Treatment of the spectrum of squeezing based on the modes of the universe. II. Applications

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In the preceding paper we developed a formalism based on the true modes of the universe containing a leaky cavity to analyze the relationship of quantum noise inside that cavity to that outside. We illustrate this formalism here by studying several active systems. We analyze both an ordinary laser and a laser operating in the phase-locked regime. We calculate the extracavity quadrature variances for the phase-locked laser and find that the spectrum of noise reduction in the squeezed quadrature is Lorentzian. Our formalism is also applied to a two-photon correlated-spontaneous-emission laser. We find the familiar result that with 50% squeezing in the phase quadrature of the field *inside* the cavity, one has nearly 100% phase squeezing *outside* the cavity. However, under different conditions the phase quadrature of the *intracavity* field is nearly perfectly squeezed, but the *extracavity* field shows almost no squeezing in its phase quadrature. We also analyze the effect of finiteness of measurement time on the extracavity quadrature variances.

I. INTRODUCTION

In the preceding paper[1] (henceforth referred to as paper I), we have developed a formalism to relate the quantum noise in the optical field inside a leaky cavity to the noise in the output field. This formalism is based on the rigorously defined modes of a larger, perfect cavity (which we call the "universe" in the limit of infinite volume) that encloses the leaky cavity completely and has perfectly reflecting ends. In this formalism, the inside and the outside of the cavity of interest are quite naturally coupled to each other via the modes of the universe. We have shown how one may derive in this formalism the spectrum of squeezing of the output field. We also presented a simple physical picture that encompasses the two essential elements of the inside-outside problem: the correlation of the input field and the intracavity field arising at the leaky mirror and the fact that the output field is the sum of the transmitted cavity field and the reflected input field.

In a recent letter, some of us have shown that a two-photon correlated-spontaneous-emission laser[2] (CEL) can generate bright squeezed light.[3] The analysis involved coherently pumped cascade three-level active atoms interacting with a single mode of the radiation field inside a high-Q cavity, which allows multiple passes and thus a large effective interaction length necessary for high gain and squeezing. However, the question of how the degree of squeezing of that intracavity mode relates to that of the field observed outside at a detector was left open. We address such a question in this paper, basing its answer on the analysis of paper I.

In this paper we further discuss the general formalism of paper I by analyzing some properties of an ordinary laser and a laser exhibiting phase locking, such as the two-photon CEL. We shall also address the issue of signal-to-noise ratio for the specific case of measurement of a very small phase difference. For this case we shall also derive some useful expressions to calculate the level of noise reduction in the output of a broad class of laser devices in which the intracavity field is described by a Fokker-Planck or master equation. We shall see that in certain situations the degree of intracavity squeezing can, in fact, exceed that of squeezing of the detected field, which may seem surprising at first glance. However, we shall see that this result may be understood, in accordance with our physical picture, in terms of the unusually high decay rates of intracavity phase correlations in such regimes of operation.

In potential applications, such as gravitational-wave detection, the total measurement time is often quite limited. We shall take into account here the finiteness of measurement time, which modifies the detected "spectrum of squeezing" as well as the signal-to-noise ratio of measured quantities. We shall see that for measurement times comparable to or lower than the intracavity correlation time, the degree of observed squeezing can be quite substantially lower than possible with infinitely long measurement time.

The layout of the paper is as follows. We shall analyze in Sec. II measurements of small phase or frequency changes for an ordinary laser and calculate the extracavity phase noise for a phase-locked laser. These analyses will be based on the mean values and normally ordered variances of quantum operators for which classical Langevin equations may be written down. The classical Langevin equation formalism will be replaced in Sec. III by the alternative Fokker-Planck formalism for the calculation of the spectrum of squeezing. In Sec. IV we apply this general Fokker-Planck formalism to the two-photon CEL. We shall see that by dispensing with one-photon resonance and initial atomic coherences involving the middle level, the maximum squeezing of the intracavity mode is 50% while the detected field can be almost perfectly squeezed. Almost the exact reverse holds, however, if exact one-photon resonance and initial atomic coherences involving the middle level are present. In particular, the intracavity field may be perfectly squeezed while the outside field is then not only unsqueezed in the same quadrature but has, in fact, increased noise in the conjugate quadrature. In Sec. V we shall briefly analyze the effect of finite measurement time on quadrature variances. Finally, in Sec. VI the conclusions of the paper will be presented.

II. PHASE MEASUREMENTS: SIGNAL-TO-NOISE RATIO INSIDE AND OUTSIDE THE CAVITY

In many situations (e.g., ring laser gyros, gravitational wave detection) we are interested in measuring a small phase change. If the field quadratures a_1 and a_2 are defined relative to the initial phase ϕ_0 (for a full discussion of phase and amplitude fluctuations in terms of the a_1 and a_2 quadratures, see Ref. 4),

$$\langle a(t=0)\rangle=\sqrt{n_0}e^{i\phi_0}\ , \tag{2.1a}$$

$$a_1=\tfrac{1}{2}(ae^{-i\phi_0}+a^\dagger e^{i\phi_0})\ , \tag{2.1b}$$

$$a_2=\frac{1}{2i}(ae^{-i\phi_0}-a^\dagger e^{i\phi_0})\ , \tag{2.1c}$$

then the small phase change $\delta\phi$ mostly changes $\langle a_2\rangle$

$$\delta\langle a_2\rangle\simeq\langle a_1\rangle\delta\phi=\sqrt{n_0}\delta\phi\ , \tag{2.2}$$

assuming that the amplitude of the field is well stabilized and large ($(\langle\Delta a_1^2\rangle^{1/2}\ll\langle a_1\rangle)$). A similar equation then relates the fluctuations in a_2 to the fluctuations in ϕ. The signal-to-noise ratio is then

$$\frac{\delta\phi}{(\langle\Delta\phi^2\rangle)^{1/2}}=\frac{\delta\langle a_2\rangle}{(\langle\Delta a_2^2\rangle)^{1/2}}\ . \tag{2.3}$$

From Eqs. (I.3.25) and (I.3.20) (equations in paper I are referred to here by affixing I to the equation numbers) we can relate the photocurrent noise for the phase quadrature at the cavity line center, which is $\langle\Delta\tilde{A}_2^2(0)\rangle$ from Eq. (I.3.27), to the phase fluctuations in the cavity

$$\langle\Delta\tilde{A}_2^2(0)\rangle=\frac{1}{4}+\frac{2\Gamma n_0}{T}\times\int_0^T dt'\int_0^T dt''\langle:\Delta\phi(t')\Delta\phi(t''):\rangle_{\text{cav}}\ , \tag{2.4}$$

where $n_0\simeq\langle a_1\rangle^2$ inside the cavity. Equation (I.3.17) may in turn be used to calculate the signal $\delta\langle\tilde{A}_2(\delta\omega)\rangle$ at the detector as a function of the intracavity $\delta\langle a_2\rangle$. In this section, we restrict our discussion to the on-resonance mode $\delta\omega=0$ of the detected field. We shall see in Sec. III that this mode has, in general, the largest noise reduction. However, modes with $\delta\omega\neq0$ may also be treated in a very similar way.

Consider the example of an ordinary laser, in which a small frequency change is to be measured. This measurement is limited in precision by phase noise. The phase change inside the cavity is

$$\delta\phi(t)=(\Delta\nu)t\ . \tag{2.5}$$

Using this expression in Eq. (22) and then substituting into Eq. (I.3.17), which is

$$\tilde{A}_{\text{out}_\theta}(\delta\omega)=\left[\frac{2\Gamma}{T}\right]^{1/2}\int_0^T a_\theta(t)e^{i\delta\omega t}dt+\tilde{r}\left[\frac{2\epsilon_0\mathcal{A}c}{\hbar\Omega T}\right]^{1/2}\int_0^T\mathcal{E}_{\text{in}_\theta}(t)e^{i\delta\omega t}dt\ ,$$

we find a $\delta\langle\tilde{A}_2(0)\rangle$ at the detector

$$\delta\langle\tilde{A}_2(0)\rangle=\left[\frac{2\Gamma n_0}{T}\right]^{1/2}\int_0^T(\Delta\nu)t\,dt=\left[\frac{2\Gamma n_0}{T}\right]^{1/2}\frac{(\Delta\nu)T^2}{2}\ , \tag{2.6}$$

where T is the measurement time. This is the signal, and the noise is given by Eq. (2.4).

To evaluate (2.4), we note that the normally ordered expectation values of intracavity field operators can be evaluated from a Fokker-Planck equation or from the associated classical Langevin equation. The Fokker-Planck approach will be discussed later. Here we use the simpler approach, which is to write for the evolution of the phase a stochastic equation

$$\dot{\phi}=F_\phi(t)\ , \tag{2.7}$$

where the correlation function for F_ϕ is

$$\langle F_\phi(t)F_\phi(t')\rangle=2D_{\phi\phi}\delta(t-t') \tag{2.8}$$

($D_{\phi\phi}$ is half of the laser linewidth). By integrating Eq. (2.7) from 0 to t, taking its second moment, and using Eq. (2.8), one may easily show that

$$\langle:\Delta\phi(t')\Delta\phi(t''):\rangle_{\text{cav}}=2D_{\phi\phi}t_<\ , \tag{2.9}$$

where $t_<$ is the minimum of t',t''. Substitution of (2.9) into (2.4) gives the "noise"

$$\langle\Delta\tilde{A}_2^2(0)\rangle=\tfrac{1}{4}+\tfrac{4}{3}\Gamma n_0 D_{\phi\phi}T^2\ . \tag{2.10}$$

Since $D_{\phi\phi}\sim\Gamma/n_0$, the second term in (2.10) is much larger than the first provided $\Gamma T\gg1$. In this limit, the signal-to-noise ratio (2.3) becomes [using (2.6) and (2.10)]

$$\left[\frac{S}{N}\right]_{\text{det}}=\frac{\Delta\nu}{2}\left[\frac{3T}{2D_{\phi\phi}}\right]^{1/2}\ . \tag{2.11}$$

Thus by equating the signal to noise, we find that the minimum detectable $(\Delta\nu)$ is $\Delta\nu_{\min}=2\sqrt{2D_{\phi\phi}/3T}$, as expected from the phase-diffusion noise. Note that inside the cavity $\delta\phi=\Delta\nu T$ [by (2.5)] and $(\langle\Delta\phi^2\rangle)^{1/2}=[2D_{\phi\phi}T+(4n_0)^{-1}]^{1/2}$ [by (2.9)], so in the limit $\Gamma T\gg 1$ the signal-to-noise ratio inside the cavity is

$$\left[\frac{S}{N}\right]_{\text{cav}}=\Delta\nu\left[\frac{T}{2D_{\phi\phi}}\right]^{1/2}, \tag{2.12}$$

which differs from (2.11) by a factor $\sqrt{3}/2\simeq 0.87$. This is not a large difference, but it shows that the detection process, and the accompanying filtering, must be considered in every case (a point also shown by Gea-Banacloche, Scully, and Anderson[5] for the same problem with a different detector model).

The method by which (2.11) was evaluated—through a Langevin equation for a classical variable ϕ associated with the normally ordered Fokker-Planck equation—may be readily generalized to lasers and masers with phase locking (to ϕ_0). For such cases we have

$$\dot{\phi}=d_\phi(\phi)+F_\phi(\phi,t), \tag{2.13}$$

where d_ϕ is responsible for the phase locking and F_ϕ satisfies

$$\langle F_\phi(\phi,t)F_\phi(\phi,t')\rangle=2D_{\phi\phi}(\phi)\delta(t-t'). \tag{2.14}$$

Linearizing about the steady-state laser phase ϕ_0 that satisfies $d_\phi(\phi_0)=0$, we find

$$\frac{d}{dt}\Delta\phi=-\left|\frac{\partial d_\phi}{\partial\phi}\right|_{\phi_0}\Delta\phi+F_\phi(\phi_0,t). \tag{2.15}$$

Defining

$$\gamma_\phi\equiv\left|\frac{\partial d_\phi}{\partial\phi}\right|_{\phi_0}, \tag{2.16}$$

we can solve (2.15) to obtain

$$\Delta\phi(t)=\Delta\phi(0)e^{-\gamma_\phi t}+\int_0^t e^{-\gamma_\phi(t-t')}F_\phi(\phi_0,t')dt'. \tag{2.17}$$

The correlation function is then found to be

$$\begin{aligned}\langle:\Delta\phi(t')\Delta\phi(t''):\rangle_{\text{cav}}&=\frac{D_{\phi\phi}(\phi_0)}{\gamma_\phi}(e^{-\gamma_\phi|t'-t''|}-e^{-\gamma_\phi(t'+t'')})\\&\quad+\langle:\Delta\phi^2(0):\rangle_{\text{cav}}e^{-\gamma_\phi(t'+t'')},\end{aligned} \tag{2.18}$$

which is stationary if we choose $\langle:\Delta\phi^2(0):\rangle_{\text{cav}}$ to have the steady-state value $D_{\phi\phi}(\phi_0)/\gamma_\phi$.

In any case, when (2.18) is used in (2.4) one finds, for sufficiently long times ($\gamma_\phi T\gg 1$),

$$\langle\Delta\tilde{A}_2^2(0)\rangle=\frac{1}{4}+\frac{4\Gamma n_0 D_{\phi\phi}(\phi_0)}{\gamma_\phi^2}, \tag{2.19}$$

and thus when $D_{\phi\phi}(\phi_0)$ is negative, we have squeezing. The same result will be derived in Sec. III from the Fokker-Planck equation, after which it will be applied to the specific problem of the two-photon CEL in Sec. IV.

III. QUADRATURE VARIANCES OUTSIDE THE CAVITY IN THE LIMIT $T\to\infty$

In this section we will derive, in the long-measurement-time limit ($T\to\infty$) explicit expressions for the quadrature variances outside the cavity via the Fokker-Planck equation approach. The effect of finite measurement time on quadrature variances outside the cavity will be discussed in Sec. V.

As the measurement time $T\to\infty$, the normally ordered quadrature variances of the field outside the cavity are given by Eq. (I.3.26), which involves normally ordered two-time correlation functions inside the cavity. These normally ordered two-time correlation functions may be found by using the Fokker-Planck equation for a normal-ordering distribution function, say the Glauber-Sudarshan P function,[6] describing the intracavity field. For any specific problem such a Fokker-Planck equation may be obtained from the master equation for the field.

Corresponding to Eqs. (2.1b) and (2.1c) we define c-member quadrature variables as

$$\mathcal{E}_1=(\mathcal{E}e^{-i\phi_0}+\mathcal{E}^*e^{i\phi_0})/2, \tag{3.1a}$$

$$\mathcal{E}_2=(\mathcal{E}e^{-i\phi_0}-\mathcal{E}^*e^{i\phi_0})/2i. \tag{3.1b}$$

In terms of the quadrature variables $\mathcal{E}_1$ and $\mathcal{E}_2$, the Fokker-Planck equation for $P(\mathcal{E},\mathcal{E}^*,t)$ may be rewritten as

$$\begin{aligned}\frac{\partial}{\partial t}P(\mathcal{E}_1,\mathcal{E}_2,t)=\Bigg[&-\frac{\partial}{\partial\mathcal{E}_1}d_1-\frac{\partial}{\partial\mathcal{E}_2}d_2+\frac{\partial^2}{\partial\mathcal{E}_1^2}D_{11}\\&+\frac{\partial^2}{\partial\mathcal{E}_2^2}D_{22}+2\frac{\partial^2}{\partial\mathcal{E}_1\partial\mathcal{E}_2}D_{12}\Bigg]P(\mathcal{E}_1,\mathcal{E}_2,t),\end{aligned} \tag{3.2}$$

where d_1 and d_2 are the drift coefficients and D_{11}, D_{22}, and D_{12} the diffusion coefficients. According to the properties of the Glauber-Sudarshan P function,[6] normally ordered moments of the quadrature operators a_1 and a_2 may be directly evaluated from $P(\mathcal{E}_1,\mathcal{E}_2,t)$. For example, it follows from Eqs. (2.1) and (3.1) that $\langle a_j\rangle=\langle\mathcal{E}_j\rangle$ and

$$\langle:\Delta a_j^2:\rangle_{\text{cav}}=\langle(\Delta\mathcal{E}_j)^2\rangle, \tag{3.3a}$$

$$\langle:\Delta a_j(t)\Delta a_j(0):\rangle_{\text{cav}}=\langle\Delta\mathcal{E}_j(t)\Delta\mathcal{E}_j(0)\rangle, \tag{3.3b}$$

where $\Delta\mathcal{E}_j=\mathcal{E}_j-\langle\mathcal{E}_j\rangle$ and $j=1,2$. The equations of motion for the first- and normally ordered second moments of the quadrature operators are readily found by using the Fokker-Planck equation (3.2),

$$\frac{d}{dt}\langle\mathcal{E}_j\rangle=\langle d_j\rangle, \tag{3.4}$$

$$\frac{d}{dt}\langle(\Delta\mathcal{E}_j)^2\rangle=2\langle d_j\Delta\mathcal{E}_j\rangle+2\langle D_{jj}\rangle, \tag{3.5}$$

with $j=1,2$.

We shall now investigate the statistical properties of the field in the long-time limit when steady state has been more or less attained for the mean values and fluctuations of the field. It follows from Eq. (3.4) that the steady-state locking point (or mean value) $(\mathcal{E}_{10},\mathcal{E}_{20})$ satisfies the following deterministic equations:

$$d_j(\mathcal{E}_{10},\mathcal{E}_{20})=0,\quad j=1,2\ . \tag{3.6}$$

For simplicity, we assume in the following discussion

$$\frac{\partial d_1(\mathcal{E}_{10},\mathcal{E}_{20})}{\partial \mathcal{E}_{20}}=\frac{\partial d_2(\mathcal{E}_{10},\mathcal{E}_{20})}{\partial \mathcal{E}_{10}}=0\ , \tag{3.7}$$

namely, that the steady-state fluctuations in the a_1 (amplitude) and a_2 (phase) quadratures are decoupled [cf. Eq. (3.12)]. Considering small fluctuations around the locking point $(\mathcal{E}_{10},\mathcal{E}_{20})$, we expand $\mathcal{E}_j$ and d_j in Eq. (3.4) around the steady-state locking point $(\mathcal{E}_{10},\mathcal{E}_{20})$ up to first order in deviations from it. Using Eqs. (3.6) and (3.7), we find that in the long-time limit the deviation $\delta\langle \mathcal{E}_j\rangle$ of the mean value $\langle \mathcal{E}_j\rangle$ at time t from its steady-state value $\mathcal{E}_{j0}$ evolves according to the law

$$\frac{d}{dt}\delta\langle \mathcal{E}_j\rangle=\frac{\partial d_j(\mathcal{E}_{10},\mathcal{E}_{20})}{\partial \mathcal{E}_j}\delta\langle \mathcal{E}_j\rangle$$

$$=-\left|\frac{\partial d_j}{\partial \mathcal{E}_j}\right|_0\delta\langle \mathcal{E}_j\rangle\ , \tag{3.8}$$

where use has been made of the stability conditions at the locking point $(\mathcal{E}_{10},\mathcal{E}_{20})$

$$\left[\frac{\partial d_j}{\partial \mathcal{E}_j}\right]_0\equiv\frac{\partial d_j(\mathcal{E}_{10},\mathcal{E}_{20})}{\partial \mathcal{E}_j}<0,\quad j=1,2\ . \tag{3.9}$$

According to the quantum regression theorem,[7,8] the two-time correlation functions of the system obey exactly the same dynamical law of evolution as the one-time functions. Thus it follows from Eqs. (3.8) that for $t\geq t'$,

$$\frac{d}{dt}\langle \Delta\mathcal{E}_j(t)\Delta\mathcal{E}_j(t')\rangle=-\left|\frac{\partial d_j}{\partial \mathcal{E}_j}\right|_0\langle \Delta\mathcal{E}_j(t)\Delta\mathcal{E}_j(t')\rangle\ . \tag{3.10}$$

The solution of Eq. (3.10) is simply

$$\langle \Delta\mathcal{E}_j(t)\Delta\mathcal{E}_j(t')\rangle=e^{-|\partial d_j/\partial \mathcal{E}_j|_0|t-t'|}\langle (\Delta\mathcal{E}_j)^2\rangle\ , \tag{3.11}$$

where $\langle (\Delta\mathcal{E}_j)^2\rangle$ are normally ordered quadrature variances in the steady state [see Eq. (3.3a)]. These normally ordered variances can be found from Eq. (3.5) by setting $d/dt=0$ and expanding d_j and D_{jj} around the locking point $(\mathcal{E}_{10},\mathcal{E}_{20})$ up to first order in $\Delta\mathcal{E}_i=\mathcal{E}_i-\mathcal{E}_{i0}$ $(i=1,2)$,

$$\langle :\Delta a_j^2:\rangle_{\rm cav}=\langle (\Delta\mathcal{E}_j)^2\rangle=\frac{D_{jj}(\mathcal{E}_{10},\mathcal{E}_{20})}{|\partial d_j/\partial \mathcal{E}_j|_0}\ . \tag{3.12}$$

Substituting Eqs. (3.3b), (3.11), and (3.12) into Eq. (I.3.26), we obtain the spectrum of squeezing outside the cavity

$$:S_j(\delta\omega):=\gamma\int_{-\infty}^{\infty}\langle \Delta\mathcal{E}_j(t)\Delta\mathcal{E}_j(0)\rangle\cos(\delta\omega t)dt$$

$$=\frac{2\gamma|\partial d_j/\partial \mathcal{E}_j|_0}{|\partial d_j/\partial \mathcal{E}_j|_0^2+(\delta\omega)^2}\langle :\Delta a_j^2:\rangle_{\rm cav} \tag{3.13a}$$

$$=\frac{2\gamma D_{jj}(\mathcal{E}_{10},\mathcal{E}_{20})}{|\partial d_j/\partial \mathcal{E}_j|_0^2+(\delta\omega)^2},\quad j=1,2 \tag{3.13b}$$

where $\gamma=2\Gamma$ is the cavity intensity decay rate. Equations (3.13) are Lorentzian centered at the cavity mode frequency Ω (i.e., $\delta\omega=0$) with full width at half maximum (FWHM) $2|\partial d_j/\partial \mathcal{E}_j|_0$. When $D_{jj}(\mathcal{E}_{10},\mathcal{E}_{20})<0$ ($j=1$ or 2), squeezing of the jth quadrature occurs both inside and outside the cavity (see Fig. 1), and the minimum noise outside the cavity is obtained at the cavity mode frequency Ω (i.e., $\delta\omega=0$):

$$:S_j(0):=\langle :\Delta\tilde{A}_j^{\,2}(0):\rangle=\frac{2\gamma}{|\partial d_j/\partial \mathcal{E}_j|_0}\langle :\Delta a_j^2:\rangle_{\rm cav}\ . \tag{3.14}$$

For the a_2 quadrature the result is the same as Eq. (2.19), since $D_{22}(\mathcal{E}_{10},\mathcal{E}_{20})=n_0D_{\phi\phi}(\phi_0)$ and $|\partial d_2/\partial \mathcal{E}_2|_0=\gamma_\phi$. Equation (3.14) gives the relation between the normally ordered quadrature variance $\langle :\Delta a_j^2:\rangle_{\rm cav}$ inside the cavity and the minimum of that outside the cavity. They differ by a factor of $2\gamma/|\partial d_j/\partial \mathcal{E}_j|_0$, which is twice the ratio of the cavity intensity loss rate γ to the locking strength $|\partial d_j/\partial \mathcal{E}_j|_0$. In other words, whether there exists more squeezing outside the cavity than inside or not really depends on this factor. Besides Eq. (3.14), another relation connecting the inside and outside variances can be found from Eq. (3.13a)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\gamma^{-1}:S_j(\delta\omega):d(\delta\omega)=\langle :\Delta a_j^2:\rangle_{\rm cav}\ ,\quad j=1,2\ . \tag{3.15}$$

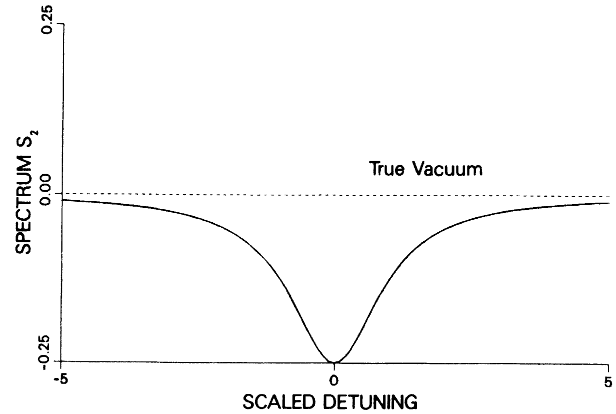

FIG. 1. Spectrum of squeezing for the a_2 quadrature vs the scaled detuning $\delta\omega/|\partial d_j/\partial \mathcal{E}_j|_0$. The dashed line represents the noise level $S_2=0$ for the true vacuum. The figure is plotted by assuming perfect extracavity squeezing, which resembles the situation of the two-photon CEL (see Sec. IV A).

In the next two sections we will only calculate the minimum in the spectrum of squeezing $\langle \Delta \tilde{A}_j^2(0)\rangle = \frac{1}{4} + :S_j(0):$.

IV. TWO-PHOTON CORRELATED-SPONTANEOUS-EMISSION LASER

In this section we apply the general formalism for the quadrature variances outside the cavity obtained in Sec. III to the two-photon CEL, i.e., a laser with coherently pumped cascade three-level atoms interacting with a single-mode radiation field (see Fig. 2). We consider the situation that the jth atom is injected into the laser cavity at time t_j with initial populations ρ_{aa}, ρ_{bb}, and ρ_{cc} and initial coherences $\rho^j_{ab}(t_j)=\rho^j_{ba}(t_j)^*=\bar{\rho}_{ab}e^{-i\nu t_j}$, $\rho^j_{bc}(t_j)=\rho^j_{cb}(t_j)^*=\bar{\rho}_{bc}e^{-i\nu t_j}$, and $\rho^j_{ac}(t_j)=\rho^j_{ca}(t_j)^*=\bar{\rho}_{ac}e^{-i2\nu t_j}$, where a, b, and c refer to the top, middle, and bottom levels, respectively, ν is the actual laser frequency, and $\bar{\rho}_{ab}$, ρ_{bc}, and $\bar{\rho}_{ac}$ are the same for all atoms. We assume perfect two-photon resonance $\omega_{ac}=2\nu$ and denote the atom-field detuning for the one-photon transition by $\Delta=\omega_{ab}-\nu=\nu-\omega_{bc}$. Here $\hbar\omega_{ij}$ is the energy difference between levels i and j ($i,j=a,b,c$). In the following, we discuss two cases (A) $\Delta\neq 0$, $\rho_{bb}=\bar{\rho}_{ab}=\bar{\rho}_{bc}=0$ and (B) $\Delta=0$, $\bar{\rho}_{ab}\neq 0$, $\bar{\rho}_{bc}\neq 0$, separately.

A. Off-resonant two-photon CEL

When the two-photon CEL is off-resonant with the one-photon transition ($\Delta\neq 0$) and the middle level b is not populated (i.e., $\rho_{bb}=\bar{\rho}_{ab}=\bar{\rho}_{bc}=0$), the master equation[2] for the reduced density operator of the intracavity field of the two-photon CEL in the linear approximation may be obtained by using the density matrix approach of the quantum theory of the laser,[9]

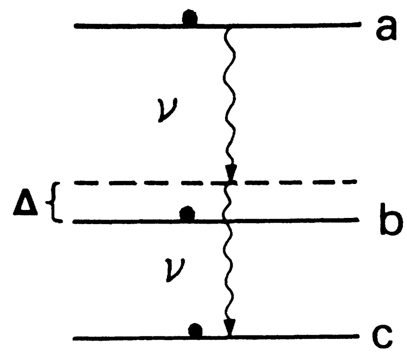

FIG. 2. Energy-level diagram for the two-photon correlated-spontaneous-emission laser. Δ is the atom-field detuning for one-photon transition.

$$\dot{\rho}=\{-\tfrac{1}{2}\alpha[\rho_{aa}\mathcal{L}(aa^\dagger\rho-a^\dagger\rho a)+\rho_{cc}\mathcal{L}^*(\rho a^\dagger a-a\rho a^\dagger) +\rho_{ca}\mathcal{L}(aa\rho-a\rho a)+\rho_{ca}\mathcal{L}^*(\rho aa-a\rho a)] -\tfrac{1}{2}\gamma(\rho a^\dagger a-a\rho a^\dagger)+\text{H.c.}\}-i(\Omega-\nu)[a^\dagger a,\rho]\ , \tag{4.1}$$

with the linear gain coefficient $\alpha=2r_a g^2/\Gamma^2$, the cavity (intensity) loss rate γ, and $\mathcal{L}=\Gamma/(\Gamma-i\Delta)$. Here r_a is the atomic injection rate, g the atom-field coupling constant (for simplicity, taken to be the same for the a-b and b-c transitions), γ_A the atomic decay rate (same for all levels), and Ω the cavity quasimode frequency.

Converting the master equation (4.1) into the Fokker-Planck equation (3.2) for the Glauber-Sudarshan $P(\mathcal{E}_1,\mathcal{E}_2,t)$ function, one finds explicit expressions for the drift and diffusion coefficients

$$d_1=\tfrac{1}{2}\mathcal{E}_1\{\alpha|\mathcal{L}|^2[\rho_{aa}-\rho_{cc}+2(\Delta/\gamma_A)|\bar{\rho}_{ac}|\sin(\theta_{ac}-2\phi_0)]-\gamma\} +\tfrac{1}{2}\mathcal{E}_2\{\alpha|\mathcal{L}|^2(\Delta/\gamma_A)[\rho_{aa}+\rho_{cc}-2|\bar{\rho}_{ac}|\cos(\theta_{ac}-2\phi_0)]+2(\Omega-\nu)\}\ , \tag{4.2a}$$

$$d_2=\tfrac{1}{2}\mathcal{E}_2\{\alpha|\mathcal{L}|^2[\rho_{aa}-\rho_{cc}-2(\Delta/\gamma_A)|\bar{\rho}_{ac}|\sin(\theta_{ac}-2\phi_0)]-\gamma\} -\tfrac{1}{2}\mathcal{E}_1\{\alpha|\mathcal{L}|^2(\Delta/\gamma_A)[\rho_{aa}+\rho_{cc}+2|\bar{\rho}_{ac}|\cos(\theta_{ac}-2\phi_0)]+2(\Omega-\nu)\}\ , \tag{4.2b}$$

$$D_{11}=\tfrac{1}{4}\alpha|\mathcal{L}|^2\{\rho_{aa}-|\bar{\rho}_{ac}/\mathcal{L}|\cos[\theta_{ac}-2\phi_0+\arctan(\Delta/\gamma_A)]\}\ , \tag{4.3a}$$

$$D_{22}=\tfrac{1}{4}\alpha|\mathcal{L}|^2\{\rho_{aa}+|\bar{\rho}_{ac}/\mathcal{L}|\cos[\theta_{ac}-2\phi_0+\arctan(\Delta/\gamma_A)]\}\ , \tag{4.3b}$$

$$D_{12}=-\tfrac{1}{4}\alpha|\bar{\rho}_{ac}\mathcal{L}|\sin[\theta_{ac}-2\phi_0+\arctan(\Delta/\gamma_A)]\ , \tag{4.3c}$$

where $\theta_{ij}=\arg\bar{\rho}_{ij}$.

There exist two stable laser phases[1] ϕ_0 satisfying

$$\theta_{ac}-2\phi_0=\tfrac{1}{2}\pi\,\text{sgn}\Delta \bmod 2\pi\ . \tag{4.4}$$

Assuming that the injected atomic coherence does not affect linear mode pulling, we find

$$\nu=\Omega+\tfrac{1}{2}\alpha|\mathcal{L}|^2\Delta/\gamma_A\ . \tag{4.5}$$

Substituting Eqs. (4.4) and (4.5) into Eqs. (4.2), one sees that Eq. (3.7) is satisfied in the current linear theory. Consequently, we may use results obtained in Sec. III.

The linear gain G of the two-photon CEL may be identified from Eq. (4.2a) and is approximately equal to the cavity loss γ in the current linear theory, i.e.,

$$G\equiv\alpha|\mathcal{L}|^2(\rho_{aa}-\rho_{cc})+2\alpha|\mathcal{L}|^2|\bar{\rho}_{ac}\Delta|/\gamma_A\approx\gamma\ . \tag{4.6}$$

Using Eqs. (4.4) and (4.6) in Eqs. (4.2b) and (4.3b), we have

$$\frac{\partial d_2}{\partial \mathcal{E}_2}=\tfrac{1}{2}\alpha|\mathcal{L}|^2(\rho_{aa}-\rho_{cc}-2|\bar{\rho}_{ac}\Delta|/\gamma_A)-\tfrac{1}{2}\gamma$$

$$=-2\alpha|\mathcal{L}|^2|\bar{\rho}_{ac}\Delta|/\gamma_A<0\ , \qquad (4.7a)$$

$$D_{22}=\tfrac{1}{4}\alpha|\mathcal{L}|^2(\rho_{aa}-|\bar{\rho}_{ac}\Delta|/\gamma_A)\ . \qquad (4.7b)$$

Substituting Eqs. (4.7) into Eq. (3.12) we find the intracavity variance in the phase quadrature of the laser field to be

$$\langle \Delta a_2^2\rangle_{\rm cav}=\langle :\Delta a_2^2:\rangle_{\rm cav}+\tfrac{1}{4}$$

$$=\tfrac{1}{8}(1+\rho_{aa}\gamma_A/|\bar{\rho}_{ac}\Delta|)\ , \qquad (4.8)$$

which is the same as $n_0\langle(\delta\phi)^2\rangle$ found in Ref. 2. Substituting Eqs. (4.7) into Eq. (3.14) and using Eq. (4.6), we find the quadrature variance outside the cavity to be

$$\langle \Delta \tilde{A}_2^2(0)\rangle=\langle :\Delta \tilde{A}_2^2(0):\rangle+\tfrac{1}{4}$$

$$=\frac{\rho_{aa}+\rho_{cc}+\rho_{aa}(\rho_{aa}-\rho_{cc})\gamma_A/|\bar{\rho}_{ac}\Delta|}{8|\bar{\rho}_{ac}\Delta|/\gamma_A}\ . \qquad (4.9)$$

In the one-photon far-off-resonant limit $|\Delta|/\gamma_A\gg 1$, we obtain simultaneously minimum noise in the phase quadrature both inside and outside the cavity, $\langle\Delta a_2^2\rangle_{\rm cav}=\frac{1}{8}$, $\langle\Delta\tilde{A}_2^2(0)\rangle=\gamma_A/(8|\bar{\rho}_{ac}\Delta|)\ll\frac{1}{4}$. Namely, we find 50% squeezing inside the cavity and nearly 100% squeezing outside the cavity for the two-photon CEL (see Fig. 1).

To gain physical insight into this result, it is important to note that (i) this result is obtained near threshold[10] since Eq. (4.6) is used, and (ii) $|\partial d_2/\partial\mathcal{E}_2|\approx G=\gamma$ in the limit $|\Delta|/\gamma_A\gg1$ as is evident in Eqs. (4.6) and (4.7a). Consequently, this result follows directly from Eq. (3.14) since the ratio $2\gamma/|\partial d_2/\partial\mathcal{E}_2|_0$ in Eq. (3.14) is 2. In terms of the physical picture developed in paper I for the inside-outside problem, this regime of operation of the two-photon CEL is exactly identical to that obtained for the degenerate parametric oscillator just below threshold,[11] with the ϵ parameter in Eq. (I.4.6) taking on the value $\frac{1}{2}$.

B. Resonant two-photon CEL

When the one-photon resonance condition is satisfied (i.e., $\Delta=0$) and $\rho_{bb}\neq0$, $\bar{\rho}_{ab}\neq0$, $\bar{\rho}_{bc}\neq0$, there is no frequency pulling $\nu=\Omega$ from symmetry considerations. There exists only one stable laser phase. When $\theta_{ab}=\theta_{bc}$, the stable phase is[2]

$$\phi_0=\theta_{ab}-\tfrac{1}{2}\pi=\tfrac{1}{2}(\theta_{ac}-\pi)\ . \qquad (4.10)$$

The drift coefficients can be found as

$$d_1=\tfrac{1}{2}[\alpha(\rho_{aa}-\rho_{cc})-\gamma]\mathcal{E}_1+(r_a g/\Gamma)(|\bar{\rho}_{ab}|+|\bar{\rho}_{bc}|)\ , \qquad (4.11a)$$

$$d_2=\tfrac{1}{2}[\alpha(\rho_{aa}-\rho_{cc})-\gamma]\mathcal{E}_2\ , \qquad (4.11b)$$

and the relevant diffusion coefficient is

$$D_{22}=\tfrac{1}{4}\alpha(\rho_{aa}+\rho_{bb}-|\bar{\rho}_{ac}|)\ . \qquad (4.12)$$

Obviously Eq. (3.7) is satisfied and the results in Sec. III apply here too.

When the linear gain is smaller than the cavity loss, i.e., $G\equiv\alpha(\rho_{aa}-\rho_{cc})<\gamma$, steady-state laser amplitudes $(\mathcal{E}_{10},\mathcal{E}_{20})$ can be found by substituting Eqs. (4.11) into Eq. (3.6) ($\mathcal{E}_{20}=0$), and the locking is stable [satisfying Eq. (3.9)]. In this case the laser intensity $n_0\simeq\mathcal{E}_{10}^2\gg1$ is maintained by the atomic coherences $\bar{\rho}_{ab}$ and $\bar{\rho}_{bc}$ involving the middle level b, so that the resonant two-photon CEL is still an active device. It follows from Eqs. (3.12), (4.11b), and (4.12) that the intracavity variance of the phase quadrature of the field is

$$\langle \Delta a_2^2\rangle_{\rm cav}=\langle :\Delta a_2^2:\rangle_{\rm cav}+\tfrac{1}{4}$$

$$=\frac{\rho_{aa}+2\rho_{bb}+\rho_{cc}-2|\bar{\rho}_{ac}|+\gamma/\alpha}{4(\rho_{cc}-\rho_{aa}+\gamma/\alpha)}\ . \qquad (4.13)$$

The variance outside the cavity is found from Eqs. (3.14), (4.11b), and (4.13) as

$$\langle \Delta \tilde{A}_2^2(0)\rangle=\tfrac{1}{4}+\langle :\Delta \tilde{A}_2^2(0):\rangle$$

$$=\frac{1}{4}+\frac{2(\rho_{aa}+\rho_{bb}-|\bar{\rho}_{ac}|)\gamma/\alpha}{(\rho_{cc}-\rho_{aa}+\gamma/\alpha)^2}\ . \qquad (4.14)$$

For the initial atomic populations

$$\rho_{aa}=\tfrac{1}{2}\{1-[2(2\lambda+1)\gamma/\alpha]^{1/2}-(\lambda-1)\gamma/\alpha\}\ ,$$

$$\rho_{bb}=\lambda\gamma/\alpha\ , \qquad (4.15)$$

$$\rho_{cc}=\tfrac{1}{2}\{1+[2(2\lambda+1)\gamma/\alpha]^{1/2}-(\lambda+1)\gamma/\alpha\}\ ,$$

and coherences $|\bar{\rho}_{ij}|=(\rho_{ii}\rho_{jj})^{1/2}$ $(i,j=a,b,c)$ with $(\gamma/\alpha)^{1/2}\ll1$, the noise in the phase quadrature inside and outside the cavity becomes

$$\langle \Delta a_2^2\rangle_{\rm cav}\simeq\tfrac{1}{4}[2(2\lambda+1)\gamma/\alpha]^{1/2}\ll\tfrac{1}{4}\ , \qquad (4.16a)$$

$$\langle \Delta \tilde{A}_2^2(0)\rangle\simeq\tfrac{1}{4}-\left[\frac{\gamma}{2(2\lambda+1)\alpha}\right]^{1/2}\ . \qquad (4.16b)$$

In other words, there is nearly perfect intracavity squeezing, but a very small amount of squeezing outside the cavity. This is a somewhat surprising result in that there is more squeezing inside than outside. This is a result of the unusual conditions of this example, namely, (i) although $G<0$, there is no real threshold here since the initial atomic coherences $\bar{\rho}_{ab}$ and $\bar{\rho}_{ac}$ act as a driving force [see Eq. (4.11a)], and (ii) $|\partial d_2/\partial\mathcal{E}_2|\simeq\frac{1}{2}|G|=[(\lambda+\frac{1}{2})\alpha\gamma]^{1/2}\gg\gamma$, so that the ratio $2\gamma/|\partial d_2/\partial\mathcal{E}_2|_0$ in Eq. (3.14) is much smaller than 1. In terms of the physical picture developed in paper I, this result may be understood as follows. Since the cavity fluctuations damp out much faster than input vacuum fluctuations enter the cavity, the reflected input vacuum (unsqueezed) fluctuations will dominate the transmitted cavity fluctuations in the output-field fluctuations, regardless of the degree of cavity quasimode squeezing. Thus one expects hardly any squeezing of the output field. The ϵ parameter in Eq. (I.4.6) takes on a value close to zero here.

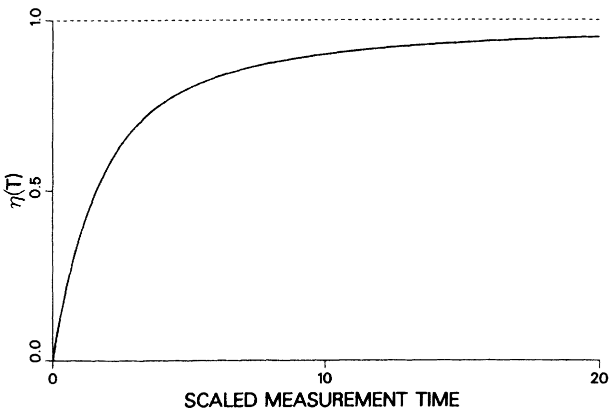

FIG. 3. Correction factor $\eta(T)$ due to finite measurement time T as a function of the scaled measurement time $T|\partial d_j/\partial \mathcal{E}_j|_0$. The dashed line indicates the long-measurement-time limit.

V. EFFECT OF FINITENESS OF MEASUREMENT TIME

In this section we study the effects of finite measurement time T on the degree of squeezing achievable at the detector. We will focus on the minimum in the spectrum of squeezing $\langle :\Delta \tilde{A}_2^2(0): \rangle$ found in the limit $T \to \infty$ in Sec. III.

It follows from Eqs. (I.3.22), (3.3b), and (3.11) for finite measurement time T and for a stationary field

$$\langle :\Delta \tilde{A}_j^2(0): \rangle = \gamma T^{-1} \int_0^T \int_0^T \langle \delta\mathcal{E}_j(t)\delta\mathcal{E}_j(t') \rangle dt\, dt' = \frac{2\gamma}{|\partial d_j/\partial \mathcal{E}_j|_0} \langle :(\Delta a_j)^2: \rangle_{\text{cav}} \eta(T) , \quad (5.1)$$

where

$$\eta(T) = 1 - \frac{1 - e^{-T|\partial d_j/\partial \mathcal{E}_j|_0}}{T|\partial d_j/\partial \mathcal{E}_j|_0} \quad (5.2)$$

is a correction factor due to finiteness of the measurement time T [see Eq. (3.14)]. We plot $\eta(T)$ in Fig. 3. One sees that the finiteness of the measurement time tends to reduce the amount of outside squeezing. To obtain larger squeezing we have to increase the measurement time T. To approach the largest outside squeezing given by Eq. (3.14), the condition $T|\partial d_j/\partial \mathcal{E}_j|_0 \gg 1$ must be met.

VI. CONCLUSIONS

We have further studied in this paper the spectrum of squeezing of a one-sided leaky cavity from the general formalism developed in the preceding paper. For measuring a small phase change by using an ordinary laser, we have calculated the signal-to-noise ratios both inside and outside the cavity via classical Langevin equation approach. The extracavity quadrature variances of the lasers exhibiting phase locking are studied by using the Fokker-Planck equations (also by using the classical Langevin equations) describing the intracavity field. When the intracavity variances in the amplitude and phase quadratures are uncorrelated, the spectra of squeezing for the two quadratures of the output field are found to be Lorentzian, one of which is an inverted one if there exists squeezing. We have applied the general formalism for the spectrum of squeezing to the two-photon CEL. For the off-resonant two-photon CEL studied in Sec. IV A, for which there exists a threshold, maximum intracavity squeezing (50% in the phase quadrature) transforms to nearly perfect phase squeezing outside the cavity. For the resonant two-photon CEL discussed in Sec. IV B, for which there is no threshold as the initial atomic coherences play the role of a driving force, nearly perfect intracavity phase squeezing amounts to a very small amount of squeezing outside the cavity. Both these results may easily be understood in terms of the physical picture developed in paper I, as we have seen here. Finally, we have analyzed the effect of finite measurement time T on the detected degree of squeezing. We find that to achieve large squeezing one needs to increase the measurement time T such that $T|\partial d_j/\partial \mathcal{E}_j|_0 \gg 1$.

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