Cooperative behavior of atoms irradiated by broadband squeezed light

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We consider the dynamics of a collection of atoms interacting with a coherent field and a broadband squeezed vacuum. We obtain an *exact* solution of the master equation and study in detail the types of nonequilibrium steady states that can be generated. We show that in the absence of coherent drive the atoms are in a state whose properties are similar to those of the squeezed vacuum for photons. We demonstrate that the steady state for certain discrete values of the external field strength and detuning is a pure state which is the eigenstate of the non-Hermitian operator $\cosh(|\xi|)S^- + \sinh(|\xi|)S^+$, where ξ is the squeezing parameter associated with the input radiation field. These eigenstates play a very fundamental role in the theory and satisfy the equality sign in the Heisenberg uncertainty relation $\Delta S_x \, \Delta S_y \geq 1/2|\langle S_z \rangle|$. We also present detailed numerical results for the characteristics of the field generated by the collective system.

I. INTRODUCTION

The interaction of squeezed radiation with atomic systems results in some unusual properties: for example, Gardiner[1] has shown that the two components of the dipole moment associated with a single atom in the field of broadband squeezed radiation decay very differently. This difference leads to the phase sensitive characteristics of the resonance fluorescence spectrum.[2] This in turn results in the narrowing of the central component of the Mollow spectrum. The two-photon absorption rate by an atom in the field of squeezed radiation has some unusual properties like linear dependence of the rate on the intensity.[3] The squeezed radiation has also been shown to affect considerably the nature of the vacuum field Rabi splittings,[4] etc. Much of the existing work deals with the interaction of a *single* atom with squeezed radiation. In what follows we discuss the interaction of a collection of atoms[5,6] with squeezed radiation. The atoms may, in addition, be driven by a classical field. The organization of this paper is as follows. In Sec. II we discuss the model and derive the basic equation describing the dynamics of the atomic system. In Secs. III and IV we obtain the exact solution for the atomic state in the long-time limit for no drive, for resonant driving field, and for off-resonant driving field. We also discuss briefly the case when the frequency of the field pumping the squeezed bath and that of the atom are different. A very important role in the theory is played by the eigenstates of the non-Hermitian operator $\cosh(|\xi|)S^- + \sinh(|\xi|)S^+$. In Appendix A we give the detailed properties of such states. We present numerical results for the atomic inversion, phase sensitive properties of the atomic system, and for the radiation which would be emitted by the atomic system irradiated by broadband squeezed radiation.

II. BASIC EQUATIONS

Consider a system of N identical two-level atoms of frequency ω_0 interacting with broadband squeezed radiation and with an external field of frequency ω_1. The total interaction Hamiltonian can be written as

$$\begin{aligned} H = {} & \hbar\omega_0 S^z + \hbar \int d\omega\, a^\dagger(\omega) a(\omega) \\ & - [(\mathbf{d}\cdot\mathbf{E}/\hbar) S^+ \exp(-i\omega_1 t) + \text{H.c.}] \\ & + \hbar \int d\omega [g(\omega) S^+ a(\omega) + \text{H.c.}] , \end{aligned} \tag{2.1}$$

where $\mathbf{d}$ is the atomic dipole moment and $\mathbf{E}$ is the electric field. Here $S^\pm$, S^z are the angular momentum operators corresponding to the spin value $N/2$. The atom-field interaction is given by $g(\omega)$. The annihilation and creation operators $a(\omega)$ and $a^\dagger(\omega)$ satisfy the usual commutation relations $[a(\omega_1), a^\dagger(\omega_2)] = \delta(\omega_1 - \omega_2)$. The field is in the squeezed vacuum state defined by[7,8]

$$\begin{aligned} |\{0\}\rangle_{\text{sq}} = \exp \Bigg\{ \frac{1}{2} \int [& a^\dagger(\omega_p + \epsilon) a^\dagger(\omega_p - \epsilon) \xi(\epsilon) \\ & - a(\omega_p + \epsilon) a(\omega_p - \epsilon) \xi^*(\epsilon)] \Bigg\} |\{0\}\rangle , \end{aligned} \tag{2.2}$$

where $|\{0\}\rangle$ is the normal vacuum, $a(\omega_p)|\{0\}\rangle = 0$ for all ω_p, and where $\xi(\epsilon)$ will give the amount of squeezing. The squeezed vacuum is such that the field modes corresponding to $\omega_p + \epsilon$ and $\omega_p - \epsilon$ are correlated. Note that $\xi(\epsilon)$ is a symmetric function of ϵ. Using (2.2) one can prove the following properties:

$$\langle a(\omega) \rangle = {}_{\text{sq}}\langle \{0\} | a(\omega) | \{0\} \rangle_{\text{sq}} = 0 , \tag{2.3}$$

$$\langle a(\omega_1)a(\omega_2)\rangle=\cosh[|\xi(\omega_1-\omega_p)|]\sinh[|\xi(\omega_1-\omega_p)|]\times\xi(\omega_1-\omega_p)/|\xi(\omega_1-\omega_p)|\times\delta(\omega_1+\omega_2-2\omega_p)\ , \tag{2.4}$$

$$\langle a^\dagger(\omega_1)a(\omega_2)\rangle=\sinh^2[|\xi(\omega_1-\omega_p)|]\delta(\omega_1-\omega_2)\ , \tag{2.5}$$

$$\langle a(\omega_2)a^\dagger(\omega_1)\rangle=\cosh^2[|\xi(\omega_1-\omega_p)|]\delta(\omega_1-\omega_2)\ . \tag{2.6}$$

We next eliminate the degrees of freedom associated with the squeezed vacuum and derive the equation for the reduced density matrix of the atomic system. This can be done by using the standard master equation methods.[9] To derive the master equation we first drop the external field terms and write (2.1) in the interaction picture:

$$H_1(t)=[S^+B(t)+\text{H.c.}]\ ,\qquad B(t)=\int d\omega\{g(\omega)a(\omega)\exp[-i(\omega-\omega_0)t]\}\ . \tag{2.7}$$

In order to derive the master equation we will make the Born and Markov approximations assuming that the interaction is weak and that the field $B(t)$ is broadband. The master equation for the reduced density matrix can be written in the form

$$\dot\rho=-\Gamma_-(S^+S^-\rho-S^-\rho S^+-\rho S^-S^++S^+\rho S^-)-\Gamma_+(S^+S^-\rho-S^-\rho S^++\rho S^-S^+-S^+\rho S^-)-\Gamma_0(S^+S^+\rho-2S^+\rho S^++\rho S^+S^+)+\text{H.c.}\ , \tag{2.8}$$

where Γ's are the correlation functions for the squeezed vacuum defined by

$$\Gamma_\mp=\lim_{t\to\infty}\left[\frac{1}{2}\int_0^\infty\langle[B(t+\tau),B^\dagger(t)]_\mp\rangle d\tau\right]\ , \tag{2.9}$$

$$\Gamma_0=\lim_{t\to\infty}\left[\frac{1}{2}\int_0^\infty\langle[B(t+\tau),B(t)]_+\rangle d\tau\right]\ . \tag{2.10}$$

Here $[\ ]_\pm$ stands for the anticommutator or commutator of the operators in the bracket. On substituting (2.4)–(2.6) in Eqs. (2.9) and (2.10) and on ignoring the small principal part contributions, we get

$$\begin{aligned}\Gamma_-&=(\pi/2)|g(\omega_0)|^2=\Gamma/2\ ,\\ \Gamma_+&=(\Gamma/2)\cosh[2|\xi(\omega_0-\omega_p)|]\ ,\\ \Gamma_0&=(\pi/2)\exp[-2i(\omega_p-\omega_0)t]g(\omega_0)g(2\omega_p-\omega_0)\times\sinh[2|\xi(\omega_0-\omega_p)|]\xi(\omega_0-\omega_p)/|\xi(\omega_0-\omega_p)|\ .\end{aligned} \tag{2.11}$$

Note that $g(\omega)$ is essentially a flat function of ω. On substituting (2.11) in (2.8) we can write the master equation in the form

$$\begin{aligned}\dot\rho=&-\Gamma(1+\bar n)(S^+S^-\rho-2S^-\rho S^++\rho S^+S^-)-\Gamma\bar n(S^-S^+\rho-2S^+\rho S^-+\rho S^-S^+)\\&-\Gamma m(S^+S^+\rho-2S^+\rho S^++\rho S^+S^+)\exp[-2i(\omega_p-\omega_0)t]\\&-\Gamma m^*/(S^-S^-\rho-2S^-\rho S^-+\rho S^-S^-)\exp[2i(\omega_p-\omega_0)t]\ ,\end{aligned} \tag{2.12}$$

where

$$\bar n=\sinh^2(|\xi|),\quad |m|=\sinh(|\xi|)\cosh(|\xi|)\ , \tag{2.13}$$

and the phase of m is related to the phase of $\xi(\omega)$.

We can now write the complete density matrix equation taking into account the interaction with the external field ϵ. We also work in a frame rotating with the frequency ω_1 of the external field. Thus the final master equation becomes

$$\begin{aligned}\dot\rho=&-i\Delta[S^z,\rho]+i[[(\mathbf{d}\cdot\epsilon/\hbar)S^++\text{H.c.}],\rho]-\Gamma(1+\bar n)(S^+S^-\rho-2S^-\rho S^++\rho S^+S^-)\\&-\Gamma\bar n(S^-S^+\rho-2S^+\rho S^-+\rho S^-S^+)-\Gamma m(S^+S^+\rho-2S^+\rho S^++\rho S^+S^+)\exp[-2i(\omega_p-\omega_1)t]\\&-\Gamma m^*|(S^-S^-\rho-2S^-\rho S^-+\rho S^-S^-)\exp[2i(\omega_p-\omega_1)t],\quad \Delta=\omega_0-\omega_1\ .\end{aligned} \tag{2.14}$$

We now absorb the phase of m in the definition of $S^\pm$, i.e., if $m=|m|\exp(i\phi)$ then we can introduce new S^+ related to old by $S^+\exp(i\phi/2)$. Also, on writing $-\mathbf{d}\cdot\epsilon/\hbar=|\Omega|\exp(i\chi)$ we can rewrite Eq. (2.14) as

$$\begin{aligned}\dot\rho=&-i\Delta[S^z,\rho]+i[|\Omega|\exp(i\psi)S^++|\Omega|\exp(-i\psi)S^-,\rho]-\Gamma(1+\bar n)(S^+S^-\rho-2S^-\rho S^++\rho S^+S^-)\\&-\Gamma\bar n(S^-S^+\rho-2S^+\rho S^-+\rho S^-S^+)-\Gamma|m|(S^+S^+\rho-2S^+\rho S^++\rho S^+S^+)\exp[-2i(\omega_p-\omega_1)t]\\&-\Gamma|m|(S^-S^-\rho-2S^-\rho S^-+\rho S^-S^-)\exp[2i(\omega_p-\omega_1)t],\quad \psi=\chi-\phi/2\ .\end{aligned} \tag{2.15}$$

This is the key equation which describes the dynamics of a collection of atoms interacting with broadband squeezed radiation and with an external drive. In the following sections we discuss the steady-state solution of (2.15). We will assume for simplicity that $\omega_p=\omega_1$ (or $\omega_p=\omega_0$ if there is no external drive). Some results for the case when $\omega_p\neq\omega_0$ will be given in Sec. VI. Note that even in the absence of a drive the master equation (2.12) implies that the populations $\langle S,m|\rho|S,m\rangle$ are coupled to the two-photon coherences $\langle S,m-2|\rho|S,m\rangle$,

$\langle S,m|\rho|S,m+2\rangle$, etc., i.e., the squeezed vacuum two-photon coherences. Clearly the transition probabilities now depend on two-photon coherences.

III. EXACT STEADY-STATE SOLUTION IN THE ABSENCE OF COHERENT DRIVE

We first consider the type of steady state that can be generated by a collection of two-level atoms interacting with broadband squeezed radiation. In the absence of the coherent drive Eq. (2.15) reduces to

$$\begin{aligned}\dot\rho=&-\Gamma(1+\bar n)(S^+S^-\rho-2S^-\rho S^++\rho S^+S^-)\\&-\Gamma\bar n(S^-S^+\rho-2S^+\rho S^-+\rho S^-S^+)\\&-\Gamma|m|(S^+S^+\rho-2S^+\rho S^++\rho S^+S^+)\\&-\Gamma|m|(S^-S^-\rho-2S^-\rho S^-+\rho S^-S^-)\ ,\end{aligned}\tag{3.1}$$

where we have assumed that the frequency of the field pumping the squeezed bath is the same as the atomic frequency. We treat the case of idealized squeezed radiation whence

$$|m|^2=\bar n(\bar n+1)\ ,\tag{3.2}$$

and set

$$\bar n=\sinh^2(|\xi|)\ .\tag{3.3}$$

For $\xi=0$, we have the interaction with the vacuum of the radiation field. The master equation (3.1) can now be written in the form

$$\dot\rho=-2\Gamma(R_z^\dagger R_z\rho-2R_z\rho R_z^\dagger+\rho R_z^\dagger R_z)\sinh(2|\xi|)\ ,\tag{3.4}$$

where the non-Hermitian operator R_z is defined by

$$\begin{aligned}&R_z=([S^-\cosh(|\xi|)+S^+\sinh(|\xi|)]/\sqrt{2\sinh(2|\xi|)}\ ;\\&S^-=[R_z\cosh(|\xi|)-R_z^\dagger\sinh(|\xi|)]\sqrt{2\sinh(2|\xi|)}\ .\end{aligned}\tag{3.5}$$

Clearly the steady-state solution of (3.4) depends on the existence of the inverse of the operator R_z. The steady state is given by

$$\rho=D(R_z^{-1})(R_z^\dagger)^{-1}\ ,\tag{3.6}$$

provided that the determinant of the operator matrix R_z is nonzero. Thus if the eigenvalue equation

$$R_z|\Psi_p\rangle=\lambda_p|\Psi_p\rangle\tag{3.7}$$

has a solution $|\Psi_0\rangle$ for $\lambda=0$, then the solution of (3.4) is

$$\rho=|\Psi_0\rangle\langle\Psi_0|\ .\tag{3.8}$$

Thus the steady state can be a mixed state or a pure state depending on the existence or otherwise of the eigenstate $|\Psi_0\rangle$ of the operator R_z. The properties of the eigenstates of R_z have been studied in the literature[10,11] and for completeness we list some important results in Appendix A. Equation (A7) shows that if N is even then

$$|\Psi_0\rangle=A_0\exp(\theta S^z)\exp(-i\pi S^y/2)|0\rangle\ ,\tag{3.9}$$

where $|0\rangle$ is the eigenvalue of S^z corresponding to the eigenvalue $m=0$ and A_0 is the normalization constant. For an odd number of atoms the solution is given by (3.6) which can be written in terms of eigenstates of $|\psi_p\rangle$ as follows:

$$\rho=D(R_z^{-1})\sum_{p=-S}^{S}|\Psi_p\rangle\langle\Phi_p|(R_z^\dagger)^{-1}\sum_{q=-S}^{S}|\Phi_q\rangle\langle\Psi_q|\ ,\tag{3.10}$$

where Eq. (A4) has been used. On simplification (3.10) reduces to

$$\rho=D\sum_{p,q}\frac{\langle\Phi_p|\Phi_q\rangle|\Psi_p\rangle\langle\Psi_q|}{pq}\ ,\qquad -N/2\le p,\quad q\le N/2,\quad N=\text{odd}\ .\tag{3.11}$$

It should be borne in mind that the overlap $\langle\Phi_p|\Phi_q\rangle$ is nonzero for all p and q.

Let us now compute the matrix elements of (3.9)

$$\begin{aligned}\Psi_{0m}&\equiv\langle m|\Psi_0\rangle\\&=A_0\exp(m\theta)\langle m|\exp(-i\pi S^y/2)|0\rangle\\&=A_0\exp(m\theta)d_{m0}^{(S)}(\pi/2)\ ,\end{aligned}\tag{3.12}$$

where the coefficient $d_{m0}^{(S)}(\pi/2)$ is given by[12]

$$d_{m0}^{(S)}(\pi/2)=\frac{[(S+m)!(S-m)!S!S!]^{1/2}}{2^S}\sum_{p=m}^{S-m}\frac{(-1)^p}{(S-p)!p!(p-m)!(S+m-p)!}\ .\tag{3.13}$$

Note that by changing the variable of summation p to $S+m-p$ and adding the resulting expression to (3.13) we can write

$$d_{m0}^{(S)}(\pi/2)=\frac{[(S+m)!(S-m)!S!S!]^{1/2}}{2^{S+1}}\sum_{p=m}^{S-m}\frac{(-1)^p[1+(-1)^{S+m-2p}]}{(S-p)!p!(p-m)!(S+m-p)!}\ .\tag{3.14}$$

Clearly

$$d_{m0}^{(S)}(\pi/2)=0\quad\text{if }S+m\text{ is odd}\ ,\tag{3.15}$$

and thus

$$\psi_{0m}=0\quad\text{if }S+m\text{ is odd}\ .\tag{3.16}$$

Thus we have proved that for even N the steady state is such that the states $|S,m\rangle$, $m=-S+(2p+1)$, $p=0,1,2,\ldots$ are unoccupied, i.e., the squeezed bath leads to pairwise excitation of atoms.

Note further that the states $|\psi_p\rangle$ have the property that

$$(\Delta S^x)^2(\Delta S^y)^2=|\langle S^z\rangle|/2\ ,\tag{3.17}$$

i.e., these states lead to equality sign in the Heisenberg uncertainty relation. Other mean values can be obtained from (3.12); e.g., the inversion is given by

$$\langle S^z \rangle = A_0^2 \sum_{m=-S}^{S} [m|d_{m0}^{(S)}(\pi/2)|^2 \exp(2m\theta)] . \quad (3.18)$$

Thus for even N we have proved the following results.

(i) The steady state is an eigenstate of the operator $\cosh(|\xi|)S^- + \sinh(|\xi|)S^+$ with zero eigenvalue.

(ii) The x component S^x is squeezed.

(iii) The atoms are excited in pairs. This provides two-photon coherences

$$\rho_{m,m+2} = \Psi_{0m}\Psi_{0,m+2}^* \neq 0 . \quad (3.19)$$

These properties of a collection of atoms are equivalent to the corresponding properties of the squeezed vacuum of the photons,

$$|\xi\rangle = \exp[(-\xi a^{+2} + \xi^* a^2)/2]|0\rangle , \quad (3.20)$$

which is an eigenstate of $\cosh(|\xi|)a + \sinh(|\xi|)a^\dagger$ and which leads to squeezing in the x quadrature. Further, (3.20) leads to two-photon excitations[13] since ρ is a superposition of the states $|0\rangle$, $|2\rangle$, $|4\rangle, \ldots$, etc. In view of the foregoing we can interpret $|\psi_0\rangle$ as the squeezed vacuum for a collection of an even number of two-level atoms. The foregoing analysis also shows that the interaction of a collection of two-level atoms with broadband squeezed radiation prepares the atoms in the squeezed vacuum state.

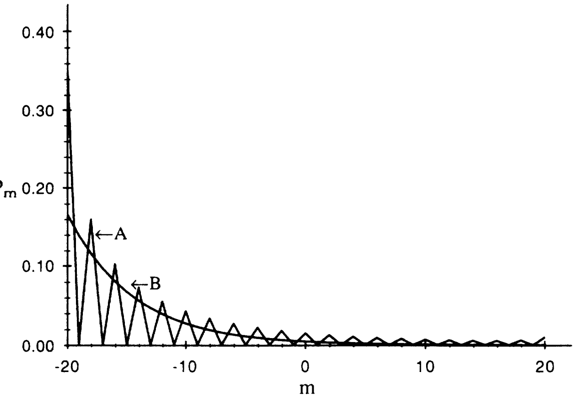

FIG. 1. The probability p_m of occupation of collective atomic states $|m\rangle$ as a function of m for no external drive and for $\Delta=0$. Curve A is for the atoms interacting with squeezed bath with $\bar{n}=5$ and curve B is for the atoms interacting with an ordinary thermal reservoir having a mean number of five photons.

The situation is different for an odd number of atoms. For example, for $N=1$, the steady state is the same as in the absence of squeezed broadband radiation. The matrix elements of ρ in the basis $|S,m\rangle$ can be obtained by using

$$\langle m|\Psi_p\rangle = A_m \exp(m\theta) d_{mp}^{(S)}(\pi/2) \quad (3.21)$$

where[12]

$$d_{mp}^{(S)}(\pi/2) = \frac{[(S+m)!(S-m)!(S+p)!(S=p)!]^{1/2}}{2^S} \sum_{q=-S}^{S} \frac{(-1)^q}{(S-p-q)!q!(q+p-m)!(S+m-q)!} . \quad (3.22)$$

In Fig. 1 we exhibit the nature of the atomic excitation probabilities p_m defined by

$$p_m = \langle S,m|\rho|S,m\rangle . \quad (3.23)$$

Note that the range of permitted values of m is $-S < m < S$. We compare the probability distribution (3.23) with that obtained by replacing the squeezed radiation by thermal radiation, i.e., by setting $m=0$ in Eq. (3.1) in which case

$$\rho_{\text{th}} \propto \exp(-\beta S^z), \quad \bar{n} = [(\exp(\beta)-1)]^{-1} \quad (3.24)$$

and the corresponding probability distribution is given by[13]

$$p_m = \left[\frac{\bar{n}}{\bar{n}+1}\right]^{S+m} \left\{(1+\bar{n})\left[1-\left[\frac{\bar{n}}{\bar{n}+1}\right]^{2S+1}\right]^{-1}\right\} . \quad (3.25)$$

The figure shows clear distinction between the steady states obtained in the two cases: in the case of the atoms interacting with squeezed vacuum p_m exhibits oscillations. Similar oscillations have been predicted for the case of the squeezed states of a photon field.[14,15] Note that Fig. 1 is for an even number of atoms. However, the plot of p_m as a function of m for an odd number of atoms has a behavior similar to that shown in Fig. 1 of Ref. 5.

IV. EXACT STEADY STATE FOR A SYSTEM DRIVEN BY COHERENT FIELD ON RESONANCE

We next examine the type of nonequilibrium steady states that can result if the atomic system is driven by a coherent field. On introducing the operators (3.5), Eq. (2.15) with $\Delta=0$ can be written as

$$\dot{\rho} = -i\sqrt{2}\sinh(2|\xi|)|\Omega|[[\cosh(|\xi|)\exp(-i\psi) - \sinh(|\xi|)\exp(i\psi)]R_z + \text{H.c.}, \rho] - 2\Gamma(R_z^\dagger R_z \rho - 2R_z \rho R_z^\dagger + \rho R_z^\dagger R_z)\sinh(2|\xi|) , \quad (4.1)$$

where ψ is the phase difference

$$\psi=\chi-\phi/2, \quad m=|m|\exp(i\phi), \quad \Omega=|\Omega|\exp(i\chi) . \tag{4.2}$$

We can also rewrite (4.1) in the following useful form:

$$\dot{\rho}=-2\Gamma\sinh(2|\xi|)(\Lambda^{\dagger}\Lambda\rho-2\Lambda\rho\Lambda^{\dagger}+\rho\Lambda^{\dagger}\Lambda) , \tag{4.3}$$

where Λ is the operator defined by

$$\Lambda=R_z+i|\Omega|[\cosh(|\xi|)\exp(i\psi)-\sinh(|\xi|)\exp(-i\psi)]/[\Gamma\sqrt{2\sinh(2|\xi|)}] . \tag{4.4}$$

The density matrix equation is now in a form whose steady-state solution can be written by inspection as

$$\rho=D\Lambda^{-1}(\Lambda^{+})^{-1} \tag{4.5}$$

provided that Λ^{-1} exists. The existence of Λ^{-1} depends on the eigenvalues of Λ which are given by

$$m+i|\Omega|[\cosh(|\xi|)\exp(i\psi)-\sinh(|\xi|)\exp(-i\psi)]/[\Gamma\sqrt{2\sinh(2|\xi|)}] .$$

If there exist values such that

$$m_0+i|\Omega|[\cosh(|\xi|)\exp(i\psi)-\sinh(|\xi|)\exp(-i\psi)]/[\Gamma\sqrt{2\sinh(2|\xi|)}]=0 , \tag{4.6}$$

then the steady-state solution will be

$$\rho=D|\Psi_{m_0}\rangle\langle\Psi_{m_0}| . \tag{4.7}$$

For $\psi=\pm\pi/2$, (4.6) leads to

$$m_0=\pm|\Omega|\exp(|\xi|)/[\Gamma\sqrt{2\sinh(2|\xi|)}] . \tag{4.8}$$

Thus the condition (4.6) can be satisfied for $|\Omega|$ such that

$$|\Omega|\le S\Gamma\sqrt{2\sinh(2|\xi|)}\exp(-|\xi|) . \tag{4.9}$$

Therefore we have proved that the steady state is given by (4.5) except for a set of $N+1$ discrete values of $|\Omega|$ given by (4.8). The eigenfunction expansion of (4.5) is

$$\rho=D\sum_{p,q=-S}^{S}(p+if)^{-1}(q-if^{*})^{-1}|\Psi_p\rangle\langle\Psi_q|\langle\Phi_p|\Phi_q\rangle , \tag{4.10}$$

where

$$f=|\Omega|[\cosh(|\xi|)\exp(i\psi) -\sinh(|\xi|)\exp(-i\psi)]/\Gamma\sqrt{2\sinh(2|\xi|)} . \tag{4.11}$$

For $\xi=0$ the solution (4.5) goes over to the standard result.[16–18] The expectation values of the dipole moment operator and the inversion operator can be obtained by expressing these in terms of $R_{\pm}$, R_z, $R_z^{\dagger}$, etc. For example, using (3.5) we get

$$\langle S^{-}\rangle=D\sum_{p,q=-S}^{S}(p+if)^{-1}(q-if^{*})^{-1}\langle\Psi_q|\Psi_p\rangle \times\langle\Phi_p|\Phi_q\rangle\sqrt{2\sinh(2|\xi|)} \times[p\cosh(|\xi|)-q\sinh(|\xi|)] . \tag{4.12}$$

Discussion of the numerical results will be given in Sec. V.

V. EXACT STEADY STATE FOR A SYSTEM DRIVEN BY COHERENT FIELD OFF RESONANCE

In this section we derive the exact steady-state solution of Eq. (2.15) when the external field is off resonance with the atoms. Unlike the preceding case such a solution is more difficult to derive. The full density matrix in terms of the R operators reads

$$\dot{\rho}=i(\Gamma\Delta_0/2)[R_{+}+R_{-},\rho]-i\sqrt{2\sinh(2|\xi|)}|\Omega|[[\cosh(|\xi|)\exp(-i\psi)-\sinh(|\xi|)\exp(i\psi)]R_z+\text{H.c.},\rho] -2\Gamma(R_z^{\dagger}R_z\rho-2R_z\rho R_z^{\dagger}+\rho R_z^{\dagger}R_z)\sinh(2|\xi|), \quad \Delta_0=\Delta/\Gamma . \tag{5.1}$$

We expand the steady-state solution in terms of the eigenstates $|\Psi_m\rangle$ and $|\Phi_m\rangle$,

$$\rho=\sum_{m,n=-S}^{S}C_{mn}|\Psi_m\rangle\langle\Psi_n|\langle\Phi_m|\Phi_n\rangle . \tag{5.2}$$

We will next derive a recursion relation for C_{mn}'s. The details are given in Appendix B where it is shown that

$$C_{mn}=C_mC_n^{*} , \tag{5.3}$$

where

$$(m-1+if^{*}-i\Delta_0/2)C_{m-1}=(m+if^{*}+i\Delta_0/2)C_m , \tag{5.4}$$

$$(m+1+if^{*}+i\Delta_0/2)C_{m+1}=(m+if^{*}-i\Delta_0/2)C_m . \tag{5.5}$$

These recursion relations are easily solved in terms of one

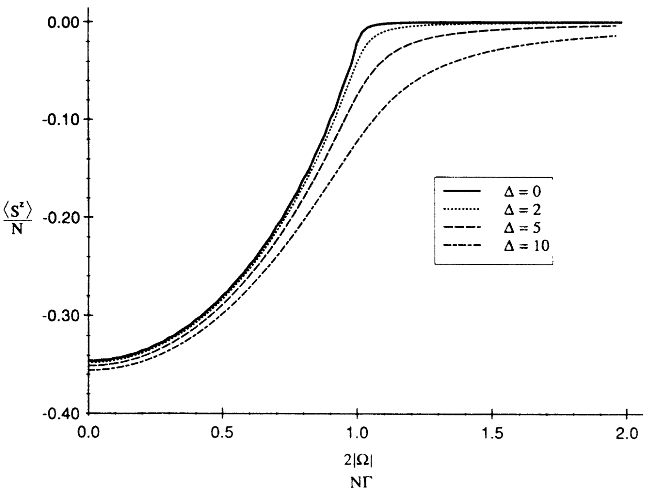

FIG. 2. The atomic population inversion $\langle S^z \rangle / N$ as a function of $2|\Omega|/N\Gamma$ for $\psi=\pi/2$, $\bar{n}=5$, and for different values of Δ.

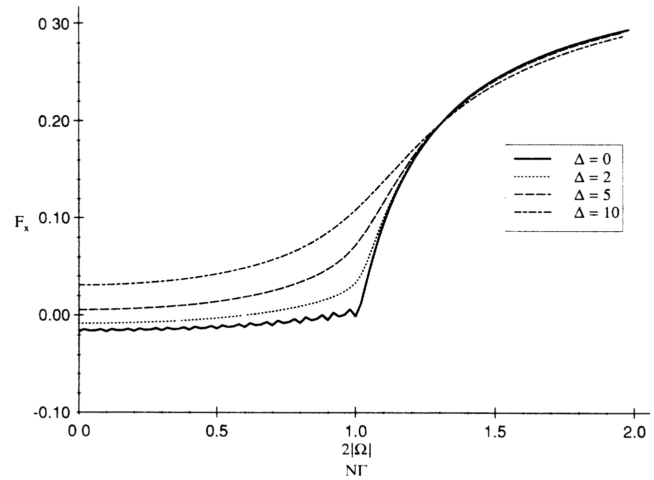

FIG. 4. The squeezing in the x component, F_x, of the emitted radiation as a function of $2|\Omega|/N\Gamma$ for $\psi=\pi/2$, $\bar{n}=5$, and for different values of Δ.

unknown which is fixed by the normalization condition

$$1=\sum_{m,n=-S}^{S} C_{mn}\langle \Psi_n|\Psi_m\rangle\langle \Phi_m|\Phi_n\rangle \ . \tag{5.6}$$

Note that if the phase ψ, field strength $|\Omega|$, squeezing parameter ξ, and the detuning are chosen such that for $m=p=$integer,

$$p+\mathrm{Im}(f)=0, \quad \mathrm{Re}(f)=\Delta_0/2 \ , \tag{5.7}$$

then

$$C_m=0, \quad \forall m>p \ . \tag{5.8}$$

Note further that for certain integer values of m one may encounter a pole in C_m with a small imaginary part, i.e.,

$$m+1+\mathrm{Im}(f)\approx 0, \quad \mathrm{Re}(f)\approx -\Delta_0/2 \ . \tag{5.9}$$

Clearly each of the conditions (5.7) and (5.9) leads to a set of discrete points in the $(\Delta_0, |\Omega|)$ plane.

In Figs. 2–4 we show the detailed characteristics of the steady-state solution (5.2)–(5.5) for given values of $\bar{n}(=5)$, $\psi(=\pi/2)$, and different values of Δ. We evaluate a number of expectation values like the mean inversion $\langle S^z\rangle/N$ (Fig. 2), the intensity $I=\langle S^+S^-\rangle/N^2$ (Fig. 3) of the emitted radiation, the parameters F_x (Fig. 4) and F_y which characterize the amount of squeezing in the two quadratures of the field emitted by the collective system

$$F_x=[(\Delta S_x)^2-|\langle S^z\rangle|/2]/S^2 \ ,$$
$$F_y=[(\Delta S_y)^2-|\langle S^z\rangle|/2]/S^2 \ . \tag{5.10}$$

It may be mentioned that the degree of squeezing may also be given in terms of parameter $D_i=2(\Delta S_i)^2/|\langle S_z\rangle|$ $(i=x,y)$ in which case $D_i<1$ implies squeezing in the ith quadrature. The oscillations are observed around the values of $|\Omega|$ for which condition (4.8) is satisfied. Note

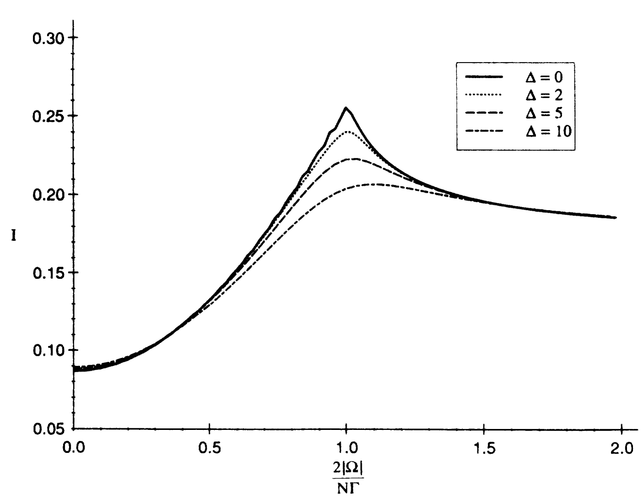

FIG. 3. The intensity I of the emitted radiation as a function of $2|\Omega|/N\Gamma$ for $\psi=\pi/2$, $\bar{n}=5$, and for different values of Δ.

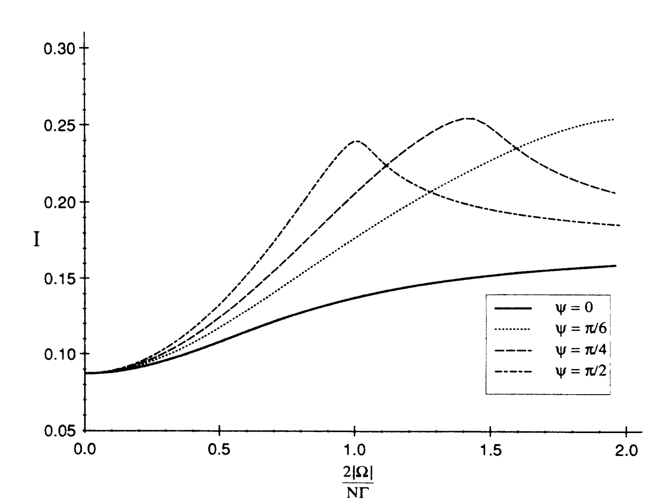

FIG. 5. The intensity I of the emitted radiation as a function of $2|\Omega|/N\Gamma$ for $\Delta=2$, $\bar{n}=5$, and for different values of ψ.

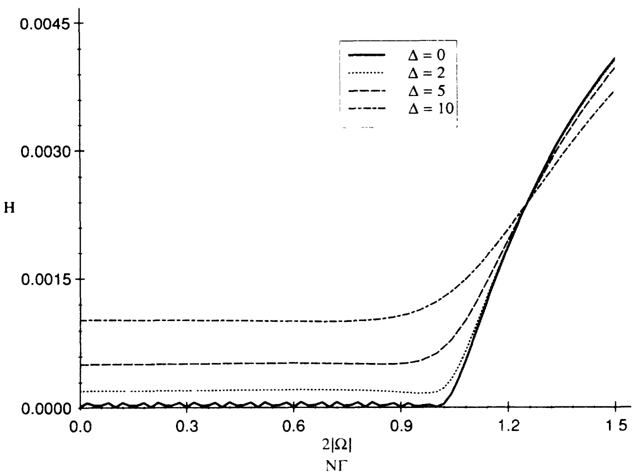

FIG. 6. The parameter $H=[(\Delta S^x)^2(\Delta S^y)^2-|\langle S^z\rangle|^2/4]/S^2$ as a function of $2|\Omega|/N\Gamma$ for $\psi=\pi/2$, $\bar{n}=5$, and for different values of Δ.

that the population inversion, intensity of emitted radiation, and its squeezing reduce with an increase in the value of Δ. In Fig. 5 we show the behavior of I for $\bar{n}=5$, $\Delta=2$, and for different values of ψ. In Fig. 6 we have plotted the quantity

$$H=[(\Delta S^x)^2(\Delta S^y)^2-|\langle S^z\rangle|^2/4]/S^2\ , \tag{5.11}$$

which shows the extent of deviation from the equality in the Heisenberg uncertainty relation for the uncertainty in the simultaneous measurement of S^x and S^y. Note that if $H=0$ then the equality sign holds in the Heisenberg uncertainty relation. This would be the case if the atomic system happens to be in a state $|\psi_p\rangle$ which is an eigenstate of R_z defined by (3.5).

VI. EXACT STEADY-STATE SOLUTION OF EQ. (2.12) FOR $\omega_p \neq \omega_0$

We finally consider the emission from atoms interacting with squeezed vacuum *alone*. We generalize the results of Sec. III to the case when the frequency of the field pumping the squeezed vacuum and the frequency of the atoms are different: $\omega_p\neq\omega_0$. It should be borne in mind that (2.12) is written in the interaction picture with respect to the unperturbed atomic Hamiltonian $\omega_0 S^z$. We now write the equation in the interaction picture obtained by choosing the unperturbed Hamiltonian as $\omega_p S^z$. The density matrix equation now reads

$$\begin{aligned}\dot{\rho}=&-i\Gamma\delta_0[S^z,\rho]-\Gamma(1+\bar{n})(S^+S^-\rho-2S^-\rho S^++\rho S^+S^-)-\Gamma\bar{n}(S^-S^+\rho-2S^+\rho S^-+\rho S^-S^+)\\&-\Gamma|m|(S^+S^+\rho-2S^+\rho S^++\rho S^+S^+)-\Gamma|m|(S^-S^-\rho-2S^-\rho S^-+\rho S^-S^-),\quad \delta_0=(\omega_0-\omega_p)/\Gamma\ ,\end{aligned} \tag{6.1}$$

which can be written in terms of R_z operators as

$$\begin{aligned}\dot{\rho}=&i\Gamma\delta_0[R_++R_-,\rho]\\&-2\Gamma[R_z^\dagger R_z\rho-2R_z\rho R_z^\dagger+\rho R_z^\dagger R_z)\sinh(2|\xi|)\ .\end{aligned} \tag{6.2}$$

Note that this is a special case of (5.1) with

$$\Delta_0\rightarrow\delta_0,\quad |\Omega|=0\ , \tag{6.3}$$

and hence its solution can be written as

$$\rho=\sum_{m,n}C_mC_n^*|\Psi_m\rangle\langle\Psi_n|\langle\Phi_m|\Phi_n\rangle\ , \tag{6.4}$$

with

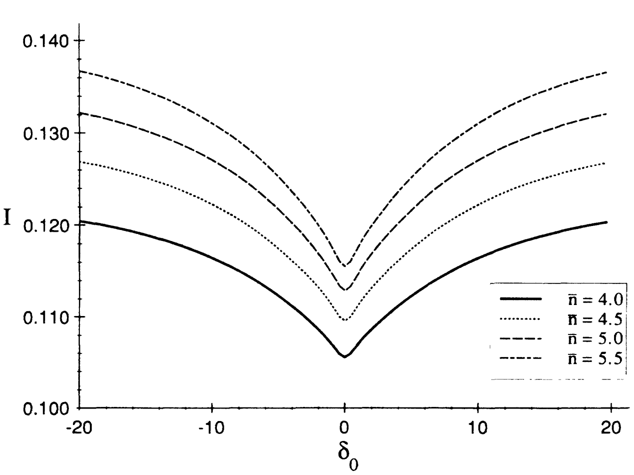

FIG. 7. The intensity I of the emitted radiation for no external drive as a function of the detuning δ_0 between the atomic transition frequency and the frequency of the pump driving the squeezed bath for different values of $\bar{n}$.

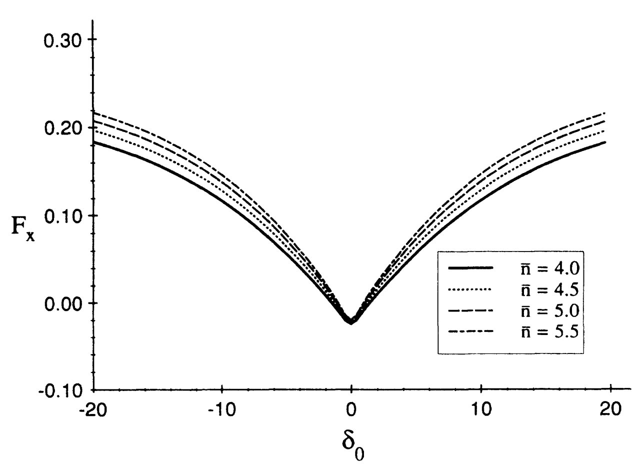

FIG. 8. The squeezing F_x in the x component of the emitted radiation for no external drive as a function of the detuning δ_0 between the atomic transition frequency and the frequency of the pump driving the squeezed bath for different values of $\bar{n}$.

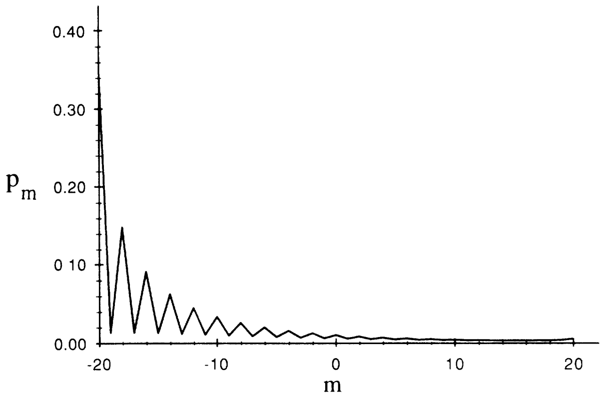

FIG. 9. The probability p_m of occupation of collective atomic states $|m\rangle$ as a function of m for no external drive and for $\delta_0=2$ where δ_0 is the detuning between the atomic transition frequency and the frequency of the pump driving the squeezed bath.

$$(m-1-i\delta_0/2)C_{m-1}=(m+i\delta_0/2)C_m \; . \tag{6.5}$$

In Figs. 7 and 8 we show some typical properties of nonequilibrium steady state [(6.4), (6.5)]. We have plotted the intensity of emitted radiation I and the squeezing F_x as a function of δ_0 for different values of $\bar{n}$. With an increase in $\bar{n}$ there is a decrease in I and F_x. In Fig. 9 we exhibit the behavior of p_m, the probability of occupation of different collective atomic states, as a function of m for a nonzero value of $\delta_0=2$. Note that in this case p_m exhibits oscillations but there is no pairwise excitation. This is in contrast with the case of $\delta_0=0$ (Fig. 1) for which there is pairwise excitation of the collective atomic states.

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APPENDIX A: SOLUTION OF THE EIGENVALUE EQUATION (3.7)

Consider the non-Hermitian eigenvalue equations

$$R_z|\Psi_p\rangle=\lambda_p|\Psi_p\rangle \; , \tag{A1}$$

$$R_z^\dagger|\Phi_p\rangle=\tilde{\lambda}_p|\Phi_p\rangle \; . \tag{A2}$$

From (A1) and (A2) it immediately follows that

$$\lambda_p=\tilde{\lambda}_p^*; \quad \langle\Phi_p|\Psi_q\rangle=0 \;\text{ if } \lambda_p\neq\lambda_q \; . \tag{A3}$$

Thus $|\Psi_p\rangle$ and $|\Phi_p\rangle$ form a biorthogonal set and these sets are complete, i.e.,

$$\sum_p|\Psi_p\rangle\langle\Phi_p|=\sum_p|\Phi_p\rangle\langle\Psi_p|=1 \; . \tag{A4}$$

Rashid[11] has shown that the eigenstates $|\psi_p\rangle$ and $|\phi_p\rangle$ can be written in terms of the eigenstates $|S,m\rangle$ of S^2 and S^z as follows:

$$|\Psi_m\rangle=A_m\exp(\theta S^z)\exp(-i\pi S^y/2)|m\rangle \; , \tag{A5}$$

$$|\Phi_m\rangle=B_m\exp(-\theta S^z)\exp(-i\pi S^y/2)|m\rangle \; , \tag{A6}$$

$$\lambda_m=m=\tilde{\lambda}_m, \quad m=-S,-S+1,\ldots,S-1,S \; , \tag{A7}$$

$$\exp(2\theta)=\tanh(2|\xi|), \quad |S,m\rangle\equiv|m\rangle \; , \qquad S^y=(S^+-S^-)/2i \; . \tag{A8}$$

Using (A5) and (A6) and the completeness relation for the states $|S,m\rangle$ the relation (A4) is easily verified.

It is clear from the above that the zero eigenvalue exists if S is an integer. Since $S=N/2$, where N is the number of atoms, the zero eigenvalue exists if N is even.

It is also possible to introduce the raising and lowering operators

$$R_\pm=\mp[\cosh(|\xi|)-\sinh(|\xi|)]S_x \pm iS^y\sqrt{2}\sinh(2|\xi|)-S^z, \quad R_-\neq R_+^\dagger \tag{A9}$$

and their adjoints

$$R_\pm^\dagger=\mp[\cosh(|\xi|)-\sinh(|\xi|)]S_x \mp iS^y/\sqrt{2}\sinh(2|\xi|)-S^z \; . \tag{A10}$$

Note that R_z, $R_\pm$ satisfy the angular momentum commutation algebra

$$[R_+,R_-]=2R_z, \quad [R_z,R_\pm]=\pm R_\pm \; . \tag{A11}$$

The operators (A9) and (A10) have the properties similar to those of $S^\pm$:

$$R_\pm|\Psi_m\rangle=\sqrt{(S\mp m)(S\pm m+1)}|\Psi_{m\pm1}\rangle \; , \tag{A12}$$

$$R_\pm^\dagger|\Phi_m\rangle=\sqrt{(S\pm m)(S\mp m+1)}|\Phi_{m\mp1}\rangle \; . \tag{A13}$$

The states $|\Psi_m\rangle$ satisfy the equality sign in Heisenberg uncertainty relation

$$(\Delta S^x)^2(\Delta S^y)^2=\tfrac{1}{4}|\langle S^z\rangle|^2 \; . \tag{A14}$$

As a matter of fact,

$$(\Delta S^x)^2-\tfrac{1}{2}|\langle S^z\rangle|=-\tfrac{1}{2}|\langle S^z\rangle|[1-\exp(-|\xi|)]\equiv F_x \; , \tag{A15}$$

$$(\Delta S^y)^2-\tfrac{1}{2}|\langle S^z\rangle|=-\tfrac{1}{2}|\langle S^z\rangle|[1-\exp(|\xi|)]\equiv F_y \; , \tag{A16}$$

which shows that the x quadrature is squeezed ($F_x<0$).

APPENDIX B: DERIVATION OF EQUATIONS (5.3)–(5.5)

Here we present briefly the derivation of the results (5.3)–(5.5). In order to derive the equations obeyed by C_{mn}, we have to know the action of the operators R_z, $R_z^\dagger$

on $|\psi_m\rangle$. Note that $R_z|\psi_m\rangle = m|\psi_m\rangle$. However, the action of $R_z^\dagger$ on $|\psi_m\rangle$ is rather complicated. We express $R_z^\dagger$ in terms of $R_\pm$ and R_z:

$$R_z^\dagger = [R_+ - R_- + 2\cosh(2|\xi|)R_z]/[2\sinh(2|\xi|)] , \quad \text{(B1)}$$

and hence on using (A12) we get

$$R_z^\dagger|\Psi_m\rangle = \frac{1}{2\sinh(2|\xi|)}[\sqrt{(S-m)(S+m+1)}|\Psi_{m+1}\rangle - \sqrt{(S-m)(S+m+1)}|\Psi_{m-1}\rangle + 2m\cosh(2|\xi|)|\Psi_m\rangle] . \quad \text{(B2)}$$

Thus using (5.2) and (B2) we get the equation

$$R_z^\dagger\rho = \frac{1}{2\sinh(2|\xi|)}\sum_{m,n}[\sqrt{(S+m)(S-m+1)}C_{m-1,n}\langle\Phi_{m-1}|\Phi_n\rangle - C_{m+1,n}\langle\Phi_{m+1}|\Phi_n\rangle\sqrt{(S-m)(S+m+1)} + 2m\cosh(2|\xi|)\langle\Phi_m|\Phi_n\rangle C_{mn}]|\Psi_m\rangle\langle\Psi_n| . \quad \text{(B3)}$$

Note further that

$$\sqrt{(S+m)(S-m+1)}\langle\Phi_{m-1}|\Phi_n\rangle = \langle\Phi_m|R_+|\Phi_n\rangle , \quad \text{(B4)}$$

$$\sqrt{(S-m)(S+m+1)}\langle\Phi_{m+1}|\Phi_n\rangle = \langle\Phi_m|R_-|\Phi_n\rangle , \quad \text{(B5)}$$

which follows from Eq (A13). We can further simplify (B4) and (B5) by using

$$R_+ = -\cosh(2|\xi|)R_z + \sinh(2|\xi|)R_z^\dagger - S^z , \quad \text{(B6)}$$

$$R_- = \cosh(2|\xi|)R_z - \sinh(2|\xi|)R_z^\dagger - S^z , \quad \text{(B7)}$$

which follow from (A9) and the definitions of R_z and $R_z^\dagger$. The matrix elements (B4) and (B5) are now found to be

$$\langle\Phi_m|R_+|\Phi_n\rangle = [-\cosh(2|\xi|)m + \sinh(2|\xi|)n] \times \langle\Phi_m|\Phi_n\rangle - \langle\Phi_m|S^z|\Phi_n\rangle , \quad \text{(B8)}$$

$$\langle\Phi_m|R_-|\Phi_n\rangle = [\cosh(2|\xi|)m - \sinh(2|\xi|)n] \times \langle\Phi_m|\Phi_n\rangle - \langle\Phi_m|S^z|\Phi_n\rangle . \quad \text{(B9)}$$

On using (B4), (B5), (B8), and (B9), Eq. (B3) can be written in terms of $|\Psi_m\rangle\langle\Psi_n|$, $\langle\Phi_m|\Phi_n\rangle$, and the matrix elements $\langle\Phi_m|S^z|\Phi_n\rangle$. The term $R_z\rho R_z^\dagger$ in the master equation leads to a simple expression

$$R_z\rho R_z^\dagger = \sum_{m,n}[mnC_{mn}\langle\Phi_m|\Phi_n\rangle|\Psi_m\rangle\langle\Psi_m|] . \quad \text{(B10)}$$

Thus we can simplify the external field and the squeezed radiation terms in (5.1). We next consider the detuning term. As an example we calculate

$$\begin{aligned} R_+\rho &= \sum_{m,n}[C_{mn}\sqrt{(S-m)(S+m+1)}\langle\Phi_m|\Phi_n\rangle|\Psi_{m+1}\rangle\langle\Psi_n|] \\ &= \sum_{m,n}[C_{m-1,n}\sqrt{(S+m)(S-m-1)}\langle\Phi_{m-1}|\Phi_n\rangle|\Psi_m\rangle\langle\Psi_n|] \\ &= \sum_{m,n}(C_{m-1,n}\langle\Phi_m|R_+|\Phi_n\rangle|\Psi_m\rangle\langle\Psi_n|) \end{aligned} \quad \text{(B11)}$$

where (B4) was used. On using (B8), (B11) reduces to

$$R_+\rho = \sum_{m,n}(C_{m-1,n}\langle\Phi_m|\Phi_n\rangle|\Psi_m\rangle\langle\Psi_n|)[-m\cosh(2|\xi|) + n\sinh(2|\xi|)] - \sum_{m,n}(C_{m-1,n}\langle\Phi_m|S^z|\Phi_n\rangle|\Psi_m\rangle\langle\Psi_n|) . \quad \text{(B12)}$$

On using (B3), (B10), (B12), and their complex conjugates in the master Eq. (5.1), we get the equation

$$\begin{aligned} &\sum_{m,n}(mC_{mn}\langle\Phi_m|\Phi_n\rangle|\Psi_m\rangle\langle\Psi_n|)\{-2m\cosh(2|\xi|) + 2n\sinh(2|\xi|) - 2i\sinh(2|\xi|)[f + \coth(2|\xi|)f^*]\} \\ &- \sum_{m,n}(\langle\Phi_m|\Phi_n\rangle|\Psi_m\rangle\langle\Psi_n|)\{C_{m-1,n}(m-1+if^*-i\Delta_0/2)[-2m\cosh(2|\xi|) + 2n\sinh(2|\xi|)]\} \\ &- \sum_{m,n}(\langle\Phi_m|\Phi_n\rangle|\Psi_m\rangle\langle\Psi_n|)\{C_{m+1,n}(m+1+if^*+i\Delta_0/2)[-2m\cosh(2|\xi|) + 2n\sinh(2|\xi|)]\} \\ &+ \sum_{m,n}(\langle\Phi_m|S^z|\Phi_n\rangle|\Psi_m\rangle\langle\Psi_n|)[C_{m-1,n}(m-1+if^*-i\Delta_0/2) - C_{m+1,n}(m+1+if^*+i\Delta_0/2)] + \text{H.c.} = 0 . \end{aligned} \quad \text{(B13)}$$

Clearly, Eq. (B13) is satisfied if we choose C_{mn}'s to satisfy

$$(m-1+if^{*}-i\Delta_0/2)C_{m-1,n}=(m+if^{*}+i\Delta_0/2)C_{mn}\ , \tag{B14}$$

$$(m+1+if^{*}+i\Delta_0/2)C_{m+1,n}=(m+if^{*}-i\Delta_0/2)C_{mn}\ , \tag{B15}$$

and the equations that follow from complex conjugation. These equations are equivalent to Eqs. (5.3)–(5.5).

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