Nonequilibrium open-system theory for continuous photodetection processes: A probability-density-functional description

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A nonequilibrium open-system theory for continuous photodetection processes is developed using a probability-density functional combined with the path-integral technique. Nonunitary time evolution of the system-environment density operator due to measurement back action and continuous measurement is exactly described for three familiar photodetection processes. New exact photocounting formulas are obtained for these three processes; in particular, Mollow's photocounting formula is generalized for a nonequilibrium open system. Distinctive features between a closed system and an open system are presented; for example, the probability distribution for the number of counts is shown to contain complete information about photoelectron statistics for a closed system, whereas it does not for an open system. The obtained formulas are applied to photodetection of a single-mode photon field that interacts linearly with a pump source consisting of a single harmonic oscillator and a reservoir consisting of an infinite number of harmonic oscillators. In the former case, increasing the ratio of the source-field coupling constant to the field-detector coupling constant causes the photon field to cross over from a closed attenuating field to an open stationary field. In the latter case, a quantum-mechanical fluctuation-dissipation theorem in the continuous photodetection context is discussed.

I. INTRODUCTION

Photoelectric phenomena have revealed several unique features of quantum mechanics. Einstein[1] found that light behaves like particles during photoelectric conversion, while Hanbury Brown and Twiss[2] demonstrated that the wave nature of photons is still preserved in photoelectric correlation. New phenomena such as photon bunching[3,4] and antibunching[5,6] were found from experiments of this type. Photoelectric correlation experiments have also served as an experimental means of testing the fundamentals of quantum mechanics, such as Bell's inequality,[7] quantum jumps,[8–11] nonlocal effects,[12] and photon localizability.[13–15] There are a number of theoretical articles which describe how to extract information concerning the original photon statistics from the observed photoelectron statistics.[16–20] Recently the quantum photodetection process has aroused renewed interest from measurement-theoretical points of view. That is, the photon field experiences state reduction caused by measurement back action[21] and it evolves nonunitarily as a result of continuous measurement of photon number.[22–26] This paper discusses the last two effects in the context of a nonequilibrium open system.

Two intrinsic quantum-mechanical effects should be considered in continuous photodetection processes. One is the uncertainty in photoelectric conversion which is ascribed to the essentially statistical nature of light and is represented by the density operator of the photon field. If we have only to consider this effect, we can adopt a conventional picture of photodetection processes, i.e., unitary evolution of the total density operator, followed by von Neumann's projection postulate[27] or projection using an operation-valued measure. This picture correctly incorporates the effect of state reduction caused by measurement back action, and is used in a number of articles.[21,28–31] The other effect arises from the fact that a photodetector performs continuous measurement; we can, in principle, read out the information concerning registration of a photocount in *real time* throughout a measurement period. The readout information at each time indicates that the photon field has experienced either *discontinuous* or *continuous* collapse, according to whether or not the detector registers a photocount.[25,32] The photon field thus develops nonunitarily in two different ways depending on the real-time readout information concerning the registration of a photocount. The crucial observation here is that the time-developed new density operator at an infinitesimally later time determines the photocount probability density at that time, but that, whether or not photoelectric conversion actually occurs is again uncertain owing to the essentially quantum-statistical nature of light. This situation is schematically illustrated in Fig. 1 for a nonequilibrium open system. The conventional picture imposes quantum-mechanical state reduction only at the end of the measurement process, while continuous measurement imposes state reduction throughout the measurement period. Therefore, to give complete quantum-statistical information about photoelectron statistics, we must have some functional which reflects a continuously infinite number of collapses of the photon field.

With this in mind, we applied a new concept, which we define as the *probability-density-functional*,[33] to develop a new framework for continuous photodetection process-

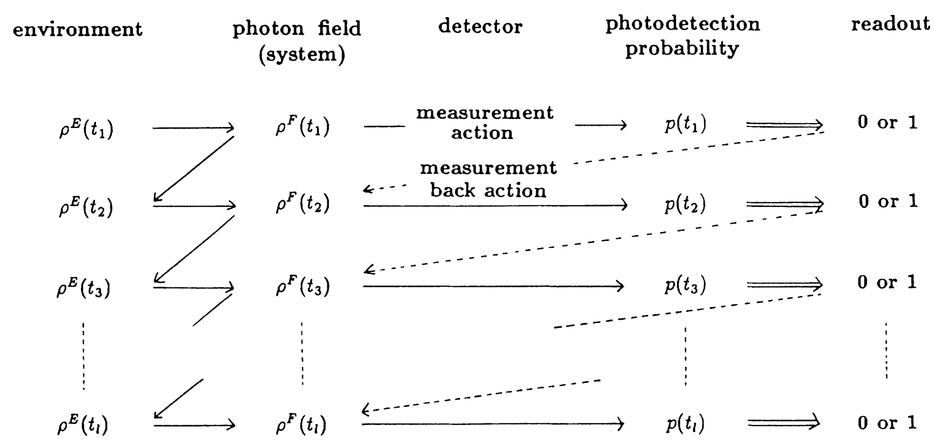

FIG. 1. Schematic illustration of nonunitary state evolution of the photon field in an open-system-theory context. The photodetection probability at a time is determined by the density operator of the photon field at the same time (measurement action), while an actual readout (i.e., information concerning the no-count or one-count event) produces the density operator of the photon field at an infinitesimally later time via nonunitary state reduction (measurement back action). Throughout the measurement period the photon field is regularly pumped or randomly disturbed by its environment (a pump source or a reservoir).

es[25] in which the probability-density functional combined with the path-integral technique replaces the conventional picture of unitary evolution of the total density operator followed by von Neumann's projection postulate.[27] It is demonstrated that the probability-density functional combined with the path-integral technique serves as a fundamental mathematical tool for the description of continuous photodetection processes; the probability-density functional incorporates the effects of a continuously infinite number of collapses of the photon field due to continuous measurements and gives complete information concerning photoelectron statistics.[25] In Ref. 25, however, the author treated a closed system in which the photon field only interacts with the photodetector. The present paper generalizes the framework to a nonequilibrium open system where, while being measured by a detector, the photon field interacts linearly with its environment, which may be a pump source or a reservoir. New exact formulas are obtained for both photocounting distributions and photon density operators immediately after the three familiar measurement processes. In particular, Mollow's photocounting formula is generalized for a nonequilibrium open system. The obtained formulas are applied to two typical nonequilibrium open-system problems: For a single-mode photon field being pumped by a single-harmonic-oscillator source, it is shown that increasing the ratio of the field-source coupling constant to the field-detector coupling constant causes the photon field to cross over from a closed attenuating field to an open stationary field. For a photon field interacting with a reservoir consisting of an infinite number of harmonic oscillators, a quantum-mechanical fluctuation-dissipation theorem is discussed in a continuous photodetection context.

This paper is organized as follows. Section II briefly reviews a probability-density-functional description of a random-point process, since it will serve as a fundamental mathematical tool for the theory of continuous measurement. This section derives a new expression for the probability-density functional which establishes that the probability-density functional gives the probability density for the real-time change or the "path" of photoelectric probability. Using the obtained expression, the relationship of our description to a standard theory of a random-point process is delineated. Section III develops a general formalism for a nonequilibrium open-system theory of continuous photodetection processes. A model is employed in which the photon field interacts linearly with an environment of harmonic oscillators. New exact formulas are obtained for both photocounting distributions and corresponding nonunitary time evolution of the system-environment density operator. The following two sections are devoted to applications of the obtained formulas to two typical nonequilibrium open-system problems. Section IV applies the obtained formulas to a case in which the photon field is pumped by a single-harmonic-oscillator source. It is shown that, when the field-detector coupling constant λ is much larger than the source-field coupling constant g ($\lambda \gg g$), the detector sees a photon-field correlation, while in the opposite limit ($g \gg \lambda$) the detector sees a source correlation. This observation is crucial when we prepare various kinds of quantum states such as a single-photon number state. Section V applies the obtained formulas to a case in which the photon field interacts with a reservoir consisting of an infinite number of harmonic oscillators. Finally, the obtained results are discussed and summarized in Sec. VI.

II. PROBABILITY-DENSITY FUNCTIONAL AND ITS FUNDAMENTAL PROPERTIES

This section defines the probability-density functional and discusses its fundamental properties because it serves as a fundamental mathematical tool for the description of continuous photodetection processes. A new expression for the probability-density functional is obtained. This expression clarifies the significance of the probability-density functional as the probability-density "path" and its relationship to a standard theory of a random-point process.

A. Definition of the probability-density functional

Suppose that observation begins at time t_a and ends at time t_b. Let this time interval be divided into l subintervals $[t_1=t_a,t_2)$, $[t_2,t_3),\ldots,[t_l,t_{l+1}=t_b]$, and let $\Delta_{\max}\equiv\max\{\Delta t_1,\Delta t_2,\ldots,\Delta t_l\}$ be made so small that the probability of more than one count being registered in any subinterval is negligible, where $\Delta t_j\equiv t_{j+1}-t_j$ $(j=1,2,\ldots,l)$. The probability-density function $p(t)$ is defined to give the probability $p(t_j)\Delta t_j$ of a count being registered in $[t_j,t_j+\Delta t_j)$. If $p(t)$ is a unique function of time, our information about photoelectron statistics is complete if all $p(t_1),p(t_2),\ldots,p(t_l)$ are known to us. However, in the precess of photoelectric conversion, $p(t)$ itself is a random function; we can get only the probability distribution for values which $p(t)$ takes on at each time. Therefore the best information we can get is the l-fold joint probability distribution

$$\mathcal{P}[p(t_1),p(t_2),\ldots,p(t_l)]\prod_{j=1}^{l}dp(t_j)$$

which means that the probability of a count being registered in $[t_j,t_j+\Delta t_j)$ $(j=1,2,\ldots,l)$ takes on values between $p(t_j)\Delta t_j$ and $[p(t_j)+dp(t_j)]\Delta t_j$. We observe that this joint probability distribution gives the probability distribution for the probability-density function $p(t)$. This function represents the uncertainty of photoelectric conversion which reflects the essentially quantum-statistical nature of light, while the probability distribution for $p(t)$ represents fluctuations of the density operator of the photon field due to measurement back action and interaction with the environment. Here we assume unit quantum efficiency for simplicity. When $\Delta_{\max}$ is infinitesimal, complete information is given by

$$\mathcal{P}[p(t_1),p(t_2),\ldots,p(t_l)]\prod_{j=1}^{l}dp(t_j)$$

with $l\to\infty$.[34] We denote complete information as

$$\mathcal{P}[p(c)]\delta p(c)\equiv\lim_{l\to\infty}\mathcal{P}[p(t_1),p(t_2),\ldots,p(t_l)]\times\prod_{j=1}^{l}dp(t_j)\,, \tag{1}$$

where $p(c)$ denotes the whole curve of $p(t)$ for $t\in[t_a,t_b)$. Figure 2 illustrates two examples of the probability-density path for initially sub-Poissonian and super-Poissonian states, where t_j $(j=1,2,3)$ denote the times at which single photons were registered. Note that for an initially super-Poissonian state the probability-density function *increases* by our extracting one photon from the photon field.[26] This Hanbury Brown-Twiss–like effect is a consequence of nonunitary state reduction described in Sec. III. We call $\mathcal{P}[p(c)]$ the *probability-density* func-

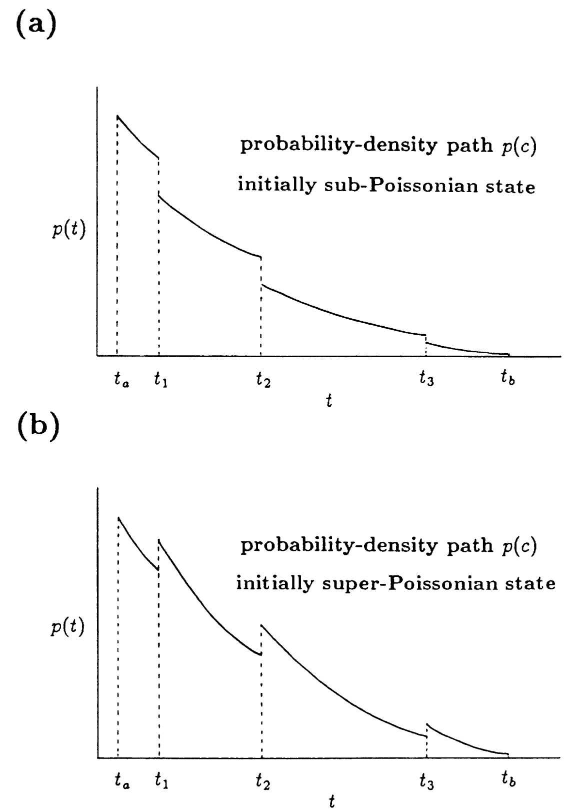

FIG. 2. Distinction between $p(c)$ and $p(t)$: $p(c)$ denotes the whole curve or the "path" of the probability-density function $p(t)$ for $t\in[t_a,t_b)$. A probability-density path typical of an initially sub-Poissonian state (a) and of an initially super-Poissonian state (b).

tional. The probability-density functional gives the probability density for the "path"; that is, $\mathcal{P}[p(c)]\delta p(c)$ gives the probability that the path $p(c)$ lies in the hatched region in Fig. 3. Thus, we find that the probability-density functional gives complete information concerning photoelectron statistics. Once we find the probability-density functional, the ensemble average of an arbitrary time-

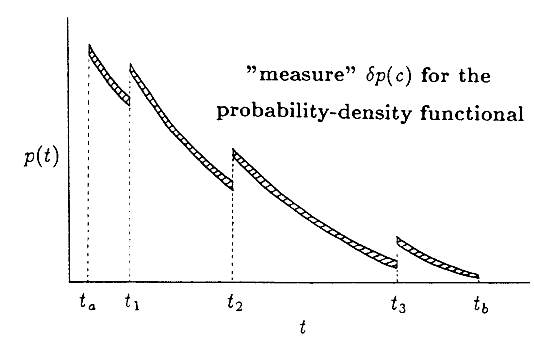

FIG. 3. Probability-density functional as the probability density for the "path" $p(c)$: $\mathrm{P}[p(c)]\delta p(c)$ gives the probability that a path $p(c)$ lies in the hatched region or "measure" $\delta p(c)$.

distributed quantity can be obtained by our carrying out the path integration of the quantity multiplied by the probability-density functional.[33] For this purpose it is necessary and sufficient to give the expressions of the probability-density functional $\mathcal{P}[p(c)]$ and its measure $\delta p(c)$ separately. They are given by[33]

$$\mathcal{P}[p(c)]=\lim_{l\to\infty}\mathcal{P}[p(t_1),p(t_2),\ldots,p(t_l)]\prod_{j=1}^{l}(\Delta t_j)^{-1/2}\ , \tag{2}$$

and

$$\delta p(c)=\lim_{l\to\infty}\prod_{j=1}^{l}dp(t_j)(\Delta t_j)^{1/2}\ . \tag{3}$$

B. Relations of the probability-density functional to three familiar joint probability distributions

The probability distribution of multicoincidence (PDM), $\lambda(t_1,t_2,\ldots,t_m;t_a,t_b)$, gives the probability density per (unit time)^m, where one count is registered at m distinct times $t_1,t_2,\ldots,t_m$ with no further conditions imposed. Here t_a and t_b denote the start and end of the observation time, respectively. The PDM can be expressed in terms of the probability-density functional as

$$\lambda(t_1,t_2,\ldots,t_m;t_a,t_b)=\iint p(t_1)p(t_2)\ldots p(t_m)\mathcal{P}[p(c)]\delta p(c)\ , \tag{4}$$

where the symbol $\iint$ denotes the path integration.[35,36] The generating functional of the PDM, $G^{(1)}[u(c)]$, is defined as

$$G^{(1)}[u(c)]\equiv\sum_{m=0}^{\infty}\frac{i^m}{m!}\int_{t_a}^{t_b}dt_1\int_{t_a}^{t_b}dt_2\cdots\int_{t_a}^{t_b}dt_m\,\lambda(t_1,t_2,\ldots,t_m;t_a,t_b)u(t_1)u(t_2)\ldots u(t_m)\ . \tag{5}$$

The PDM is obtained by functional differentiation of $G^{(1)}[u(c)]$:

$$\lambda(t_1,t_2,\ldots,t_m;t_a,t_b)=\frac{1}{i^m}\frac{\delta^m G^{(1)}}{\delta u(t_1)\delta u(t_2)\ldots\delta u(t_m)}\bigg|_{u=0} \tag{6}$$

An expression of $G^{(1)}[u(c)]$ in terms of the probability-density functional is obtained if we substitute $\lambda(t_1,t_2,\ldots,t_m;t_a,t_b)$ from Eq. (4) into the right-hand side (rhs) of Eq. (5):

$$G^{(1)}[u(c)]=\iint\exp\left[i\int_{t_a}^{t_b}dt\,p(t)u(t)\right]\mathcal{P}[p(c)]\delta p(c)\ . \tag{7}$$

From this equation we find that the probability-density functional is the generalized inverse Fourier transform of the generating function of the PDM. That is,

$$\mathcal{P}[p(c)]=\iint\exp\left[-i\int_{t_a}^{t_b}dt\,p(t)u(t)\right]\times G^{(1)}[u(c)]\delta(u(c)/2\pi)\ . \tag{8}$$

Thus we find that complete statistical properties of photoelectrons can be obtained from the set of all PDM's.

The probability distribution of forward recurrence times (PDF) or that of waiting times, $P_m^{(\mathrm{forward})}(t_1,t_2,\ldots,t_m;t_a,t_b)$, gives the probability density per (unit time)^m, where one count is registered at m distinct times $t_1,t_2,\ldots,t_m$ and no counts are registered in between. This can be expressed in terms of the probability-density functional as

$$P_m^{(\mathrm{forward})}(t_1,t_2,\ldots,t_m;t_a,t_b)=\iint p(t_1)p(t_2)\ldots p(t_m)\exp\left[-\int_{t_a}^{t_b}dt\,p(t)\right]\times\mathcal{P}[p(c)]\delta p(c)\ . \tag{9}$$

The generating functional of the PDF, $G^{(2)}[u(c)]$, is defined as

$$G^{(2)}[u(c)]\equiv\sum_{m=0}^{\infty}\frac{i^m}{m!}\int_{t_a}^{t_b}dt_1\int_{t_a}^{t_b}dt_2\cdots\int_{t_a}^{t_b}dt_m\,P_m^{(\mathrm{forward})}(t_1,t_2,\ldots,t_m;t_a,t_b)u(t_1)u(t_2)\ldots u(t_m)\ . \tag{10}$$

The PDF is obtained by functional differentiation of $G^{(2)}[u(c)]$:

$$P_m^{(\mathrm{forward})}(t_1,t_2,\ldots,t_m;t_a,t_b)=\frac{1}{i^m}\frac{\delta^m G^{(2)}[u(c)]}{\delta u(t_1)\delta u(t_2)\ldots\delta u(t_m)}\bigg|_{u=0}\ . \tag{11}$$

An expression of $G^{(2)}[u(c)]$ in terms of the probability-density functional is obtained if we substitute $P_m^{(\mathrm{forward})}(t_1,t_2,\ldots,t_m;t_a,t_b)$ from Eq. (9) into the rhs of Eq. (10):

$$G^{(2)}[u(c)]=\iint \exp\left[i\int_{t_a}^{t_b} dt\, p(t)[u(t)+i]\right]\mathcal{P}[p(c)]\delta p(c)\ . \tag{12}$$

This equation can be inverted to give

$$\mathcal{P}[p(c)]=\iint \exp\left[-i\int_{t_a}^{t_b} dt\, p(t)[u(t)+i]\right]G^{(2)}[u(c)]\delta(u(c)/2\pi)\ . \tag{13}$$

Thus we find that complete statistical properties of photoelectrons can be obtained from the set comprising all the PDF's. Moreover, we find that the set comprising all the PDM's has the same (complete) amount of information about photoelectron statistics as all the PDF's. In fact, from Eqs. (7) and (12), the following equation holds:

$$G^{(1)}[u(c)]=G^{(2)}[u(c)-i]\ . \tag{14}$$

The probability distribution for the number of counts (PDN), $P(m;t_a,t_b)$, gives the probability of m counts being registered between t_a and t_b. This can be expressed in terms of the probability-density functional as

$$P(m;t_a,t_b)=\iint [(\int_{t_a}^{t_b} dt\, p(t))^m/m!]\exp\left[-\int_{t_a}^{t_b} dt\, p(t)\right]\mathcal{P}[p(c)]\delta p(c)\ . \tag{15}$$

The generating functional of the PDF, $G^{(3)}[u(c)]$, is defined as

$$G^{(3)}[u(c)]\equiv \sum_{m=0}^{\infty}\frac{i^m}{m!}\int_{t_a}^{t_b}dt_1\int_{t_a}^{t_b}dt_2\cdots\int_{t_a}^{t_b}dt_m P\left[m;t_a,\frac{t_1+t_2+\cdots+t_m}{m}\right]u(t_1)u(t_2)\ldots u(t_m)\ . \tag{16}$$

The PDN is obtained by functional differentiation of $G^{(3)}[u(c)]$:

$$P(m;t_a,t)=\frac{1}{i^m}\frac{\delta^m G^{(3)}[u(c)]}{\delta u(t)^m}\Bigg|_{u=0}\ . \tag{17}$$

An expression of $G^{(3)}[u(c)]$ in terms of the probability-density functional is obtained if we substitute $P(m;t_a,(t_1+t_2+\cdots+t_m)/m)$ from Eq. (15) into the rhs of Eq. (16):

$$\begin{aligned}G^{(3)}[u(c)]=\sum_{m=0}^{\infty}\frac{i^m}{m!}\int_{t_a}^{t_b}dt_1\int_{t_a}^{t_b}dt_2\cdots\int_{t_a}^{t_b}dt_m u(t_1)u(t_2)\ldots u(t_m)\\ \times\iint\left[\left[\int_{t_a}^{(t_1+t_2+\cdots+t_m)/m}dt\,p(t)\right]^m/m!\right]\\ \times\exp\left[-\int_{t_a}^{(t_1+t_2+\cdots+t_m)/m}dt\,p(t)\right]\mathcal{P}[p(c)]\delta p(c)\ .\end{aligned} \tag{18}$$

In contrast to the PDM and the PDF, this equation cannot be inverted because the rhs includes essentially only two independent time parameters t_a and $(t_1+t_2+\cdots+t_m)/m$, i.e., information concerning internal times of photocount registration is averaged out. Therefore only the first two distributions of the PDM and PDF can be obtained from the PDN.[37] The interrelations between the three joint probability distributions, their generating functionals, and the probability-density functional are summarized schematically in Fig. 4. The solid arrows show the existence of general expressions and the dashed arrows show where only special expressions exist.[37]

C. An expression of the probability-density functional: relationship to the standard theory of a random-point process

This section finishes by deriving a new explicit δ-functional-type expression of the probability-density functional because this expression clarifies the meaning of the probability-density functional. Let us start by noting the relation

$$\lambda(t_1,t_2,\ldots,t_m;t_a,t_b)=\langle p(t_1)p(t_2)\ldots p(t_m)\rangle\ , \tag{19}$$

where the angle brackets denote the process of ensemble averaging. Substituting Eq. (19) into the rhs of Eq. (5) yields

$$G^{(1)}[u(c)]=\left\langle \exp\left[i\int_{t_a}^{t_b}dt\,p(t)u(t)\right]\right\rangle\ . \tag{20}$$

An explicit expression of the probability-density functional is obtained if we substitute $G^{(1)}[u(c)]$ from Eq. (20) into the rhs of Eq. (8):

$$\mathcal{P}[p(c)]=\langle\delta[p(c)-p'(c)]\rangle p'\ , \tag{21}$$

where $\delta[p(c)]$ is the generalized δ function or the δ functional defined by

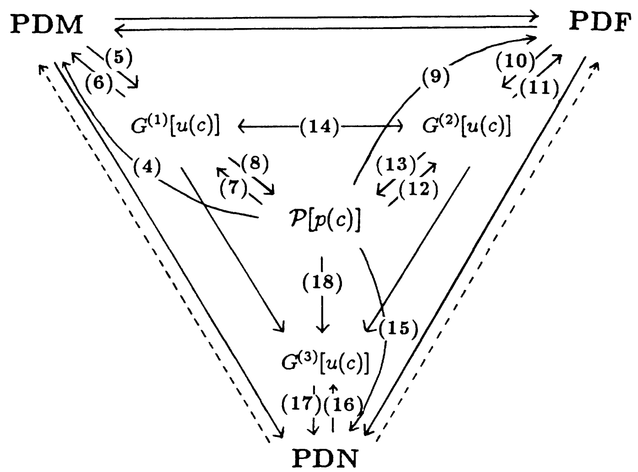

FIG. 4. Various levels of "information triangle" concerning a random-point process and interrelations between three familiar joint probability distributions (PDM, PDF, and PDN), their generating functionals ($G^{(1)}[u(c)], G^{(2)}[u(c)], G^{(3)}[u(c)]$), and the probability-density functional. The solid arrows indicate the existence of a general relationship where the figures indicate the equation numbers in the text. The remaining interrelations are shown in Ref. 33.

$$\delta[p(c)] = \iint \exp\left[\pm i \int_{t_a}^{t_b} dt\, p(t) u(t)\right] \delta\left[\frac{u(c)}{2\pi}\right], \tag{22}$$

and the subscript p' denotes that ensemble averaging is carried out with respect to $p'(c)$. Equation (21) clearly shows that the probability-density functional gives the probability density for the "path" $p(c)$.

The expression (21) is reminiscent of the standard description of a random-point process.[38-40] According to the standard theory, the joint probability density of a random function $\xi(t)$ is given by

$$\omega_m(\xi_1,\ldots,\xi_m;t_1,\ldots,t_m) = \langle \delta(\xi_1-\xi(t_1)) \cdots \delta(\xi_m-\xi(t_m)) \rangle, \tag{23}$$

where $\xi_1,\ldots,\xi_m$ are a set of realizations of random function $\xi(t)$ at times $t_1,\ldots,t_m$, and the angle brackets denote the process of ensemble averaging with respect to $\xi(t_j)$ ($j=1,2,\ldots,m$). Equation (21) shows that the probability-density functional is a generalization of Eq. (23) to a continuously infinite-time-point expression. The joint probability density $\omega_m(\xi_1,\ldots,\xi_m;t_1,\ldots,t_m)$ satisfies the normalization condition

$$\int d\xi_1 \cdots \int d\xi_m \omega_m(\xi_1\ldots,\xi_m;t_1,\ldots,t_m) = 1 . \tag{24}$$

The corresponding expression for the probability-density functional reads as

$$\iint \mathcal{P}[p(c)]\delta p(c) = 1 . \tag{25}$$

The ensemble average of a function of $f(\xi_1,\ldots,\xi_r)$ can be obtained by multiple integration of this function multiplied by the joint probability density. That is,

$$\langle f(\xi_1,\ldots,\xi_m)\rangle = \int d\xi_1 \cdots \int d\xi_m f(\xi_1,\ldots,\xi_m) \times \omega_m(\xi_1,\ldots,\xi_m;t_1,\ldots,t_m) . \tag{26}$$

For continuous measurement, a measurable quantity like the number of photocounts in a time interval is a function of the probability-density "path" $p(c)$. In general, such a time-distributed quantity can be denoted as $f[p(c)]$. Therefore the ensemble average and generating (or characteristic) functional of a quantity distributed in time can be defined in the most natural way using the probability-density functional. The process of ensemble averaging of a probability function, $f[p(c)]$, is equivalent to that of path integration of this quantity multiplied by the probability-density functional. Thus we obtain a basic formula

$$\langle f[p(c)]\rangle = \iint f[p(c)]\mathcal{P}[p(c)]\delta p(c) . \tag{27}$$

This formula reduces to Eq. (26) when the relevant quantity is distributed only at discrete times. Equations such as Eqs. (4) and (7) are special applications of this general formula.

III. GENERAL FORMALISM FOR A NONEQUILIBRIUM OPEN-SYSTEM THEORY OF CONTINUOUS PHOTODETECTION PROCESSES

This section develops a general formalism for a nonequilibrium open-system theory of continuous photodetection processes, and derives new exact formulas for both photocounting probability distributions and photon field (system)—environment density operators immediately after three different measurement processes.

A. An exact mathematical model of the continuous photodetection process for a nonequilibrium open system

Our model consists of a photon field (system, Hamiltonian H_F), its environment (Hamiltonian H_E), and a photodetector (see Fig. 5). The interaction between the photon field and its environment is represented by H_I. The environment may be a pump source or a reservoir. The interaction between the photon field and the photo-

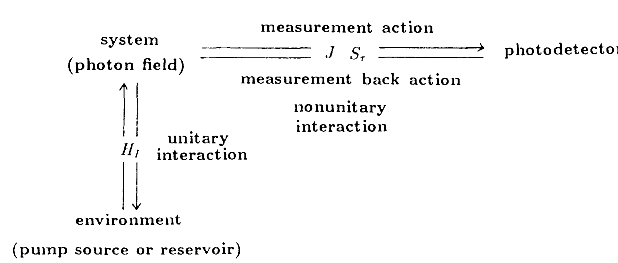

FIG. 5. Schematic diagram of the open-system photodetection model: The system (photon field) is interacting with its environment (pump source or reservoir) via a unitary evolution while continuous measurement of photon number is being performed by a photodetector.

detector is characterized by two operators J and S_τ,[22] which describe the one-count and no-count processes, respectively. These two operators describe a continuous photodetection process and give both the result of measurement (photoelectron statistics) and the system-environment density operator immediately after the measurement process.

Let $\rho(t)$ be the system-environment density operator at time t. Then the operator J gives the probability

$$\mathrm{Tr}[J\rho(t)]dt \tag{28}$$

of one count being registered in $[t,t+dt)$, and gives the density operator immediately after the measurement process as

$$\rho(t^+)=\frac{J\rho(t)}{\mathrm{Tr}[J\rho(t)]}, \tag{29}$$

where the symbol t^+ denotes a time infinitesimally later than t. Since such a *one-count process* annihilates one photon in the photon field, it is natural to assume that

$$J\rho(t)=\lambda a\rho(t)a^\dagger , \tag{30}$$

where a and $a^\dagger$ are the annihilation and creation operators of the photon field, and λ denotes the reciprocal expectation value of waiting times[22] and is therefore real.

The operator S_τ gives the probability

$$\mathrm{Tr}[S_\tau\rho(t)] \tag{31}$$

of no counts being registered in $[t,t+\tau)$, and gives the density operator immediately after the measurement process as

$$\rho(t+\tau)=\frac{S_\tau\rho(t)}{\mathrm{Tr}[S_\tau\rho(t)]} . \tag{32}$$

The action of the operator S_τ is determined as follows. First, it is postulated that

$$S_\tau\rho(t)=e^{Y\tau}\rho(t)e^{Y^\dagger\tau} . \tag{33}$$

Since we are considering a regular-point process[41] in which the no-count and one-count processes form an exclusive exhaustive set of events occurring in any infinitesimal time interval, it follows from Eqs. (30) and (33) that

$$\mathrm{Tr}[J\rho(t)+Y\rho(t)+\rho(t)Y]=0 . \tag{34}$$

When the detector is switched off ($\lambda=0$), the photon field must evolve in time according to the Hamiltonian $H_E+H_F+H_I$ of the photon field and its environment. Therefore, Eq. (34) is satisfied if the operator Y assumes the form

$$Y=-\frac{i}{\hbar}(H_E+H_F+H_I)-\frac{\lambda}{2}a^\dagger a . \tag{35}$$

The operator $N_\tau(m)$ gives the probability

$$\mathrm{Tr}[N_\tau(m)\rho(0)] \tag{36}$$

of m counts being registered in an interval $[0,\tau)$, and gives the density operator immediately after the measurement process as

$$\rho_m^{\mathrm{QPN}}(\tau)=\frac{N_\tau(m)\rho(0)}{\mathrm{Tr}[N_\tau(m)\rho(0)]} . \tag{37}$$

Since such an *m-count process* annihilates m photons in the photon field but does not register the times at which they are annihilated, we have

$$N_\tau(m)=\int_0^\tau dt_m\int_0^{t_m}dt_{m-1}\cdots \times\int_0^{t_2}dt_1 S_{\tau-t_m}JS_{t_m-t_{m-1}}\cdots JS_{t_1} . \tag{38}$$

The operator T_τ is defined to describe a time evolution of the system-environment density operator when the detector is performing a continuous measurement in $[t,t+\tau)$ without any reference to its results. Since such a *nonreferring measurement process* does not register even the number of photocounts, we have

$$T_\tau=\sum_{m=0}^{\infty}N_\tau(m) . \tag{39}$$

From this equation we have

$$\mathrm{Tr}[T_\tau\rho(t)]=1 . \tag{40}$$

The density operator immediately after the measurement is therefore

$$\rho(t+\tau)=T_\tau\rho(t) . \tag{41}$$

It should be noted that the two operators J and S_τ play both roles of measurement action and back action. Measurement action produces photoelectron statistics, as in Eqs. (28) and (31), while the measurement back action produces a new state via nonunitary state reduction, as in Eqs. (29) and (32). This is a quantum-mechanical version of Newton's third law of motion, although the outlook differs greatly from the classical law.

We assume that the photon field has a single mode

$$H_F=\hbar\omega_a a^\dagger a , \tag{42}$$

and its environment consists of harmonic oscillators

$$H_E=\sum_j \hbar\omega_j c_j^\dagger c_j , \tag{43}$$

where $\hbar$ is the Planck constant divided by 2π and zero-point energies are dropped since they have no bearing on the following discussion. We assume that the photon field and the environment interact linearly with each other as

$$H_I=\sum_j \hbar g_j(c_j^\dagger a+a^\dagger c_j) , \tag{44}$$

and that the commutation relations of these operators obey

$$[a,a^\dagger]=1 ,$$
$$[c_i,c_j^\dagger]=\delta_{ij} , \tag{45}$$

and

$$[a,c_i]=[a,c_i^\dagger]=0\ , \tag{46}$$

where δ_{ij} is Kronecker's delta. Then the operator Y is given by

$$Y=-\alpha a^\dagger a-\sum_j \beta_j c_j^\dagger c_j-\sum_j \gamma_j(c_j^\dagger a+a^\dagger c_j)\ , \tag{47}$$

where

$$\alpha=i\omega_a+\frac{\lambda}{2},\quad \beta_j=i\omega_j,\quad \text{and}\ \gamma_j=ig_j\ . \tag{48}$$

B. Nonunitary time evolution of the system-environment density operator by continuous measurement of photon number

Let us investigate nonunitary time evolution of the system-environment density operator for three different quantum photodetection processes. We shall see that nonunitary time evolution depends strongly on the way we read out or discard information concerning registration of a photocount.

1. Quantum photodetection process of forward recurrence times

Let us consider a continuous photodetection process in which we read out information concerning registration of a photocount at *every* measurement time. We refer to such a process as a quantum photodetection process of forward recurrence time (QPF). Associated with such a process is the probability distribution of forward recurrence times (PDF) which gives the probability per (unit time)^m of m photocounts being registered at m distinct times t_j $(j=1,2,\ldots,m)$ in $[0,\tau)$ with no further photocounts registered in between. The PDF is expressed in terms of J and S_τ as[22]

$$P_m^{(\text{forward})}(t_1,t_2,\ldots,t_m;0,\tau)=\text{Tr}[S_{\tau-t_m}JS_{t_m-t_{m-1}}\cdots S_{t_2-t_1}JS_{t_1}\rho(0)]\ . \tag{49}$$

Substituting Eqs. (30) and (33) into the rhs of Eq. (49) yields

$$S_{\tau-t_m}JS_{t_m-t_{m-1}}\cdots S_{t_2-t_1}JS_{t_1}\rho(0)=\lambda^m e^{Y\tau}\left[\prod_{i=1}^m(e^{-Yt_i}ae^{Yt_i})\right]\rho(0)\left[\prod_{j=1}^m(e^{Y^\dagger t_j}a^\dagger e^{-Y^\dagger t_j})\right]e^{Y^\dagger\tau}\ , \tag{50}$$

$$=\lambda^m e^{Y\tau}\left[\prod_{i=1}^m A(t_i)\right]\rho(0)\left[\prod_{j=1}^m A^\dagger(t_j)\right]e^{Y^\dagger\tau}\ , \tag{51}$$

where

$$A(t)\equiv e^{-Yt}ae^{Yt}\ . \tag{52}$$

The time evolution of the operator $A(t)$ is determined from

$$\frac{d}{dt}A(t)=e^{-Yt}[a,Y]e^{Yt}=-\alpha A(t)-\sum_j\gamma_jC_j(t)\ , \tag{53}$$

where Eq. (47) was used, and

$$C_j(t)\equiv e^{-Yt}c_je^{Yt}\ . \tag{54}$$

Since $A(0)=a$ and $C_j(0)=c_j$, we find that operator $A(t)$ contains only annihilation operators. This observation greatly facilitates calculation, as we shall see below. From Eq. (51) we have

$$P_m^{(\text{forward})}(t_1,t_2,\ldots,t_m;0,\tau)=\lambda^m\text{Tr}\left[\rho(0)\left[\prod_{i=1}^m A^\dagger(t_i)\right]e^{Y^\dagger\tau}e^{Y\tau}\left[\prod_{j=1}^m A(t_j)\right]\right]\ . \tag{55}$$

The system-environment density operator immediately after the QPF, $\rho^{\text{QPF}}(t_1,t_2,\ldots,t_m;0,\tau)$, is given by[25]

$$\rho^{\text{QPF}}(t_1,t_2,\ldots,t_m;0,\tau)=\frac{S_{\tau-t_m}JS_{t_m-t_{m-1}}\cdots S_{t_2-t_1}JS_{t_1}\rho(0)}{\text{Tr}[S_{\tau-t_m}JS_{t_m-t_{m-1}}\cdots S_{t_2-t_1}JS_{t_1}\rho(0)]}\ . \tag{56}$$

Substituting Eqs. (51) and (55) into the rhs of (56) yields

$$\rho^{\mathrm{QPF}}(t_1,t_2,\ldots,t_m;0,\tau)=\frac{e^{Y\tau}\left[\prod_{i=1}^{m}A(t_i)\right]\rho(0)\left[\prod_{j=1}^{m}A^{\dagger}(t_j)\right]e^{Y^{\dagger}\tau}}{\mathrm{Tr}\left[\rho(0)\left[\prod_{i=1}^{m}A^{\dagger}(t_i)\right]e^{Y^{\dagger}\tau}e^{Y\tau}\left[\prod_{j=1}^{m}A(t_j)\right]\right]} . \tag{57}$$

To obtain an expression of the generating functional of the PDF, let us rewrite Eq. (55) as

$$P_m^{(\mathrm{forward})}(t_1,t_2,\ldots,t_m;0,\tau)=\lambda^m\mathrm{Tr}\left[\rho(0)e^{Y^{\dagger}\tau}\left[\prod_{i=1}^{m}A^{\dagger}(t_i-\tau)\right]\left[\prod_{j=1}^{m}A(t_j-\tau)\right]e^{Y\tau}\right] , \tag{58}$$

where Eq. (52) is used. Note that $A(t)$ contains only annihilation operators and $A^{\dagger}(t)$ contains only creation operators. This observation facilitates the operator calculation because then the relevant operators in Eq. (58) are normally ordered. The generating functional of the PDF is obtained if we substitute Eq. (58) into Eq. (10):

$$\begin{aligned}G^{(2)}[u(c)]&=\sum_{m=0}^{\infty}\frac{i^m}{m!}\lambda^m\int_0^{\tau}dt_1\int_0^{\tau}dt_2\cdots\int_0^{\tau}dt_m\mathrm{Tr}\left[\rho(0)e^{Y^{\dagger}\tau}:\prod_{i=1}^{m}[A^{\dagger}(t_i-\tau)A(t_i-\tau)u(t_i)]:e^{Y\tau}\right]\\&=\mathrm{Tr}\left[\rho(0)e^{Y^{\dagger}\tau}:\sum_{m=0}^{\infty}\frac{1}{m!}\left[i\lambda\int_0^{\tau}dt\,A^{\dagger}(t-\tau)A(t-\tau)u(t)\right]^m:e^{Y\tau}\right]\\&=\mathrm{Tr}\left[\rho(0)e^{Y^{\dagger}\tau}:\exp\left[i\lambda\int_0^{\tau}dt\,A^{\dagger}(t-\tau)A(t-\tau)u(t)\right]:e^{Y\tau}\right] .\end{aligned} \tag{59}$$

Equations (55) and (59) give the PDF and its generating functional for a nonequilibrium open photon field. Equation (57) gives the photon field-environment density operator immediately after the QPF. To check our results, let us consider a closed system of the photon field for which Y and $A(t)$ are given by

$$Y=-(i/\hbar)H_F-(\lambda/2)a^{\dagger}a=-[i\omega+(\lambda/2)]a^{\dagger}a \tag{60}$$

and

$$A(t)=e^{[i\omega+(\lambda/2)]a^{\dagger}at}ae^{-[i\omega+(\lambda/2)]a^{\dagger}at}=ae^{-[i\omega+(\lambda/2)]t} . \tag{61}$$

Then Eq. (55) reduces to

$$P_m^{(\mathrm{forward})}(t_1,t_2,\ldots,t_m;0,\tau)=\lambda^m\exp\left[-\lambda\sum_{i=1}^{m}t_i\right]\mathrm{Tr}[\rho(0)a^{\dagger m}e^{-\lambda a^{\dagger}a\tau}a^m] , \tag{62}$$

which is identical to the result obtained by Mollow.[21] Equation (57) reduces to

$$\rho_m^{\mathrm{QPF}}(t_1,t_2,\ldots,t_m;0,\tau)=\frac{e^{-[i\omega+(\lambda/2)]a^{\dagger}a\tau}a^m\rho(0)a^{\dagger m}e^{[i\omega-(\lambda/2)]a^{\dagger}a\tau}}{\mathrm{Tr}[\rho(0)a^{\dagger m}e^{-\lambda a^{\dagger}a\tau}a^m]} , \tag{63}$$

which is identical to the result obtained by Ueda.[25] Note that in Eq. (62) information concerning registration times contributes to the PDF only through a simple exponential factor $\exp(-\lambda\sum_{i=1}^{m}t_i)$. This is because for a closed system the photon field is coupled only to a photodetector. For an open system, the photon field is pumped or disturbed by its environment throughout a measurement period, so the PDF depends on the registration times in a more complicated way as in Eq. (55) through Eq. (53). Note also that the rhs of Eq. (63) no longer depends even on registration times. This is because for a closed system, the measurement action (which produces photoelectron statistics) and its back action (which causes nonunitary state reduction) are uniquely related so that their time dependencies are canceled out in forming the ratio of the rhs of Eq. (56). However, for an open system they are not related uniquely because of the effect of the environment, as shown in Eq. (57). An explicit expression for $A(t)$ will be given in Secs. IV and V for two typical open-system problems.

2. *Quantum photodetection process for the number of counts*

Let us consider a continuous photodetection process in which we read out only the number of counts in $[0,\tau)$ and discard information concerning the times at which they are registered. We refer to such a process as a quantum photodetection process for the number of counts (QPN). Associated with such a process is the probability distribution for the number of counts (PDN). The PDN is expressed in terms of $N_\tau(m)$ as

$$P(m;0,\tau)=\mathrm{Tr}[N_\tau(m)\rho(0)] . \tag{64}$$

It is difficult to calculate the rhs of this equation directly from Eqs. (38) and (51). Instead, we apply the probability-density-functional method developed in Sec. II. The probability-density functional is obtained if we substitute Eq. (59) into the rhs of Eq. (13):

$$\mathcal{P}[p(c)]=\mathrm{Tr}\left[\rho(0)e^{Y^\dagger\tau}:\exp\left[\int_0^\tau p(t)dt\right]\delta(p(c)-\lambda A^\dagger(c-\tau)A(c-\tau)):e^{Y\tau}\right] . \tag{65}$$

The PDN is therefore obtained by substituting Eq. (65) into Eq. (15) to give

$$P(m;0,\tau)=\mathrm{Tr}\left[\rho(0)e^{Y^\dagger\tau}:\left[\lambda\int_0^\tau dt\ A^\dagger(t-\tau)A(t-\tau)\right]^m\Big/m!:e^{Y\tau}\right] . \tag{66}$$

The system-environment density operator immediately after the QPN is obtained from Eqs. (37), (38), (51), and (66) as

$$\rho_m^{\mathrm{QPN}}(0,\tau)=\frac{\int_0^\tau dt_m\int_0^{t_m}dt_{m-1}\cdots,\int_0^{t_2}dt_1 e^{Y\tau}\left[\prod_{i=1}^m A(t_i)\right]\rho(0)\left[\prod_{j=1}^m A^\dagger(t_j)\right]e^{Y^\dagger\tau}}{\mathrm{Tr}\left[\rho(0)e^{Y^\dagger\tau}:\left[\int_0^\tau dt\ A^\dagger(t-\tau)A(t-\tau)\right]^m\Big/m!:e^{Y\tau}\right]} . \tag{67}$$

Equation (66) gives the exact formula of the PDN. It is a generalization of Mollow's photocounting formula to a nonequilibrium open photon field. For a closed photon field, Eq. (66) reduces to Mollow's formula[21]

$$P(m;0,\tau)=\mathrm{Tr}\left[\rho(0):\frac{[a^\dagger a(1-e^{-\lambda\tau})]^m}{m!}\exp[-a^\dagger a(1-e^{-\lambda\tau})]:\right] , \tag{68}$$

and Eq. (67) reduces to

$$\rho_m^{\mathrm{QPN}}(\tau)=\frac{e^{-[i\omega+(\lambda/2)]a^\dagger a\tau}a^m\rho(0)a^{\dagger m}e^{[i\omega-(\lambda/2)]a^\dagger a\tau}}{\mathrm{Tr}[\rho(0)a^{\dagger m}e^{-\lambda a^\dagger a\tau}a^m]} , \tag{69}$$

which is identical to the result obtained by Ueda.[25] It is noted that $\rho_m^{\mathrm{QPN}}(\tau)$ in Eq. (69) is identical to ρ_m^{QPF} in Eq. (63). That is, for a closed system, only the number of photocounts which are registered in a measurement period contributes to the post-measurement state. In contrast, for an open system, registration times are also relevant for the post-measurement state, as in Eq. (57), and therefore $\rho_m^{\mathrm{QPN}}(\tau)$ is not, in general, identical to ρ_m^{QPF}.

3. *Quantum photodetection process of multicoincidence*

Let us consider a continuous photodetection process in which we read out the information concerning registration of a photocount at m distinct times in a measurement period and discard all other information. We refer to such a process as a quantum photodetection process of multicoincidence (QPM). Associated with such a process is the probability distribution of multicoincidence (PDM) which gives the probability per (unit time)^m of m counts being registered at m distinct times t_j $(j=1,2,\ldots,m)$ with no further condition imposed. The PDM is expressed in terms of T_t and J as[22]

$$\lambda(t_1,t_2,\ldots,t_m;0,\tau)=\mathrm{Tr}[T_{\tau-t_m}JT_{t_m-t_m}\cdots T_{t_2-t_1}JT_{t_1}\rho(0)] . \tag{70}$$

It is difficult to calculate the rhs of Eq. (70) directly. Instead, we apply the probability-density-functional method. The PDM is readily obtained if we substitute Eq. (65) into Eq. (4):

$$\lambda(t_1,t_2,\ldots,t_m;0,\tau)=\lambda^m\mathrm{Tr}\left[\rho(0)e^{Y^\dagger\tau}:\prod_{i=1}^m A^\dagger(t_i-\tau)\exp\left[\lambda\int_0^\tau dt\ A^\dagger(t-\tau)A(t-\tau)\right]\prod_{j=1}^m A(t_j-\tau):e^{Y\tau}\right] , \tag{71}$$

$$=\lambda^m\mathrm{Tr}\left[\rho(0)\left[\prod_{i=1}^m A^\dagger(t_i)\right]e^{Y^\dagger\tau}:\exp\left[\lambda\int_0^\tau dt\ A^\dagger(t-\tau)A(t-\tau)\right]:e^{Y\tau}\left[\prod_{j=1}^m A(t_j)\right]\right] . \tag{72}$$

The system-environment density operator immediately after the QPM is given by

$$\rho^{\mathrm{QPM}}(t_1,t_2\ldots,t_m;0,\tau)=\frac{T_{\tau-t_m}JT_{t_m-t_{m-1}}\cdots T_{t_2-t_1}JT_{t_1}\rho(0)}{\mathrm{Tr}[T_{\tau-t_m}JT_{t_m-t_{m-1}}\cdots T_{t_2-t_1}JT_{t_1}\rho(0)]} , \tag{73}$$

where the denominator is given by Eq. (71); the numerator cannot, however, be simplified unless we specify more about the form of the interaction, although it can be simplified somewhat for a closed system.[25]

Equation (71) gives the PDM for open systems. For a closed system of the photon field, it reduces to

$$\lambda(t_1,t_2,\ldots,t_m;0,\tau)=\lambda^m\exp\left[-\lambda\sum_{i=1}^m t_i\right]\mathrm{Tr}[\rho(0)a^{\dagger m}a^m] . \tag{74}$$

Some comments concerning an inversion problem are appropriate here. The problem of recovering the original photon statistics from the observed photoelectron statistics has been one of the central issues in quantum optics. In semiclassical theory, the distribution $P(W)$ of integrated intensities W can be determined from the photocount distribution.[19] In full quantum theory, the problem is restated as the recovery of the diagonal matrix elements of the original density operator from the knowledge of the observed photoelectron statistics. For a closed system, this is possible. Rewriting the rhs of Eq. (68) in terms of the diagonal matrix elements of the density operator gives

$$P(m;0,\tau)=\sum_{n=m}^{\infty}\begin{bmatrix}n\\m\end{bmatrix}(1-e^{-\lambda\tau})^m(e^{-\lambda\tau})^{n-m}\rho_{nn}\ , \tag{75}$$

where

$$\begin{bmatrix}n\\m\end{bmatrix}=\frac{n!}{m!(n-m)!}\ .$$

This equation can be inverted to give

$$\rho_{mn}=\sum_{n=m}^{\infty}P(n;0,\tau)\begin{bmatrix}n\\m\end{bmatrix}\left[\frac{-e^{-\lambda\tau}}{1-e^{-\lambda\tau}}\right]^{n-m}\left[\frac{1}{1-e^{-\lambda\tau}}\right]^m\ . \tag{76}$$

However, for an open system, the corresponding equation (66) cannot, in general, be inverted. The physical reason for the difference can be explained as follows. For a closed system, the photon field evolves exclusively owing to the measurement back action of the photodetector. Since the measurement back action is uniquely related to the measurement action which produces the observed photoelectron statistics, the diagonal matrix elements of an initial density operator are completely reproducible from the observed photoelectron statistics. In contrast, for an open system, they are not uniquely related because the fluctuations from the environment side are unpredictably added to the photon field.

It is well known that the PDN cannot, in general, contain complete information about photoelectron statistics.[33] Nevertheless, Eq. (76) demonstrates that the diagonal matrix elements of the initial density operator of the photon field can be completely recovered from the PDN. Thus, we find for a closed system that the sets of the PDN, PDM, and PDF each contain complete information about photoelectron statistics. If we find the PDN, we can obtain the PDM and PDF. This is a unique feature of a closed system. It will be useful to write down the expressions of the PDM and PDF in terms of the PDN. Expressing the rhs of Eq. (74) in terms of the diagonal matrix elements of the initial density operator of the photon field, we obtain

$$\lambda(t_1,t_2,\ldots,t_m;0,\tau)=\lambda^m\exp\left[-\lambda\sum_{i=1}^{m}t_i\right]\sum_{l=m}^{\infty}\rho_{ll}\frac{l!}{(l-m)!}\ , \tag{77}$$

where $\rho_{ll}=\langle l|\rho(0)|l\rangle$. Substituting Eq. (76) into the rhs of Eq. (77), we obtain

$$\lambda(t_1,t_2,\ldots,t_m;0,\tau)=\left[\frac{\lambda}{1-e^{-\lambda\tau}}\right]^m\exp\left[-\lambda\sum_{i=1}^{m}t_i\right]\times\sum_{n=m}^{\infty}P(n;0,\tau)\frac{n!}{(n-m)!}\ . \tag{78}$$

Similarly, we obtain an expression of the PDF in terms of the PDN:

$$P^{(\text{forward})}(t_1,t_2,\ldots,t_m;0,\tau)=m!\left[\frac{\lambda}{1-e^{-\lambda\tau}}\right]^m\exp\left[-\lambda\sum_{i=1}^{m}t_i\right]P(m;0,\tau)\ . \tag{79}$$

IV. APPLICATION TO A PHOTON FIELD BEING PUMPED BY A SINGLE-HARMONIC-OSCILLATOR SOURCE

Sections IV and V are devoted to applications of our method to two typical nonequilibrium open-system problems. In the present paper, however, the scope of research is limited to the bare essentials of the problems. In this section we consider a pump source and demonstrate that increasing the ratio of the source-field coupling constant to the field-detector coupling constant causes the photon field to cross over from a closed attenuating field to an open stationary field.

Let us consider a case in which the photon field interacts linearly with a single-harmonic-oscillator environment, which may be considered as a pump source.[42] The Hamiltonians of the photon field H_F, source H_S, and their interaction H_I are

$$H_F=\hbar\omega_a a^\dagger a\ , \tag{80}$$

$$H_S=\hbar\omega_c c^\dagger c\ , \tag{81}$$

and

$$H_I=\hbar g(ac^\dagger+ca^\dagger)\ . \tag{82}$$

Then the operator Y is given by

$$Y=-\{[i\omega_a+(\lambda/2)]a^\dagger a+i\omega_c c^\dagger c+ig(ac^\dagger+ca^\dagger)\}\ . \tag{83}$$

From Eq. (A6) in Appendix A, we find that

$$A(t)=e^{-Y t}ae^{Yt}=\exp\left[-\left[i\frac{\omega_a+\omega_c}{2}+\frac{\lambda}{4}\right]t\right][(-u\sinh\omega t+\cosh\omega t)a-v\sinh\omega t c]\ , \tag{84}$$

where

$$u=\frac{i(\omega_a-\omega_c)+(\lambda/2)}{\{[i(\omega_a-\omega_c)+(\lambda/2)]^2-4g^2\}^{1/2}}\ , \tag{85}$$

$$v=\frac{2ig}{\{[i(\omega_a-\omega_c)+(\lambda/2)]^2-4g^2\}^{1/2}}\ , \tag{86}$$

and

$$\omega=\frac{\{[i(\omega_a-\omega_c)+(\lambda/2)]^2-4g^2\}^{1/2}}{2}\ . \tag{87}$$

To focus on important physical aspects of the problem, let us consider the resonant condition, where $\omega_a=\omega_c$. Under this condition, the nonunitary time evolution of the total system is classified into two cases.

(a) $4g>\lambda$. In this case, Eqs. (85), (86), and (87) reduce to

$$u=-\frac{i\lambda/4g}{[1-(\lambda/4g)^2]^{1/2}}\ , \tag{88}$$

$$v=\frac{1}{[1-(\lambda/4g)^2]^{1/2}}\ , \tag{89}$$

and

$$\omega=ig[1-(\lambda/4g)^2]^{1/2}\ , \tag{90}$$

and $A(t)$ is therefore given by

$$A(t)=e^{-[i\omega+(\lambda/4)]t}\left[\left[-\frac{\lambda/4g}{[1-(\lambda/4g)^2]^{1/2}}\sin\{[1-(\lambda/4g)^2]^{1/2}gt\}+\cos\{[1-(\lambda/4g)^2]^{1/2}gt\}\right]a\right.$$
$$\left.-\frac{i}{[1-(\lambda/4g)^2]^{1/2}}\sin\{[1-(\lambda/4g)^2]^{1/2}gt\}c\right]\ . \tag{91}$$

This result shows that the photon field and its environment interact coherently, while the measurement back action attenuates the amplitude of the photon field exponentially. In the limit of $4g\gg\lambda$, Eq. (91) reduces to

$$A(t)=e^{-[i\omega+(\lambda/4)]t}[(\cos gt)a-(i\sin gt)c]\ . \tag{92}$$

The photocount probability density at time t, $\lambda(t_1;0,\tau)$, is derived from Eq. (72) as

$$\langle p(t)\rangle=\lambda(t;0,\tau)=\lambda\,\mathrm{Tr}\left[\rho(0)A^\dagger(t)e^{Y^\dagger\tau}{:}\exp\left[\lambda\int_0^\tau dt\ A^\dagger(t-\tau)A(t-\tau)\right]{:}e^{Y\tau}A(t)\right]\ . \tag{93}$$

For $4g\gg\lambda$, the rhs of (93) can be approximated to give

$$\langle p(t)\rangle=\lambda e^{-(\lambda/2)t}\mathrm{Tr}\{\rho(0)[(\cos^2 gt)a^\dagger a+(\sin^2 gt)c^\dagger c]\}\ . \tag{94}$$

This result shows that if we identify c with a pump field which generates single photons at equal time intervals T which are much larger than $1/\lambda$, we can observe a discrete single-photon number state with jitter $1/\lambda$, as shown schematically in Fig. 6. In Fig. 6 we observe that the probability-density function traces out coherent Rabi flopping between the photon field and the pump source before a photon is actually detected by the detector at time t_i ($i=1,2,\ldots$). The exponential decrease is due to the feedback of the readout information (in this case "no count") into the immediate future photon field-source density operator.

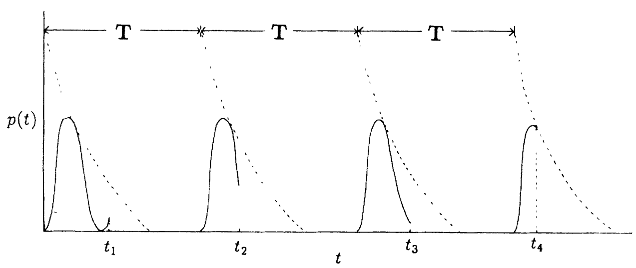

FIG. 6. Time development of the probability-density function for $4g\gg\lambda$. The probability-density function traces out a coherent Rabi flopping until a photon is actually detected by the detector at time t_i ($i=1,2,\ldots$).

Conversely, if we set T much shorter than $1/\lambda$, we can observe an approximate Poisson state whose cycle-averaged light intensity is nearly constant. In this case the photon field remains stationary while being measured.

(b) $\lambda>4g$. In this case, Eqs. (85), (86), and (87) reduce to

$$u=\frac{1}{[1-(4g/\lambda)^2]^{1/2}}\ , \tag{95}$$

$$v=\frac{4ig/\lambda}{[1-(4g/\lambda)^2]^{1/2}}\ , \tag{96}$$

and

$$\omega=(\lambda/4)[1-(4g/\lambda)^2]^{1/2}\ , \tag{97}$$

and $A(t)$ is therefore given by

$$A(t)=e^{-[i\omega+(\lambda/4)]t}\left[\left[-\frac{1}{[1-(4g/\lambda)^2]^{1/2}}\sinh\frac{[1-(4g/\lambda)^2]^{1/2}}{4}\lambda t+\cosh\frac{[1-(4f/\lambda)^2]^{1/2}}{4}\lambda t\right]a-\frac{4ig/\lambda}{[1-(4g/\lambda)^2]^{1/2}}\sinh\left[\frac{[1-(4g/\lambda)^2]^{1/2}}{4}\lambda t\right]c\right]. \quad (98)$$

In the limit of $\lambda \gg 4g$, Eq. (98) reduces to

$$A(t)=ae^{-[i\omega+(\lambda/2)]t}. \quad (99)$$

This result shows that the detector looks upon a photon field as a closed system; no effects of the pump source are included. The physical reason for this is that, owing to a strong field-detector coupling, a photon generated by a source attenuates long before coherent Rabi flopping starts. It should be noted that this attenuation is not caused by actual detection of a photon but the knowledge that *no* photon is detected until t_i ($i=1,2,\ldots$). Such an apparently paradoxical result is a unique feature of continuous measurement. In Fig. 7, the time development of the probability-density function is shown schematically for $T \gg 1/\lambda$.

Thus, we find that increasing the ratio of the source-photon field coupling constant to the field-detector coupling constant causes the photon field to cross over from a close attenuating field to an open stationary field. An exact photocounting formula can be obtained if we substitute $A(t)$ in Eq. (84) into the corresponding general formula obtained in Sec. III.

V. APPLICATION TO A PHOTON FIELD INTERACTING LINEARLY WITH A RESERVOIR OF AN INFINITE NUMBER OF HARMONIC OSCILLATORS

In this section, we consider a reservoir and discuss a quantum-mechanical fluctuation-dissipation theorem for open-system continuous photodetection. Here again we focus only on the essential features of the problem.

Let us consider a case in which the photon field interacts linearly with a reservoir consisting of an infinite number of harmonic oscillators. The Hamiltonians of the photon field H_F, reservoir H_R, and their interaction H_I are given by

$$H_F=\hbar\omega_a a^\dagger a , \quad (100)$$

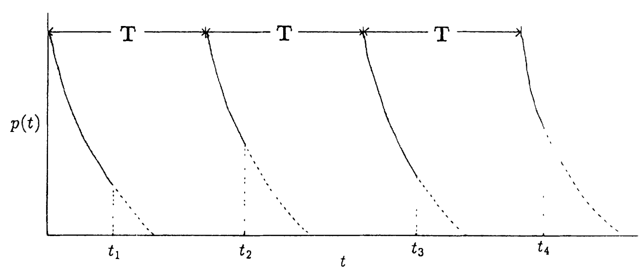

FIG. 7. Time development of the probability-density function for $\lambda \gg 4g$. The probability-density function simply decreases exponentially due to a strong field-detector coupling.

$$H_R=\sum_j \hbar\omega_j c_j^\dagger c_j , \quad (101)$$

and

$$H_I=\sum_j \hbar g_j(ac_j^\dagger+c_j a^\dagger) , \quad (102)$$

respectively. The operator Y is given by

$$Y=-\left[i\omega_a+\frac{\lambda}{2}\right]a^\dagger a-\sum_j i\omega_j c_j^\dagger c_j - \sum_j ig_j(ac_j^\dagger+c_j a^\dagger) . \quad (103)$$

If the photon field is a quasimonochromatic light, it is shown in Appendix B that the operator $A(t)\equiv e^{-Yt}ae^{Yt}$ satisfies the following Langevin-type equation:

$$\frac{d}{dt}A(t)=-\left[(i\omega_a+\Delta\omega_a)+\frac{\gamma}{2}+\frac{\lambda}{2}\right]A(t)+L(t) , \quad (104)$$

where $L(t)$ is a Langevin operator defined by (B13), and $\Delta\omega_a$ [defined by (B10)] is the Lamb shift in the center of frequency of the photon field due to interaction with a reservoir. This Langevin operator represents fluctuations which are unpredictably introduced from the reservoir into the photon field.

It can be shown that the fluctuations caused by the Langevin force exactly balance the dissipation represented by $\gamma/2$. This is a quantum-mechanical fluctuation-dissipation theorem.[43,44] Usually this theorem ensures that the commutation relation is preserved all the time. However, in our case, the commutation relation of the operator $A(t)$ decays exponentially owing to an additional decay constant $\lambda/2$. In fact, the Langevin-type equation (104) can be solved by the Laplace transform method, giving

$$A(t)=\mu(t)a+\sum_j \nu_j(t)c_j , \quad (105)$$

where

$$\mu(t)=e^{-[i(\omega_a+\Delta\omega_a)+(\gamma/2)+(\lambda/2)]t} , \quad (106)$$

and

$$\nu_j(t)=g_j e^{-i\omega_j t}\frac{1-e^{[i(\omega_j-\omega_a-\Delta\omega_a)-(\gamma/2)-(\lambda/2)]t}}{\omega_j-\omega_a-\Delta\omega_a+i(\gamma/2)+i(\gamma/2)} . \quad (107)$$

The commutation relation is therefore given by

$$[A(t), A^{\dagger}(t)] = e^{-\lambda t} . \tag{108}$$

An exact photon-counting formula for this case can be obtained if we substitute Eq. (105), together with Eqs. (106) and (107), into the corresponding general formula derived in Sec. III.

One comment is appropriate here. We showed in Sec. III that when the photon field interacts with only the detector, we can obtain complete information about photoelectron statistics. In the present case, where the photon field interacts with a reservoir as well, however, the above statement cannot hold because the fluctuations from the reservoir are unpredictably introduced into the photon field, as shown in Eq. (105), so that every internal time in the measurement process has a special meaning and cannot be averaged out.

VI. DISCUSSION AND CONCLUSIONS

The continuous photodetection process provides us with several unique features of quantum mechanics. Among other things, this paper focused on state reduction caused by measurement back action and nonunitary state evolution due to the feedback of the readout information from continuous measurement in a nonequilibrium open-system-theory context.

In general, the quantum photodetection process plays two distinct roles with respect to the past and future of the observed system. With respect to the past, it verifies the predicted photoelectron statistics by *repeated* measurements of the same quantum state. With respect to the future, it produces a new state via nonunitary state reduction determined by the result of a *single* measurement. A conventional unitary picture of a photodetection process imposes such a quantum-mechanical state reduction only once at the end of the measurement period.[21] However, our new nonunitary picture of a continuous photodetection process imposes state reduction throughout the measurement period because we can, in principle, read out information concerning registration of a photocount in *real time* throughout the measurement process. To completely describe time distributions of photoelectrons, we must therefore obtain some functional which reflects a continuously infinite number of collapses of the photon field-environment density operator. Reference 25 demonstrated that such a complete description can be achieved using the probability-density functional combined with the path-integral technique. In the present paper, this new framework for continuous photodetection processes has been generalized to a nonequilibrium open system in which, while being measured by a photodetector, the photon field interacts with its environment, which may be a pump source or a reservoir.

New exact photocounting formulas are obtained for a nonequilibrium open photon field. The obtained formulas are then applied to two typical environments: a pump source and a reservoir. For a pump source, increasing the ratio of the field-source coupling constant to the field-detector coupling constant causes the photon field to cross over from a closed attenuating field to an open stationary field. For a reservoir, a quantum-mechanical fluctuation-dissipation theorem is discussed for a nonequilibrium open system.

The time evolution of the photon field-environment density operator is discussed for three familiar quantum photodetection processes: QPF, QPN, and QPM. The obtained results present a sharp contrast to those obtained for a closed photon field. Suppose that a continuous photoelectron process starts at time 0 and ends at time τ. For a closed photon field the density operator immediately after the QPF gives exactly the same result as that after the QPN:[25]

$$\rho^{\mathrm{QPF}}(t_1, t_2, \ldots, t_m; 0, \tau) = \rho_m^{\mathrm{QPN}}(0, \tau) \tag{109}$$

for a closed system, where t_i $(i = 1, 2, \ldots, m)$ give the times at which photons were detected. This result shows that specific internal times at which photoelectric conversion occurred do not contribute to the post-measurement density operator; only the number of detected photons contribute. On the other hand, the specific internal times do contribute to the time distributions of photoelectrons. In fact we have

$$P_m^{(\mathrm{forward})}(t_1, t_2, \ldots, t_m; 0, \tau) = \frac{m!\lambda^m \exp\left[-\lambda \sum_{l=1}^{m} t_i\right]}{(1 - e^{-\lambda\tau})^m} P(m; 0, \tau) . \tag{110}$$

The physical reason for the result in Eq. (109) is that since no fluctuations are introduced into the photon field from the environment, the measurement action producing photoelectron statistics and its back action causing nonunitary state reduction are uniquely related to each other throughout the measurement process and are therefore canceled out between the denominator and numerator in Eq. (56). This is why complete information about statistical properties of photoelectrons can be obtained from the PDN. In contrast, Eq. (109) does not hold, in general, for an open photon field, as is found by comparing (57) with (67):

$$\rho^{\mathrm{QPF}}(t_1, t_2, \ldots, t_m; 0, \tau) \neq \rho^{\mathrm{QPN}}(0, \tau) \tag{111}$$

for an open system. In this case, every internal time, or more precisely the correlation of internal times, has a special meaning which reflects the statistical properties of the environment.

It should be noted that the general formalism developed in Sec. III holds true irrespective of the special form of the photon field-environment interaction Eq. (44) if $A(t)$ contains only an annihilation operator. Therefore, the following important interaction, for example, equally applies to our formalism:

$$H_I = \sum_j \hbar g (c_j^{\dagger 2} a + a^{\dagger} c_j^2) . \tag{112}$$

Physically, this Hamiltonian represents the second harmonic generation.

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APPENDIX A

Here, we describe the time evolution of the operator $A(t)$ which is defined by

$$A(t) \equiv \exp\{[\alpha a^\dagger a + \beta c^\dagger c + \gamma(ac^\dagger + ca^\dagger)]t\} \times a \exp\{-[\alpha a^\dagger a + \beta c^\dagger c + \gamma(ac^\dagger + ca^\dagger)]t\} . \quad (A1)$$

It is convenient to define the dual operator $C(t)$ associated with $A(t)$ by

$$C(t) \equiv \exp\{[\alpha a^\dagger a + \beta c^\dagger c + \gamma(ac^\dagger + ca^\dagger)]t\} \times c \exp\{-[\alpha a^\dagger a + \beta c^\dagger c + \gamma(ac^\dagger + ca^\dagger)]t\} . \quad (A2)$$

It can be shown that the two operators $A(t)$ and $C(t)$ satisfy the differential equations

$$(Dt + \alpha)A(t) + \gamma C(t) = 0 \quad (A3)$$

and

$$\gamma A(t) + (Dt + \beta)C(t) = 0 , \quad (A4)$$

where $Dt \equiv d/dt$. Multiplying $(Dt+\beta)$ from the left of Eq. (A3) and γ from the left of Eq. (A4), and subtracting both sides of the resultant equations yields the differential equation which includes the operator $A(t)$ only:

$$[(Dt+\beta)(Dt+\alpha) - \gamma^2]A(t) = 0 . \quad (A5)$$

This equation can be solved by a standard method, yielding

$$A(t) = e^{-(\alpha+\beta/2)t}[(-u \sinh\omega t + \cosh\omega t)a - (v \sinh\omega t)c] , \quad (A6)$$

where

$$u \equiv \frac{\alpha-\beta}{[(\alpha-\beta)^2+4\gamma^2]^{1/2}} , \quad (A7)$$

$$v \equiv \frac{2\gamma}{[(\alpha-\beta)^2+4\gamma^2]^{1/2}} , \quad (A8)$$

$$w \equiv \frac{[(\alpha-\beta)^2+4\gamma^2]^{1/2}}{2} . \quad (A9)$$

APPENDIX B

Let us start with definitions

$$A(t) \equiv e^{-Yt} a e^{Yt} , \quad (B1)$$

$$C_j(t) \equiv e^{-Yt} c_j e^{Yt} , \quad (B2)$$

where

$$Y \equiv -\left[i\omega_a + \frac{\lambda}{2} \right] a^\dagger a - \sum_j i\omega_j c_j^\dagger c_j - \sum_j i g_j (ac_j^\dagger + c_j a^\dagger) . \quad (B3)$$

It can be shown that $A(t)$ and $C_j(t)$ satisfy the differential equations

$$\frac{d}{dt}A(t) = -\left[i\omega_a + \frac{\lambda}{2} \right] A(t) - \sum_j i g_j C_j(t) \quad (B4)$$

and

$$\frac{d}{dt}C_j(t) = -ig_j A(t) - i\omega_j C_j(t) . \quad (B5)$$

The solution of Eq. (B5) can be written as

$$C_j(t) = C_j(0)e^{-i\omega_j t} - ig_j \int_0^t A(t')e^{i\omega_j(t'-t)}dt' . \quad (B6)$$

Substituting Eq. (B6) into the rhs of Eq. (B4) yields

$$\frac{d}{dt}A(t) = -\left[i\omega_a + \frac{\lambda}{2} \right] A(t) - \sum_j i g_j C_j(0) e^{-i\omega_j t} - \sum_j g_j^2 \int_0^t A(t')e^{i\omega_j(t'-t)}dt' . \quad (B7)$$

If we assume quasimonochromatic light, we have

$$A(t') \approx A(t)e^{i\omega_a(t-t')} . \quad (B8)$$

Then the third term on the rhs of Eq. (B7) becomes[45]

$$\sum_j g_j^2 \int_0^t A(t')e^{i\omega_j(t'-t)}dt' \approx A(t) \sum_j g_j^2 e^{-i(\omega_j-\omega_a)t} \int_0^\infty e^{i(\omega_j-\omega_a)t'}dt'$$
$$= A(t) \left[i\mathrm{P} \int_{-\infty}^{\infty} d\omega_j \rho(\omega_j)\gamma(\omega_j)^2 \frac{e^{-i(\omega_a-\omega_j)t}}{\omega_j-\omega_a} + \pi\gamma(\omega_a)^2\rho(\omega_a) \right] , \quad (B9)$$

where $\rho(\omega)$ is the density function of the reservoir oscillators, and the following identity is used:[46]

$$\int_0^\infty e^{i\omega t}dt = i\mathrm{P}(1/\omega) + \pi\delta(\omega) ,$$

where P denotes the principal value. The first term on the rhs of Eq. (B9) gives the frequency shift of the photon field due to the interaction of the reservoir

$$\Delta\omega_a \equiv \mathrm{P}\int_{-\infty}^{\infty} \frac{\rho(\omega')\gamma(\omega')^2}{\omega_a-\omega'} d\omega' , \quad (B10)$$

and the second term gives the photon lifetime due to the random action of the reservoir

$$\gamma \equiv 2\pi\rho(\omega_a)\gamma(\omega_a)^2 \ . \tag{B11}$$

Thus, we find that the operator $A(t)$ satisfies a Langevin-type equation

$$\frac{d}{dt}A(t) = -\left[i(\omega_a + \Delta\omega_a) + \frac{\gamma}{2} + \frac{\lambda}{2} \right] A(t) + L(t) \ , \tag{B12}$$

where $L(t)$ is a Langevin operator defined by

$$L(t) \equiv -i \sum_j g_j c_j e^{-i\omega_j t} \ . \tag{B13}$$

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