $C_k(t)$ as

$$dC_k(t)/dt=\begin{cases}\lambda e^{-\lambda t}C_{k-1}(t) & \text{for } k\geq 1\\ 0 & \text{for } k=0\end{cases} \tag{C7}$$

with the initial condition $C_k(0)=\delta_{0k}$. It is straightforward to show that the solution for this equation is given by

$$C_k(t)=\frac{[1-\exp(-\lambda t)]^k}{k!}\ . \tag{C8}$$

Combining Eqs. (C4), (C5), (C6), and (C8), we obtain Eq. (4.4).

APPENDIX D: PHOTON NUMBER MOMENTS IN NONREFERRING PROCESS

The kth moment of the photon number $\langle n(t+\tau)^k\rangle$ after the nonreferring measurement time τ is given using Eq. (4.4) as

$$\langle n(t+\tau)^k\rangle=\sum_{m=0}^{\infty}\frac{[1-\exp(-\lambda\tau)]^m}{m!}\mathrm{Tr}[\exp(-i\Omega a^{\dagger}a\tau)a^m\rho(t)(a^{\dagger})^m\exp(i\Omega^* a^{\dagger}a\tau)(a^{\dagger}a)^k]\ , \tag{D1}$$

where $\Omega=\omega-i\lambda/2$. With the aid of the operator identity $(a^{\dagger}a)^k a^m=a^m(n-m)^k$ with n being the eigenvalue of $a^{\dagger}a$, Eq. (D1) is rewritten as

$$\begin{aligned}\langle[n(t+\tau)]^k\rangle&=\sum_{m=0}^{\infty}\frac{[1-\exp(-\lambda\tau)]^m}{m!}\exp(m\lambda\tau)\sum_{j=0}^{\infty}\begin{pmatrix}j\\ j-m\end{pmatrix}(j-m)^k\exp(-j\lambda\tau)\\ &=\frac{\partial^k}{\partial y^k}\sum_{m=0}^{\infty}\rho_{mm}(t)[1-\exp(-\lambda\tau)+\exp(y)]^m\Big|_{y=-\lambda\tau}\ .\end{aligned} \tag{D2}$$

This is a general expression for $k=1,2,\ldots$. The first two moments result in Eqs. (4.5) and (4.6).

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