Fluctuational transitions between stable states of a nonlinear oscillator driven by random resonant force

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Fluctuations of a nonlinear oscillator driven by an intense resonant field with fluctuating phase are considered in the region of bistability. The general expression for the probability of a fluctuational transition between stable states is found to logarithmic accuracy. In the weak-damping limit the probabilities are obtained in the explicit form including both the argument of the exponential and the preexponential factor. A simple general method of determining the latter for underdamped systems is suggested. The probability of an escape from a metastable state near the bifurcation point where this state disappears is analyzed. The low-frequency susceptibility of the oscillator is shown to have a peculiar structure in the region of parameters where the stationary populations of the stable states are of the same order of magnitude. The stochastic modulation of the phase of an oscillator due to coupling to a bath is considered and its consequences are compared to those of randomness of the phase of a driving field.

I. INTRODUCTION

Nonlinear systems subjected to a strong periodic field often have several stable states that correspond to forced vibrations differing in their amplitudes, phases, periods, etc. A large number of such systems is investigated presently in nonlinear optics (optical bi- and multistability, see, e.g., Ref. 1). A simple model system displaying bistability in a periodic field is a nonlinear oscillator. The frequency of eigenoscillations of a nonlinear oscillator depends on their amplitude. This gives rise to coexistence of two "self-consistent" regimes of forced vibrations of an oscillator in a certain range of the parameters of a resonant field. In the first regime the vibration amplitude is small and the respective eigenfrequency differs comparatively strongly from the field frequency ω (self-consistently "bad" resonance), while in the second regime the amplitude is large so that the eigenfrequency is close to ω (self-consistently "good" resonance).[2]

The model of an oscillator bistable in a resonant field is actual not only for optical bistability;[3] it directly describes the motion of a free electron in a magnetic field and in an intense resonant electric field which was investigated experimentally in Ref. 4 and also some other physical systems.[5]

In actual cases the field acting on a system is not strictly periodic. Fluctuations of various origin result inevitably in a smearing of the lines in the field frequency spectrum. When fluctuations are sufficiently intense (spectral lines are sufficiently broad), so that the root-mean-square displacements from the stable states of a field-driven nonlinear system exceed the distance between the states, the bistability of the system is not manifested. In essence, bistability can be observed clearly, provided that the fluctuations are so small that there are two strongly differing characteristic times in a system: the time τ_r of relaxation to a stable state of forced vibrations in the absence of the fluctuations and the inverse probability W^{-1} of fluctuational transitions between the stable states, with

$$W\tau_r \ll 1 \ . \tag{1}$$

When this inequality is fulfilled, the system for a time τ such that $W^{-1} \gg \tau \gg \tau_r$ stays with an overwhelming probability near that stable state in whose vicinity it has been brought as a result of the variation of external parameters, and thus hysteresis can be observed.

The analysis of bistability in a fluctuating field is therefore closely related to the problem of transitions between stable states. In contrast to classical thermally equilibrium systems weakly coupled to a bath where W is given by the Arrhenius law[6] while the stationary distribution is Gibbsian, for nonequilibrium systems there are no universal expressions for W and for the stationary distribution. The expressions depend not only on the intensity but also on other characteristics of noise acting on a system (cf. Ref. 7). For a nonlinear oscillator in an external field they were obtained earlier in case of the field presenting itself a superposition of a comparatively strong resonant monochromatic field and weak white noise (in some respects this case is similar to that of an oscillator subjected to the resonant monochromatic field and coupled to a bath, with the coupling energy being proportional to the oscillator coordinate).[5,8] Transitions between stable states and stationary distribution in a periodic field and additive weak noise were considered also for other nonlinear systems.[9]

In the present paper the transition probabilities are analyzed for a nonlinear oscillator driven by a resonant external field with random phase. Randomness of the phase variation in time is a feature of the fields generated by various sources including lasers, electronic generators, etc.[10] If, on the average, the phase is homogeneously distributed over the interval $[0,2\pi]$, then the average value of the field equals zero (although the intensity is finite). Therefore for a fast phase change as compared with the oscillator relaxation time, the oscillator perceives the field as effectively weak and bistability does not arise. At the same time for slow phase variation the bistability is possible, and the dependence of the transition probabilities on the intensity of phase fluctuations (that is, on the linewidth in the field frequency spectrum) is shown below to be of the activation type. A similar dependence was found numerically for a bistable system of another type.[11]

The motion of an oscillator in a resonant field depends, in fact, not on each of the phases (those of the oscillator and of the field), but on their difference. This difference can fluctuate due to fluctuations of both the phases. Fluctuations of the oscillator phase result from the oscillator-to-bath coupling.

In the present paper the microscopic theory of stochastic modulation of the oscillator phase is developed. This modulation is shown to correspond to transverse relaxation in terms of quantum theory, that is, to damping of the off-diagonal elements of the oscillator density matrix (strictly speaking, to that contribution to the damping decrements which is not connected with the oscillator energy relaxation) and to respective broadening of the peaks in the power and susceptibility spectra of the oscillator. This relaxation is extremely important for many physical vibrational systems, in particular, for localized vibrations in solids[12] (see also the surveys[5,13]). For an oscillator bistable in a strong resonant field the respective mechanism gives rise to transitions between the stable states.

This paper is organized as follows. In Sec. II we give the stochastic equations of motion for the "slow" (as compared to the field frequency) oscillator variables. In Sec. III the general expression for the transition probability is obtained to logarithmic accuracy for the case of weak phase fluctuations, and the criterion of bistability is given. In Sec. IV the transition probability including the preexponential factor and the quasistationary distribution are found in an explicit form in the limit of weak damping. In Sec. V we analyze the probability of an escape from a metastable state near the bifurcation point where this state coalesces with an unstable steady state and disappears (the saddle-node bifurcation). In Sec. VI the features of time correlation functions and of power spectra connected with a randomness of phase of the driving field are discussed. In Sec. VII we analyze the low-frequency susceptibility and the extremely narrow peak in it arising in the range of the kinetic phase transition. The Appendix contains microscopic theory of stochastic modulation of the phase of an oscillator coupled to a bath.

II. EQUATIONS OF MOTION IN A RANDOM RESONANT FIELD

The dynamics of the classical dissipating nonlinear oscillator in the external field is described by the equation

$$\frac{d^2q}{d\tau^2}+2\Gamma\frac{dq}{d\tau}+\omega_0^2q+\gamma q^3=F\cos[\omega_F\tau+\phi_F(\tau)]\ . \quad (2)$$

This equation is written for the archetypal model,[3-5] the Duffing oscillator with a linear friction (the friction force is proportional to velocity[14]). We suppose the friction coefficient Γ to be small and the external field to be resonant,

$$\Gamma,|\delta\omega| \ll \omega_F,\quad \delta\omega=\omega_F-\omega_0\ . \quad (3)$$

In addition, we assume that the field amplitude is not too large so that the oscillator nonlinearity is relatively small, $|\gamma|\langle q^2\rangle \ll \omega_F^2$, and that the phase of the field $\phi_F(\tau)$ is a random function slowly varying over the period $2\pi\omega_F^{-1}$ (that is, the characteristic width $\sim\langle|d\phi_F/d\tau|\rangle$ of the frequency spectrum of the field is small as compared with the midband frequency ω_F).

It is convenient under these conditions to transform from the fast oscillating variables, the coordinate q and the momentum $dq/d\tau$ of the oscillator, to the smooth variables, the dimensionless squared amplitude x and the phase θ,

$$\begin{aligned} q&=\left[\frac{8\omega_F|\delta\omega|}{3|\gamma|}\right]^{1/2}x^{1/2}\cos(\omega_F\tau-\theta+\phi_F)\ ,\\ \frac{dq}{d\tau}&=-\omega_F\left[\frac{8\omega_F|\delta\omega|}{3|\gamma|}\right]^{1/2}x^{1/2}\sin(\omega_F\tau-\theta+\phi_F)\ . \end{aligned} \quad (4)$$

Dimensionless equations of motion for x,θ with allowance for (2)–(4) are of the form

$$\begin{aligned} &\dot{x}\equiv\frac{dx}{dt}=2\frac{\partial g}{\partial\theta}-2\eta x,\quad \dot{\theta}\equiv\frac{d\theta}{dt}=-2\frac{\partial g}{\partial x}+\frac{d\phi_F}{dt}\ ,\\ &g\equiv g(x,\theta)=\tfrac{1}{4}[x\,\mathrm{sgn}(\gamma/\delta\omega)-1]^2\\ &\qquad\times\mathrm{sgn}(\delta\omega)-\sqrt{\beta x}\cos\theta,\quad t=|\delta\omega|\tau\ , \end{aligned} \quad (5)$$

where

$$\eta=\Gamma/|\delta\omega|,\quad \beta=3|\gamma|F^2(32\omega_F^3|\delta\omega|^3)^{-1}\ . \quad (6)$$

The dimensionless parameters η and β (6) are seen from Eqs. (2) and (3) to characterize the relative detuning of the field frequency ω_F from ω_0 and the relative field intensity. In the spirit of the standard averaging method we have neglected fast oscillating terms $\sim\exp[\pm i(\omega_F\tau-\theta+\phi_F)]$, $\exp[\pm 3i(\omega_F\tau-\theta+\phi_F)]$ in the right-hand sides (rhs) of Eqs. (5) (when integrated over $t=|\delta\omega|\tau\sim 1$ they give a correction $\sim|\delta\omega|/\omega_F\ll 1$).

The character of the solution of Eqs. (5) for x,θ depends on the values of the involved parameters η,β and on the character of the random function $d\phi_F/dt$. When the phase of the field ϕ_F is constant Eqs. (5) have three stationary solutions in the parameter range,[2,8]

$$\begin{aligned} &\eta<1/\sqrt{3},\quad \beta_B^{(1)}(\eta)<\beta<\beta_B^{(2)}(\eta)\quad(\gamma\delta\omega>0)\\ &\beta_B^{(1,2)}(\eta)=\tfrac{2}{27}[1+9\eta^2\mp(1-3\eta^2)^{3/2}]\ . \end{aligned} \quad (7)$$

Two of the solutions (those with smallest and largest x) are stable, they correspond to stable forced vibrations of the oscillator, while the third solution is unstable. The stable solutions correspond to stable foci (or nodes) on the phase plane (x,θ), the unstable one corresponds to the saddle point. The ranges of "attraction" of the system to different stable states are separated by the separatrix which passes through the saddle point. At $\beta=\beta_B^{(1,2)}(\eta)$ one of the stable states merges with the unstable one.

The bistability is obvious from the qualitative arguments given above and from Eqs. (5) to arise only for $\gamma\cdot\delta\omega>0$. Since Eqs. (5) for $\delta\omega<0$ may be put into the form they have for $\delta\omega>0$ by changing $g\rightarrow -g$, $\theta\rightarrow -\theta$, $d\phi_F/dt\rightarrow -d\phi_F/dt$, we shall suppose for the sake of concreteness that

$$\gamma>0, \quad \delta\omega>0 \ . \tag{8}$$

Fluctuations of the phase ϕ_F result in randomness of the motion of the oscillator. We shall consider this motion assuming fluctuations of $d\phi_F/dt$ (the field frequency fluctuations) to be described by a white noise $f(t)$ over the time scale $\tau\gg\omega_F^{-1}$,

$$\frac{d\phi_F}{dt}=f(t), \quad \langle f(t)\rangle=0 \ ,$$
$$\langle f(t)f(t')\rangle=2D\delta(t-t') \ . \tag{9}$$

In the model (9) the line in the power spectrum of the field is Lorentzian in shape with the maximum at ω_F and a halfwidth $D|\delta\omega|$.[10] Fluctuations of the type in Eq. (9) result directly from fluctuations of the frequency-controlling factors of the field source (say, capacitance or inductance in a contour of a radio-frequency generator, or refractive index of the laser cavity). The model (9) is applicable provided they are Gaussian (this is fulfilled usually) and their characteristic correlation time is much smaller than $|\delta\omega|^{-1},\Gamma^{-1}$. Well above threshold of generation fluctuations of other origins which fulfill these conditions result mainly in the fluctuations of the type (9) also.[10]

Equations (5) and (9) are shown in the Appendix to describe also the dynamics of the oscillator in the important case of phase fluctuations due to interaction between the oscillator and bath. The value of D in this case is determined by the intensity of the respective fluctuations [cf. Eq. (A10)].

III. GENERAL EXPRESSION FOR THE TRANSITION PROBABILITIES

Equations (5) and (9) have the form of the dimensionless equations of motion of a Brownian particle with a "coordinate" x and "momentum" θ; its Hamiltonian function is $2g(x,\theta)$, the "friction force" is $-2\eta x$ and the diffusion coefficient is D. Note, however, that friction enters the equation for the coordinate x. This feature and the explicit form of $g(x,\theta)$ (5) cause a substantial difference between the motion of the present system and that of a "true" Brownian particle in a static potential. In particular, the stable states in the limit of small η correspond to the minimum and the local maximum of $g(x,\theta)$, while the unstable stationary state corresponds to the inflection point of $g(x,\theta)$.

The Hamiltonian function $g(x,\theta)$ contains a single parameter β. For β not too close to the bifurcational values $\beta_B^{(1,2)}$ [Eq. (7)] the distances between the stationary states on the plane (x,θ) and the differences in the values of $g(x,\theta)$, for these states are of the order of unity. Therefore the character of motion depends mainly on the ratio of D to η. For

$$D\ll\eta \tag{10}$$

the oscillator within a dimensionless time $t\sim\eta^{-1}$ with an overwhelming probability approaches one or another stable state (depending on its initial state), while the transitions between stable states do not happen practically within such time.

In calculating the transition probabilities it is of the utmost importance to find the argument of the exponential in the expression for W. Rather a general approach to this problem is based[5,8,15] on the path-integral method in the theory of random processes.[16] This approach is convenient for systems driven not only by white noise, but by an arbitrary Gaussian noise. The idea lies in writing down the expression for the transition probability as a functional integral over the noise trajectories and calculating it by the steepest-descent method. The extreme trajectory describes the evolution of a random force in the course of its optimal fluctuation which results in bringing the system from the stable state occupied initially to the saddle point (see below). The determination of $\ln W$ is reduced thus to certain variational problem.[8]

By applying this method directly to the present problem we arrive at the following expression for the probability of a transition from the stable state f:

$$W=\text{const}\times\exp(-R/D), \quad R=\min\int_0^t dt\, L(x,\theta,\dot\theta) \ ,$$
$$L(x,\theta,\dot\theta)=\tfrac{1}{4}\left[\dot\theta+2\frac{\partial g}{\partial x}\right]^2, \quad \dot x=-2\eta x+2\frac{\partial g}{\partial\theta} \ . \tag{11}$$

The functional R should be minimized with respect to the trajectories $x(t),\theta(t)$ which satisfy the given equation for $\dot x$. The trajectories go from the stable state f [a focus or a node with the coordinates (x_f,θ_f) on the phase plane] to the saddle point s with the coordinates (x_s,θ_s) (see below),

$$x(0)\approx x_f, \quad \theta(0)\approx\theta_f, \quad x(t)\approx x_s, \quad \theta(t)\approx\theta_s \tag{12}$$

[the appropriate equalities (12) should be fulfilled with an accuracy to $\sim D^{1/2}$]. The minimum in Eq. (11) is taken also with respect to the duration of the motion t. Equation (11) is valid to logarithmic accuracy in D.

According to Eq. (11) the dependence of W on the phase diffusion coefficient D is of the "activation" type, and R plays a role of a characteristic "activation energy of the transition." The structure of the expression (11) may be understood if one takes into account that the probability density functional $P[f]$ which gives the probabilities of various realizations of the random force $f(t)$ determining the phase of the field ϕ_F [Eq. (9)] for $f(t)$ presenting a white noise is of the form[16]

$$P[f]=\exp\left[-\frac{1}{4D}\int dt\, f^2(t)\right] \ .$$

According to Eqs. (5) and (7) $(1/4D)f^2(t)$

$=(1/D)L(x,\theta,\dot{\theta})$, and thus the condition (11) that $\int dt\, L(x,\theta,\dot{\theta})$ be minimal is just the condition of the most probable (optimal) realization of $f(t)$ for the motion of the system from the stable state to the saddle point.

The variational problem (11) is equivalent to the problem of minimization of the functional

$$\tilde{R}=\int dt\left[L(x,\theta,\dot{\theta})+\lambda(t)\left[\dot{x}+2\eta x-2\frac{\partial g}{\partial \theta}\right]\right],$$

with respect to $x(t)$, $\theta(t)$ (their variations are now independent); $\lambda(t)$ is an undetermined coefficient which can be found from the Euler equations and from Eq. (11) for $\dot{x}$. The respective variational problem corresponds to the problem of the motion of a certain auxiliary two-dimensional mechanical system. The Hamiltonian function for this system is of the form

$$H(p_x,p_\theta,x,\theta)=p_\theta^2-2p_\theta\frac{\partial g}{\partial x}-2p_x\left[\eta x-\frac{\partial g}{\partial \theta}\right], \tag{13}$$

where x,θ are the generalized coordinates and p_x,p_θ are the respective generalized momenta of this system; $L(x,\theta,\dot{\theta})$ is its Lagrangian and R is its mechanical action. The condition that R in Eq. (11) be extremal with respect to t reduces (cf. Ref. 2) to the equality

$$H(p_x,p_\theta,x,\theta)=0\ . \tag{14}$$

Equation (14) may be obtained in the limit of small D from the Fokker-Planck equation for the stationary distribution function of the system $\rho(x,\theta)$, with $p_x=-D\partial\ln\rho/\partial x$, $p_\theta=-D\partial\ln\rho/\partial\theta$. For Markov systems the approach to the problem of transition probabilities and of stationary distribution based on equations of the type (14) was used in a number of papers, cf. Ref. 17.

Equation (14) has been used, in fact, to show that the extreme path finishes in the vicinity of the saddle point [see Eq. (12)]. For the considered case of a δ-correlated noise it is sufficient when calculating $\ln W$ to find the probability of a system to reach the separatrix [the latter divides the phase plane (x,θ) into ranges of attraction to different stable states]. Indeed, the system placed on the separatrix goes to the stable state needed with a probability $\sim 1/2$ due to small fluctuations, so the large optimal fluctuation of the force is "switched off" when the respective path of the system given by Eq. (11) reaches the separatrix. Obviously, the minimum in Eq. (11) should be taken with respect to the position of the end point $(\bar{x}_e,\bar{\theta}_e)$ of this optimal path on the separatrix.[8] The variation of R as a function of $(\bar{x}_e,\bar{\theta}_e)$ is[2]

$$\delta R=p_x\delta\bar{x}_e+p_\theta\delta\bar{\theta}_e\ .$$

When the point $(\bar{x}_e,\bar{\theta}_e)$ is shifted along the separatrix which is a "regular" path (that in the absence of a random force),

$$\delta\bar{x}_e=-\delta\bar{\theta}_e[(\partial g/\partial\theta)-\eta x](\partial g/\partial x)^{-1}\ ,$$

cf. Eqs. (5). Substituting this relation into the expression for δR and taking Eq. (14) into account we obtain that $\delta R=\frac{1}{2}p_\theta^2\delta\bar{\theta}_e/(\partial g/\partial x)$. Since on the regular path $\delta\theta(\partial g/\partial x)<0$ [cf. Eqs. (5)] we see that R decreases as the end point moves along the separatrix to the saddle point, and therefore the latter is the "true" end point of the extreme path.

The criterion of the applicability of Eqs. (11)–(14) for W is the inequality

$$R\gg D\ . \tag{15}$$

This inequality provides also the fulfillment of the bistability criterion (1).

IV. TRANSITION PROBABILITIES FOR WEAK DAMPING

Equations (11)–(14) make it possible to find numerically the activation energies R of the transitions for any concrete values of two dimensionless parameters of the oscillator β and η. Explicit expressions for the transition probabilities may be obtained in some limiting cases. We now consider the limit of weak damping (or of the large frequency detuning),

$$\eta=\Gamma/|\delta\omega|\ll 1\ . \tag{16}$$

To zeroth order in η,D the motion of the oscillator in the (x,θ) representation is a conservative one with the effective energy $2g(x,\theta)$ (in quantum theory g is the quasienergy of the oscillator in the resonant field[18,19]). This motion is described by the periodic functions $X(g,\psi),\Theta(g,\psi)$ which obey the equations

$$\begin{aligned}&\frac{\partial X}{\partial\psi}=2\omega^{-1}(g)\frac{\partial g}{\partial\Theta},\quad \frac{\partial\Theta}{\partial\psi}=-2\omega^{-1}(g)\frac{\partial g}{\partial X}\ ,\\ &\frac{d\psi}{dt}=\omega(g),\quad g\equiv g(X,\Theta)=\tfrac{1}{4}(X-1)^2-\sqrt{\beta X}\,\cos\Theta\ ,\end{aligned} \tag{17}$$

where $\omega(g)$ is the dimensionless frequency of the motion with a given g,

$$\omega(g)=\pi\left|\oint dX\left(\frac{\partial g(X,\Theta)}{\partial\Theta}\right)^{-1}\right|^{-1},$$

ψ is the phase of the motion. The phase trajectories described by Eq. (17) are shown in Fig. 1. Since the states of the oscillator differing in Θ by 2π are identical, the trajectory pattern is periodic in Θ. The shape of a trajectory depends on the value of g, and there are both closed and open trajectories.

In the range of β where bistability occurs, $0<\beta<\frac{4}{27}$, the g axis is split into three parts by the values g_{f_1}, g_{f_2}, and g_s which g takes on in the stable states (the foci) f_1,f_2 and in the saddle point s (we assume the foci f_1 and f_2 to correspond to the smallest and to the largest amplitudes of the forced vibrations; then $g_{f_1}>g_s>g_{f_2}$). In the range $g>g_{f_1}$ all trajectories $X(\Theta)$ are open and lie above the separatrix. In the range $g_{f_1}>g>g_s$ there coexist two types of the trajectories: those above the separatrix and those in the attraction range of the state f_1. The latter either surround the point f_1 and are thus closed or (for $\beta<\frac{2}{27}$, see Fig. 1) are open in part and lie below the separatrix. For $g_{f_2}<g<g_s$ the trajectories lie in the at-

traction range of the state f_2. At $\beta<\frac{2}{27}$ they surround the point f_2 (see Fig. 1), while at $\beta>\frac{2}{27}$ they may be open (for not too small $g-g_{f_2}$) and lie then below the separatrix (at $\beta>\frac{2}{27}$ the separatrix loops around the point f_1 in contrast to the case $\beta<\frac{2}{27}$ shown in Fig. 1 where the separatrix loop surrounds the point f_2).

For $\eta, D \ll \omega(g)$ the main manifestation of friction and noise is slow drift and diffusion of the oscillator on g. The respective random process can be described (cf. Refs. 6, 18, and 20) by a one-dimensional Fokker-Planck equation for the distribution function $\rho(x,\theta,t)$ which is of the following form in the case under consideration

$$\rho(x,\theta,t)\approx\bar{\rho}(g(x,\theta),t)\ ,$$
$$\frac{\partial\bar{\rho}}{\partial t}=\omega(g)\frac{\partial}{\partial g}\left[-\eta I(g)\bar{\rho}+DM(g)\frac{\partial\bar{\rho}}{\partial g}\right]\ , \tag{18}$$

where

$$I(g)=\frac{1}{2\pi}\int_0^{2\pi}d\psi\, X(g,\psi)\frac{\partial\Theta(g,\psi)}{\partial\psi}\ ,$$
$$M(g)=\frac{\omega(g)}{8\pi}\int_0^{2\pi}d\psi\left[\frac{\partial X(g,\psi)}{\partial\psi}\right]^2 . \tag{19}$$

In the range $g_{f_1}\geq g\geq g_s$ the function $\bar{\rho}(g,t)$ and the coefficients $I(g)$,$M(g)$ are ambiguous. We denote their values as $\bar{\rho}^{(1)}$, $I^{(1)}$, $M^{(1)}$, and $\bar{\rho}^{(2)}$,$I^{(2)}$,$M^{(2)}$ for x,θ lying in the attraction range of the stable state f_1 and above the separatrix in Fig. 1, respectively.

Near the separatrix, where

$$|g-g_s|\lesssim[DM(g_s)]^{1/2},\quad \eta|I(g_s)|\ , \tag{20}$$

slow and fast motions cannot be decoupled and Eq. (18) is inapplicable [the rhs of (20) gives the change of $g\approx g_s$ per unit time due to diffusion and drift; Eq. (18) is valid provided the change of g per the period $2\pi\omega^{-1}(g)$ is small, but $\omega^{-1}(g)$ diverges logarithmically when $g\to g_s$]. In the range (20) the solutions $\bar{\rho}^{(1,2)}(g,t)$ for $g>g_s$ match one another and the solution $\bar{\rho}(g,t)$ for $g<g_s$.

A. Argument of the exponential

In the case of the small diffusion coefficient $D\ll\eta$, within a dimensionless time t such that $\eta^{-1}\ll t\ll|\delta\omega|W^{-1}$ a quasistationary distribution of the oscillator over g is worked out in the neighborhood of the stable state f occupied initially. According to Eq. (18) it is quasi-Gibbsian,

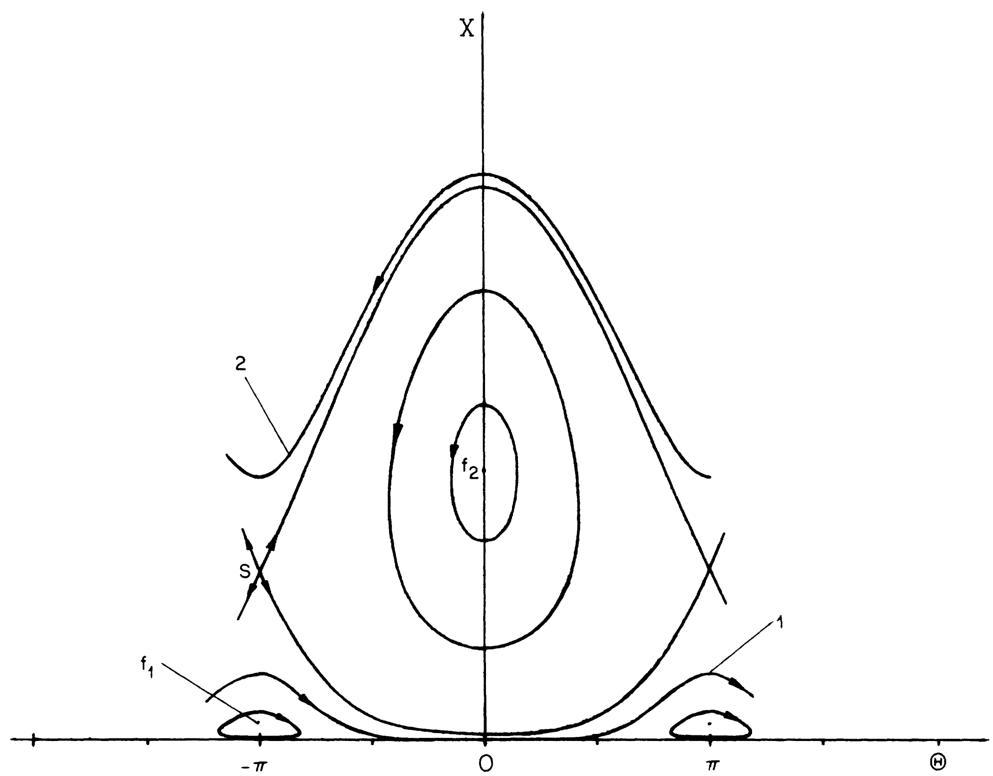

FIG. 1. Topology of phase trajectories (17) of the oscillator in the zero-friction limit for $\beta<\frac{2}{27}$; f_1,f_2 are equilibrium positions (foci for finite damping), s is the saddle point. The trajectory passing through the saddle point is separatrix. The trajectories 1 and 2 refer to the same value of the quasienergy g (the concrete value of β taken is 0.045; the respective values of X in the points f_1, s, f_2 are approximately 0.05, 0.76, 1.19).

$$\bar{\rho}_G(g,t)=\bar{\rho}_f \exp\left[-\frac{1}{D}R(g)\right],$$
$$R(g)=-\eta\int_{g_f}^{g} dg\, I(g)/M(g), \tag{21}$$

with an effective "temperature" $-DM(g)/\eta I(g)$ depending on the quasienergy g [the sign of $I(g)$ is obvious from Eq. (19) and from Fig. 1 to be such that $R(g)>0$]. The quantity $\bar{\rho}_f$ varies over the time $\propto W^{-1}$.

To logarithmic accuracy the probability W of the escape from the state f equals $\exp[-R(g_s)/D]$ according to Eq. (21). The quantity $R(g_s)$ coincides with the value of R which may be obtained independently from the general formulas (11)–(14) for $\eta \ll 1$ with the aid of the approach developed in Ref. 8. As is obvious from (21), the activation energy of the transition $R \propto \eta = \Gamma/|\delta\omega|$. This gives the dependence of R on the friction coefficient of the oscillator Γ. The quantity R/η depends on the rest of the parameters of the system only in terms of their combination $\beta \propto F^2|\gamma/(\delta\omega)^3|$ [Eq. (6)]. The respective dependences for the activation energies R_1 and R_2 of the transitions from the stable states f_1 and f_2 are shown in Fig. 2.

Both R_1 and R_2 decrease with increasing dimensionless field intensity β. In weak fields one obtains from Eqs. (5), (19), and (21) that R_1 diverges with decreasing β,

$$R_1 \approx \tfrac{1}{3}\eta\beta^{-1} \quad (\beta \ll 1) \tag{22}$$

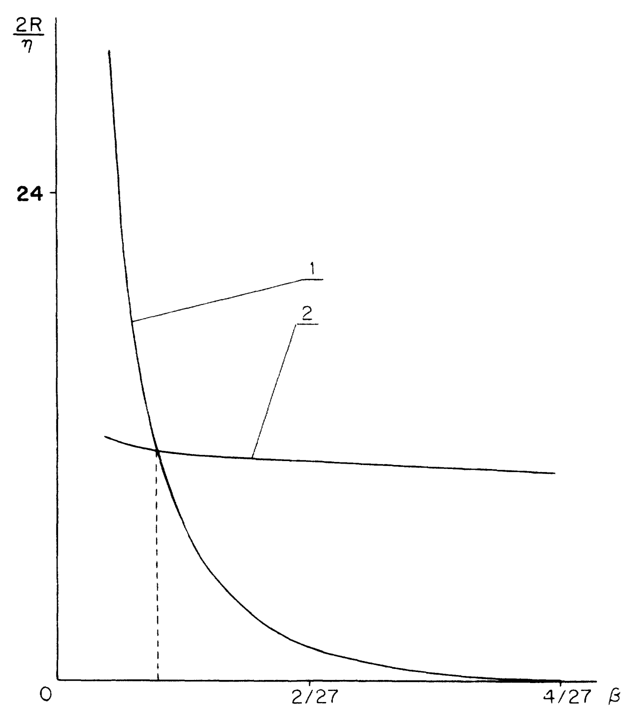

FIG. 2. Activation energies R_1/η and R_2/η vs β for the transitions $f_1 \rightarrow f_2$ and $f_2 \rightarrow f_1$ in the case of small damping, $\eta \ll 1$. The phase-transition point, where $R_1 = R_2$ lies at $\beta \approx \beta_0 \approx 0.030$.

while R_2 remains finite in spite of the values of g_{f_2} and g_s approaching one another as $\beta \rightarrow 0$ [however, $R_2 \rightarrow 0$ when $\beta \rightarrow \beta_B^{(1)}(\eta) \approx \eta^2$, see Sec. V]. Such a behavior results from the fact that the fluctuations under consideration are due to the external field. Therefore when the field decreases the effective fluctuation intensity decreases also and the probability of a large fluctuation necessary for the escape falls rapidly. For $\beta \rightarrow \frac{4}{27}$, when g_{f_1} approaches g_s, the value of R_1, as obvious from Fig. 2, vanishes.

B. Preexponential factor

In the case of weak damping, $\eta < 1$, the preexponential factor in the expression for the transition probability W depends on the ratio of two small quantities in the rhs of (20), that is, on D/η^2 [note that $D \ll \eta$, cf. (10)]. We shall consider it supposing the diffusion coefficient to be not too small,

$$\eta^2 \ll D \ll \eta . \tag{23}$$

We note first that parallel to solution (21), Eq. (18) has one more quasistationary solution, $\bar{\rho}_{dr}(g)$,

$$\bar{\rho} \approx \bar{\rho}_G + \bar{\rho}_{dr}, \quad \bar{\rho}_{dr}(g)=C/I(g), \quad |g-g_f| \gg D/\eta . \tag{24}$$

The solution $\bar{\rho}_{dr}$ corresponds to the quasistationary flow ηC along the quasienergy axis [a quantum analog to (24) for a particle moving in a static potential and coupled to a thermostat was found in Ref. 21]. At $t \gg \eta^{-1}$ the flow is due to the transitions between the ranges of attraction to different stable states, and thus it is exponentially small, $C \ll 1$.

The probability of the transition from the state f is just determined by the relative change in the population

$$2\pi\int_{g_f}^{g} dg\, \omega^{-1}(g)\bar{\rho}(g,t)$$

of the respective attraction range per unit time. According to Eqs. (18), (21), and (24), in the "real" time scale

$$W \approx |\delta\omega| \frac{d}{dt}\ln\left[\int_{g_f}^{g} dg\, \omega^{-1}(g)\bar{\rho}(g,t)\right]$$
$$\approx |\delta\omega| \frac{8\eta^2}{D}(C_f/\bar{\rho}_f)(\partial^2 g/\partial x^2)_f . \tag{25}$$

The subscript f on the derivative in (25) means that it is calculated in the state f, C_f is the constant in $\bar{\rho}_{dr}$ (24) for the considered attraction range. We have taken into account that W does not depend practically on the upper limit of the integral in (25),

$$\int_{g_f}^{g} dg\, \omega^{-1}(g)\frac{\partial\bar{\rho}}{\partial t} = -\eta C ,$$
$$\int_{g_f}^{g} dg\, \omega^{-1}(g)\bar{\rho} = (\bar{\rho}_f D/\eta)(\omega^{-1}M/I)_f, \quad |g-g_f| \gg D/\eta$$

and used Eq. (18) and the explicit expressions for $M(g)$, $I(g)$, $\omega(g)$ at $|g-g_f| \sim D/\eta \ll 1$ which follow from (17) and (19).

The ratio $C_f/\bar{\rho}_f$ in (25) may be found if one notices that the characteristic scale $\Delta g = D/\eta$ over which $\bar{\rho}_G(g,t)$ varies [cf. Eq. (21)] exceeds greatly the range (20) due to inequality (23). The solution $\bar{\rho}_{dr}$ varies over the scale $\Delta g \sim 1 \gg [DM(g_s)]^{1/2}$, $\eta |I(g_s)|$. Therefore the solution $\bar{\rho}$ (24) as a whole is practically constant within a layer (20), and to zeroth order in η^2/D we can neglect an effect of this layer on the distribution function. As a consequence the solutions $\bar{\rho}$ above and below g_s should coincide with each other [$\bar{\rho}$ takes on one and the same value on both sides of the separatrix in Fig. 1 or, to be more strict, of the "stripe" of a width (20) around the separatrix]. The flow should be continuous at $g = g_s$ as well (note that in the quasistationary regime there is no flow in the range above the separatrix loop in Fig. 1 where there are no stable states).

If we consider transitions from the state f_i ($i=1,2$) and thus the state f_{3-i} is not occupied practically ($t \ll |\delta\omega| W^{-1}$), then $\bar{\rho}_{f_{3-i}}=0$ and the matching conditions take the form

$$\bar{\rho}_{f_i}\exp(-R_i/D)+C_{f_i}I_i^{-1}(g_s)=C_{f_{3-i}}I_{3-i}^{-1}(g_s)\ ,\qquad C_{f_i}=C_{f_{3-i}} \tag{26}$$

[we have allowed for Eqs. (21) and (24) here]. Equations (21), (25), and (26) result in the following expression for the probability W_{ij} of the transition from the focus f_i to f_j:

$$W_{ij}=|\delta\omega|\,A_i\frac{\Gamma^2}{D}\exp\left[-\frac{R_i}{D}\right],$$
$$A_i=8\frac{|I_1(g_s)I_2(g_s)|}{|I_1(g_s)|+|I_2(g_s)|}\left[\frac{\partial^2 g}{\partial x^2}\right]_{f_i},\quad \eta^2 \ll D \ll \eta\ . \tag{27}$$

We demonstrate explicitly here the dependence of the preexponential factor on the oscillator damping Γ and on the phase diffusion coefficient D. The value of A_i depends on the parameters only in terms of $\beta \propto F^2|\gamma/(\delta\omega)^3|$. Since $R_i \propto \Gamma$ the probabilities W_{ij} increase with decreasing Γ. The proposed method is obvious to be applicable to arbitrary underdamped Markovian systems.

V. TRANSITION PROBABILITIES NEAR BIFURCATION POINTS

Near a bifurcation point, when one of the stable states is close in the phase space to the unstable steady state, in the vicinity of these states a slow one-dimensional motion of a system (a "soft mode") can be singled out.[22] This enables one to reduce the problem of fluctuations to a one-dimensional one by making use of an adiabatic approximation and to obtain in an explicit form the probability W of an escape from the respective metastable state including the preexponential factor.[23]

The slow motion of the oscillator for $\beta \approx \beta_B(\eta)$ is described by the variable $y=\sqrt{x}\,\sin\theta$.[23] The equation for y in the present case of a stochastic phase modulation is of the same form as in the case of an oscillator driven by a monochromatic field and an additive broadband noise considered in Ref. 23. The difference lies in the form of the noise intensity in the equation for y which is obvious from Eqs. (5) and (9) to equal now to

$$2Dx_B\cos^2\theta_B \equiv 2Dx_B^2(x_B-1)^2/\beta_B\ ,$$
$$x_B^{(1,2)} \equiv x_B^{(1,2)}(\eta)=\tfrac{2}{3}\pm\tfrac{1}{3}(1-3\eta^2)^{1/2} \tag{28}$$

[x_B and θ_B are the values of x and θ for the merging stable and unstable states at $\beta=\beta_B(\eta)$; note that $\eta=\Omega^{-1}$ in notations of Ref. 23]. By applying the results of Ref. 23 and allowing for (28) we arrive at the following expression for the activation energy R (11) of the transition near the bifurcation point:

$$R=H(\eta)|\beta-\beta_B|^{3/2}\ ,$$
$$H(\eta)=\frac{\sqrt{2}}{3}\beta_B^{1/2}[x_B^2(x_B-1)^2|a|^{1/2}]^{-1}\ ,$$
$$\beta_B\equiv\beta_B(\eta), x_B\equiv x_B(\eta)\ ,$$
$$a\equiv a(\eta)=x_B[5x_B-3+3(2x_B-1)^2(x_B-1)\eta^{-2}] \tag{29}$$

[the same expression may be obtained independently by solving the variational problem (11)–(14) near a bifurcation point]. The preexponential factor in the expression for W is given by

$$\frac{|\delta\omega|}{\pi}|\tfrac{1}{2}a(\beta-\beta_B)|^{1/2}[D\sqrt{\beta_B}x_B^2(x_B-1)^2]^{-1/3}\ .$$

The probability of an escape from a metastable state is obvious to increase rapidly with approaching a bifurcation point, that is, with decreasing $|\beta-\beta_B(\eta)|$. The factor $H(\eta)$ in the expression (29) for R is shown in Fig. 3. In the range of small η

$$H^{(1)}(\eta)\approx\frac{8}{3}\eta^{-3},\quad H^{(2)}(\eta)\approx\tfrac{27}{2}\eta$$

[note that $H^{(i)}(\eta)$ determines the probability of the es-

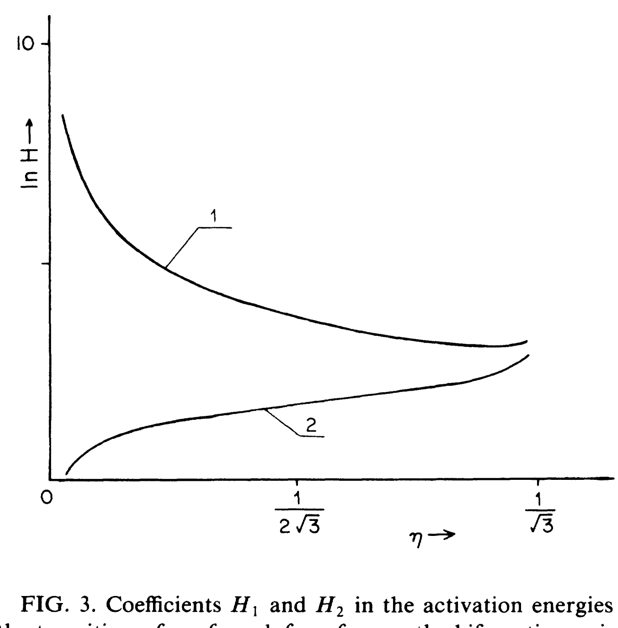

FIG. 3. Coefficients H_1 and H_2 in the activation energies of the transitions $f_2 \rightarrow f_1$ and $f_1 \rightarrow f_2$ near the bifurcation points, where f_2 and f_1, respectively, merge with the saddle point. In the critical point ($\beta=\frac{8}{27}$, $\eta=1/\sqrt{3}$) H_1 and H_2 diverge.

cape from the state $3-i$, $i=1,2$]. The function $H^{(1)}(\eta)$ decreases (except a narrow range where $\eta \approx 1/\sqrt{3}$) while $H^{(2)}(\eta)$ increases with increasing η (in contrast with the results[23] for an oscillator driven by additive white noise). They approach each other as η approaches the critical value $\eta_K = 1/\sqrt{3}$ where $\beta_B^{(1)}(\eta) = \beta_B^{(2)}(\eta) \equiv \beta_K = \frac{8}{27}$ [the spinode point on the bifurcation curve $\beta_B(\eta)$]. In the critical point, $\beta = \beta_K$, $\eta = \eta_K$, two stable states and the unstable steady state of the oscillator coalesce. Fluctuations for $\beta \approx \beta_K$, $\eta \approx \eta_K$ are described by the theory[5,23] with a properly renormalized random noise intensity.

VI. TIME CORRELATION FUNCTIONS AND NARROW PEAKS IN THE POWER SPECTRA

An important feature of systems bistable in intense periodic fields is the onset of extremely narrow peaks in the power spectra and in the spectra of response to an additional weak field. These peaks can arise in addition to much broader peaks (with a width $\sim \Gamma$, Refs. 8, 9, and 15) which are caused by vibrations about stable states. The narrow peaks lie at the strong-field frequency and its overtones and also at very low frequency. They are due to fluctuational transitions between stable states and occur within a narrow range of parameters where the transition probabilities are of the same order of magnitude $W_{12} \sim W_{21}$ and, respectively, $w_1 \sim w_2$, where w_1, w_2 are the stationary populations of the states f_1, f_2,

$$w_1 + w_2 = 1, \quad \frac{w_1}{w_2} = \frac{W_{21}}{W_{12}} \propto \exp[-(R_2 - R_1)/D]$$

according to Eq. (11). The range of parameters where $R_1 \approx R_2$ and thus $w_1 \sim w_2$ may be called a range of a kinetic "phase transition." On the opposite sides of it the system occupies with overwhelming probability either one or another stable state, that is, either $w_1 \ll w_2 \approx 1$ or $w_2 \ll w_1 \approx 1$. For the system under consideration the condition $R_1 = R_2$ determines a curve $\beta_0(\eta)$ on parameter plane (β, η). Near the spinode point on the bifurcation curve $\beta_B(\eta)$ (a critical point), $\beta_K = \frac{8}{27}$, $\eta_K = 1/\sqrt{3}$, one can obtain by making use of the approach[8,23]

$$\beta_0(\eta) \approx \tfrac{8}{27}[1 + (3\sqrt{3}/2)(\eta - \frac{1}{\sqrt{3}})] ,$$

while in the limit $\eta \to 0$ it is obvious from Fig. 2 that $\beta_0 \to 0.030$. For intermediate η the function $\beta_0(\eta)$ may be estimated by interpolating smoothly between these limiting values [$\beta_0(\eta)$ may be obtained numerically for arbitrary η by solving the variational problem (11)–(14)].

In the case of a strong driving field with a random phase, the spectral peaks and certain other statistical properties of a system differ from those in the case of fluctuations induced by a field-independent noise source. This is connected with the following: In the latter case forced vibrations of the oscillator present themselves a nonlinear "superposition" of coherent (with a regular phase) and stochastic vibrations and are thus "inhomogeneous" in time, while in the case of stochastic field phase the vibrations are incoherent and thus are "homogeneous" in time on the average, that is, they are not time locked. We note that in this respect the effect of fluctuations of the field phase differ qualitatively from that of fluctuations of the oscillator phase due to coupling to a bath.

The dependence of the average values of dynamical variables on time are qualitatively different in these two cases. For a system driven by a high-frequency field $F\cos(\omega_F \tau + \phi_F)$ it is convenient to write down an arbitrary dynamical variable L in a stable state f_j in the form

$$L_j(\tau) = \sum_n L_{jn} \exp[in(\omega_F \tau + \phi_F)], \quad j = 1,2 \tag{30}$$

[cf. Eqs. (4) for q, $dq/d\tau$, with x,θ having been replaced by their values in a respective stable state]. This expansion may be generalized also to include subharmonics if present; it is valid for time-dependent ϕ_F provided the change in ϕ_F over a characteristic relaxation time is small.

If the phase ϕ_F is fixed and noise is weak so that the system is localized predominantly close to the stable states, then

$$\langle L(\tau) \rangle \approx \sum_{j,n} w_j L_{jn} \exp[in(\omega_F \tau + \phi_F)] \tag{31a}$$

($\langle\ \rangle$ denotes the ensemble averaging). In the opposite case of random ϕ_F

$$\langle L(\tau) \rangle \approx \sum_j w_j L_{j0} \tag{31b}$$

and thus the average values of dynamical variables are time-independent. The behavior of time-correlation function $Q_{LM}(\tau)$ of dynamical variables L,M,

$$Q_{LM}(\tau) = \lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} d\tau_1 [L(\tau+\tau_1) - \langle L(\tau+\tau_1) \rangle] \times [M(\tau_1) - \langle M(\tau_1) \rangle] , \tag{32}$$

changes respectively (note that in the case of fixed ϕ_F the system is nonergodic and $\langle L(\tau+\tau_1) M(\tau_1) \rangle$ depends on τ_1 generally speaking). We shall assume first the parameters of the system to lie far from the phase-transition region, so that $w_1 w_2 \ll 1$ and only one stable state is occupied. Then for fixed ϕ_F the main contribution to $Q_{LM}(\tau)$ comes from small fluctuations of L,M about their values in the occupied stable state, while for fluctuating ϕ_F there arise in $Q_{LM}(\tau)$ large terms $Q_{LM}^{(n)}(\tau)$ oscillating at frequency $n\omega_F$. According to (30) and (9) they are given by

$$Q_{LM}^{(n)}(\tau) = \sum_j w_j L_{jn} M_{jn}^* \exp(in\omega_F \tau - n^2 D |\delta\omega\tau|) , \quad n \neq 0, \quad w_1 w_2 \ll 1 . \tag{33}$$

The spectral density of fluctuations of the variable L (the power spectrum) is given by the Fourier transform of $Q_{LL}(\tau)$,

$$Q_{LL}(\omega) = \frac{1}{\pi} \mathrm{Re} \left[\int_0^\infty d\tau\, e^{i\omega\tau} Q_{LL}(\tau) \right] . \tag{32'}$$

According to (33) $Q_{LL}(\omega)$ contains Lorentzian peaks at frequencies $n\omega_F$ with halfwidths $D|\delta\omega|n^2$ proportional to the phase diffusion coefficient [these peaks correspond to δ-shaped ones in the Fourier transform of the correlator $\langle L(\tau+\tau_1)M(\tau_1)\rangle$ averaged over τ_1 for the case of fixed ϕ_F]. An addition $\sim n^2D|\delta\omega|$ to the halfwidths of peaks arises also in susceptibility spectra, say, in the spectrum of absorption of a weak trial field. For $w_1w_2 \ll 1$ (that is, far from the phase-transition region) these peaks are due to small-amplitude vibrations about the occupied stable state. In the considered case of the small phase diffusion coefficient, $D \ll \eta$ or $D|\delta\omega| \ll \Gamma$, the respective broadening of peaks is small as compared to that ($\sim\Gamma$) which is due to friction.

The phase-diffusion-induced broadening is, however, of primary importance for the peaks caused by fluctuational transitions between stable states in the region of the kinetic phase transition. The shape of the respective peaks in the spectral density of fluctuations may be found from Eqs. (30), (32), and (9) with allowance for the balance equation for the time-dependent state populations $w_{1,2}(\tau)$,

$$\frac{dw_1(\tau)}{d\tau} = -W_{12}w_1(\tau)+W_{21}w_2(\tau), \quad w_1(\tau)+w_2(\tau)=1 \tag{34}$$

[w_1, w_2 in (31) and (33) and below are the stationary values of $w_1(\tau), w_2(\tau)$]. The redistribution over the states described by (34) gives rise to the following term in $Q_{LL}(\tau)$ [in addition to that given by (33)]:

$$\tilde{Q}_{LL}(\tau) = w_1w_2\sum_n |L_{1n}-L_{2n}|^2 \exp[in\omega_F\tau - n^2D|\delta\omega\tau| - (W_{12}+W_{21})|\tau|] \; . \tag{35}$$

Since

$$w_1w_2 = W_{12}W_{21}/(W_{12}+W_{21})^2 \propto \exp(-|R_1-R_2|/D)$$

this term is obvious to be exponentially small everywhere except the phase-transition region, where $|R_1-R_2| \lesssim D$. Damping of $\tilde{Q}_{LL}(\tau)$ and the respective halfwidths of the peaks in its Fourier transform $\tilde{Q}_{LL}(\omega)$ are determined by $W_{12}+W_{21}+n^2D|\delta\omega|$. If fluctuational transitions are due to the fluctuations of ϕ_F, then for $D \ll \eta$ according to Eq. (11)

$$D|\delta\omega| \gg W_{ij} \propto \Gamma\exp(-R_i/D) \; .$$

Therefore the peaks at frequencies $n\omega_F$ which were exponentially narrow for $\phi_F=$const [their halfwidth was $\sim W_{12}+W_{21} \propto \Gamma\exp(-R/D) \ll \Gamma$ (Ref. 15)] turn out to be smeared at $n \neq 0$. The respective peaks in the susceptibility spectra[5,8] are smeared also.

The phase-diffusion-induced broadening is obvious from Eq. (35) to be absent for the spectral peaks at zero frequency. According to (32′) and (35) these peaks are given by

$$\tilde{Q}_{LL}(\omega) = \frac{1}{\pi}w_1w_2|L_{10}-L_{20}|^2 \times (W_{12}+W_{21})/[\omega^2+(W_{12}+W_{21})^2] \; .$$

Such exponentially narrow peaks are universal for bistable systems in the kinetic-phase-transition region[5,8,15,24] and occur both in the power spectra and in the susceptibility spectra.

VII. LOW-FREQUENCY SUSCEPTIBILITY

For thermally nonequilibrium systems the generalized susceptibilities are not expressed directly in terms of the spectral densities of fluctuations (as it is the case for equilibrium systems[25]). However, the features of the respective spectra usually correlate. Therefore, in view of the peculiar low-frequency structure of the power spectra in the range of the kinetic phase transition, it is interesting to analyze the structure of the low-frequency susceptibility for a nonequilibrium bistable system. To illustrate it we consider a nonlinear oscillator driven by the strong resonant force $F\cos[\omega_F\tau+\phi_F(\tau)]$ (2) and perturbed by a weak coordinate-dependent low-frequency force $\lambda qF'\exp(-i\omega'\tau)$. The perturbing addition to the Hamiltonian of the oscillator is of the form

$$H' = -\tfrac{1}{2}\lambda q^2F'\exp(-i\omega'\tau) \tag{36}$$

[if F' is an electromagnetic field the interaction (36) is connected with a nonlinear dependence of the dipole moment of the oscillator on its coordinate; a contribution to the low-frequency susceptibility similar to that due to (36) may be shown to result from a linear in q interaction in the presence of cubic oscillator anharmonicity].

Allowing for (4) the generalized susceptibility $\bar{\chi}(\omega')$ (Ref. 25) for $\omega' \ll \omega_F$ with an accuracy to small corrections may be written as

$$\bar{\chi}(\omega') \equiv (\langle q^2\rangle_{F'} - \langle q^2\rangle_0)/F'\exp(-i\omega'\tau) \approx \tfrac{4}{3}\lambda\omega_F\left|\frac{\delta\omega}{\gamma}\right|\chi(\omega') \; , \tag{37}$$

$$\chi(\omega') \equiv (\langle x\rangle_{F'} - \langle x\rangle_0)/\lambda F'\exp\left[-i\frac{\omega'}{|\delta\omega|}t\right], \quad \omega' \ll \omega_F$$

where $\langle\ \rangle_{F'}$ and $\langle\ \rangle_0$ denote statistical averaging in the presence and in the absence of the perturbation (36) (that is, of the trial force proportional to F'), respectively.

To find $\langle x\rangle_{F'}$ to the first order in F' one has to solve Eqs. (5) for x,θ with g having been replaced by $g+g'$,

$$g(x,\theta) \rightarrow g(x,\theta)+g'(x,\theta) \; , \qquad g' = -\frac{1}{4}\frac{\lambda}{\omega_F|\delta\omega|}xF'\exp[-i(\omega'/|\delta\omega|)t] \; . \tag{38}$$

For low intensity of the phase fluctuations $D \ll \eta$, there are two main contributions to the susceptibility $\chi(\omega')$. They come from a motion in the vicinity of the stable states f_1, f_2 of the oscillator and from fluctuational transitions between the states,

$$\chi(\omega') = \sum_{i=1,2} w_i\chi_i(\omega') + \chi_{\rm tr}(\omega') \; . \tag{39}$$

The "partial" susceptibility $\chi_i(\omega')$ may be obtained easily by linearizing Eqs. (5) and (38) near the stable states,

$$\chi_j(\omega')=\frac{|\delta\omega|}{\omega_F}x_{f_j}(1-x_{f_j})[\omega'^2-\nu_j^2+2i\Gamma\omega']^{-1}\ ,$$
$$\nu_j^2=(\delta\omega)^2[\eta^2+(1-x_{f_j})(1-3x_{f_j})]\ , \tag{40}$$

where x_{f_j} is the value of x in the stable state f_j, x_{f_1} and x_{f_2} are the smallest and the largest roots of the equation

$$x_f[(x_f-1)^2+\eta^2]-\beta=0\ . \tag{41}$$

The frequency dependence of $\chi_j(\omega')$ coincides in form with that of susceptibility of a harmonic oscillator with the eigenfrequency ν_j and the friction coefficient Γ. Respectively, the partial contribution to the absorption coefficient $\mu(\omega')$ of a trial field F',

$$\mu(\omega')=2\lambda\omega'\mathrm{Im}[\bar{\chi}(\omega')] \tag{42}$$

is described by a curve with a maximum. For weak damping $\Gamma\ll\nu_j$, it is Lorentzian near the maximum, which is placed at ν_j, and its halfwidth is Γ. Since for the small-amplitude stable state $f_1, x_{f_1}<1$ according to (41), the sign of $\mu_1(\omega')$ is obvious from (40) and (42) to correspond to amplification of a trial field F' (at the expense of the strong field F). The sign of $\mu_2(\omega')$ corresponds to absorption of F'.

Far from the kinetic-phase-transition region the state populations w_1 and w_2 differ exponentially and only one of $\chi_i(\omega')$ contributes to $\chi(\omega')$ [Eq. (39)]. In the transition region the terms with $i=1$ and $i=2$ are of the same order of magnitude. The addend $\chi_{\mathrm{tr}}(\omega')$ in Eq. (39) becomes essential here as well.

A. Extremely narrow peak of susceptibility in the region of the kinetic-phase transition

The term $\chi_{\mathrm{tr}}(\omega')$ is due to a trial-field-induced redistribution of the system over the stable states and manifests itself at very low frequencies $\omega'\ll\Gamma,|\delta\omega|$. For $\omega'\ll\Gamma,|\delta\omega|$ the trial field is practically constant within a characteristic duration $(\sim\Gamma^{-1},|\delta\omega|^{-1})$ of a transition and is obvious from Eqs. (5) and (38) to influence the transition probability only parametrically,

$$W_{ij}\rightarrow W_{ij}+v_{ij}W_{ij}\frac{\lambda F'}{\omega_F|\delta\omega|}\exp\left[-i\frac{\omega'}{|\delta\omega|}t\right]\ ,\qquad \omega'\ll\Gamma,|\delta\omega|\ , \tag{43}$$

$$v_{ij}=\frac{1}{4D}\int dt\left[\dot{\theta}+2\frac{\partial g}{\partial x}\right],\quad |v_{ij}|\sim R_i/D\gg 1\ .$$

The integral in the expression for v_{ij} is taken supposing x,θ to vary along the extreme path which goes from the focus f_i to the saddle point and is described by the variational problem (11)–(14) [Eq. (43) is obtained by linearizing the expression (11) for W in the addition to the activation energy of the transition R caused by the addition g' (38) to g].

Oscillating terms in W_{ij} give rise to the oscillating in time additions to the state populations $w_1(\tau), w_2(\tau)$ which are determined by Eqs. (34) and (43). They result in the following expression for $\chi_{\mathrm{tr}}(\omega')$:

$$\chi_{\mathrm{tr}}(\omega')=\frac{1}{\omega_F|\delta\omega|}(v_{21}-v_{12})(x_{f_1}-x_{f_2})\times\frac{W_{12}W_{21}}{(W_{12}+W_{21})^2}\left[1-i\frac{\omega'}{W_{12}+W_{21}}\right]^{-1}\ . \tag{44}$$

The frequency dependence of $\chi_{\mathrm{tr}}(\omega')$ is extremely sharp, the characteristic frequency scale is $W_{12}+W_{21}\ll\Gamma,|\delta\omega|$. The function $\mathrm{Im}[\chi_{\mathrm{tr}}(\omega')]$ has a peak at $W_{12}+W_{21}$ with the height proportional to $(\eta/D)\exp(-|R_1-R_2|/D)$. Since $\eta\gg D$, this height is large in the region of η,β, where $R_1\approx R_2$ and the kinetic phase transition occurs, but it is exponentially small out of this region. The absorption coefficient $\mu(\omega')$ [Eq. (42)] is small at $\omega'\lesssim W_{12}+W_{21}$. However, its strong dependence on frequency results in a sharp and high peak of $d\mu/d\omega'$ at $\omega'=(W_{12}+W_{21})/\sqrt{3}$.

The universal narrow peak of the low-frequency susceptibility predicted in Ref. 8 and analyzed above is related closely to the so-called "stochastic resonance."[26,27] The latter was investigated for symmetric bistable systems; the stationary populations of the stable states for such systems are equal (just the midpoint of the range of the kinetic phase transition discussed above). The phenomenon consists in a periodic modulation of noise-induced switching between stable states by a relatively weak low-frequency force [cf. Refs. 5 and 8; see also Eqs. (34) and (43) above], or (that is physically equivalent) in a strong dependence[27] on an external noise of the ratio of the force-induced periodic signal to noise at the force frequency. Such dependence is described by Eqs. (37) and (44) which just give the ratio of the periodic signal to the weak driving force whose frequency is small as compared with all eigenfrequencies of a system. For $W_{12}=W_{21}=W\propto\exp(-R/D)$ the real part of the susceptibility $\mathrm{Re}[\chi_{\mathrm{tr}}(\omega')]\propto W/(4W^2+\omega^2)$ increases exponentially (in absolute value) with increasing noise intensity D for $2W\ll\omega$, while for higher D, when $W\gg\omega$, it decreases with D.

In the case of bistability in a strong periodic field with fixed phase the trial-force-induced transitions between stable states are efficient in the phase-transition region not only for a trial force with a small frequency $\omega'\ll\omega_F$ (ω_F is the frequency of the strong field), but also when ω' is close to ω_F (or to its overtones), $|\omega'-\omega_F|\ll\omega_F$.[5,8] Therefore, in this case the high-frequency "stochastic resonance" similar to the low-frequency one can arise. To describe it the explicit expressions given in Refs. 5 and 8 may be used (we note that in[5,8] $\mathrm{Im}[\chi(\omega')]$ was given explicitly; $\mathrm{Re}[\chi(\omega')]$ is described by the same expressions with Im having been replaced by Re). In contrast to the low-frequency susceptibility (44), where

$$\mathrm{Im}[\chi_{\mathrm{tr}}(\omega')]\propto(\omega'/2W)\mathrm{Re}[\chi_{\mathrm{tr}}(\omega')]\ ,$$

so that only $|\mathrm{Re}[\chi_{\mathrm{tr}}(\omega')]|$ increases exponentially with the noise intensity D for $\omega'\gg 2W$, in the case

$2W \ll |\omega' - \omega_F| \ll \omega_F$ both $|\mathrm{Im}[\chi(\omega')]|$ and $|\mathrm{Re}[\chi(\omega')]|$ contain exponentially increasing terms in the phase-transition range.

For thermally equilibrium systems $\mathrm{Re}[\chi(\omega')]$ and $\mathrm{Im}[\chi(\omega')]$ are interconnected via the Kramers-Kronig relations; in addition

$$\mathrm{Im}[\chi(\omega')] = (\pi\omega'/T)Q(\omega') ,$$

where $Q(\omega')$ is the spectral density of fluctuations (32′) for the dynamical variable whose response to a trial force is given by $\chi(\omega')$, and T is temperature. In the region of the zero-frequency peak

$$\mathrm{Im}[\chi(\omega')] \approx (\omega'/2W)\mathrm{Re}[\chi(\omega')] ,$$

thus frequency dependence of $\mathrm{Re}[\chi(\omega')]$ is given by that of $Q(\omega')$. The zero-frequency peak of $Q(\omega')$ was considered theoretically and investigated by means of analog simulation in Refs. 5, 15, and 24, and its width was found to increase exponentially with increasing noise intensity; this just corresponds to the very interesting phenomenon of stochastic resonance.

VIII. CONCLUSIONS

The above results demonstrate the features of fluctuations in bistable systems driven by an intense periodic force with a random phase. As compared with the case of a system driven by a regular periodic force and additive noise[5,8] there are qualitative changes in the time correlation functions, in the power spectra, and the susceptibility spectra in a vicinity of the kinetic phase transition, in the dependences of the probabilities of fluctuational transitions between stable states on field intensity, etc.

In the general case, when both the types of noise are present, the predominance of one of them as a cause of fluctuational transitions is determined by their relative intensities (however, due to different dependences of the transition probabilities on the parameters of a system the transition mechanism can interchange in course of variation of the parameters). Since coupling to a bath which results in friction inevitably gives rise to a field-independent noise, fluctuations of the field phase can dominate as a source of fluctuational transitions of an oscillator provided their intensity

$$D \gg |\gamma|\Gamma T\omega_0^{-3}(\delta\omega)^{-2} \tag{45}$$

[we have used Eq. (11) and the results of Refs. 5 and 8 here, T is temperature of a bath in energy units].

Thermal fluctuations of the oscillator phase (see Appendix) are similar to fluctuations of the field phase from the viewpoint of the fluctuational transitions, but the effects of these two types of fluctuations on correlation functions and narrow spectral peaks in the kinetic-phase-transition region are shown to be qualitatively different. The oscillator phase fluctuations dominate as a transition mechanism provided the following relation between the relaxation parameters of the oscillator is fulfilled:

$$1 \gg \Gamma_m/\Gamma \gg |\gamma/\delta\omega| T\omega_0^{-3}$$

[cf. (A10), (A11), and (45)]. It has been noted,[19] however, that fluctuations of the phase can influence the transition probability extremely strongly even for rather small Γ_m/Γ (or small D) when the transitions are due to quantum fluctuations ($T \ll \hbar\omega_F$).

The collapse of bistability due to sufficiently strong noise in the driving field was observed experimentally for a relativistically anharmonic cyclotron oscillator.[4] This system is perhaps an ideal physical object for the experimental investigation of the dependence of the probabilities of fluctuational transitions on noise intensity and on other parameters, as well as of the features occurring in the kinetic-phase-transition region which were considered above.

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APPENDIX

Relaxation of an oscillator is due to coupling to a medium with many degrees of freedom which is assumed to be a thermostat. In Eqs. (2) and (5) the only relaxation taken into account is that corresponding to a friction force proportional to the oscillator velocity. In terms of quantum theory this relaxation is due to decay processes in which the oscillator switches to a neighboring energy level by exchanging an energy quantum $\approx\hbar\omega_0$ with a thermostat (cf., e.g., Ref. 5). However, there are often important relaxation processes which are not accompanied by a change in the oscillator energy. A simple mechanism resulting in such processes is quasielastic scattering of elementary excitations of a medium by the oscillator. It is actual for many physical systems.[12] This mechanism is shown below to correspond to a stochastic modulation of the slow part of the oscillator phase in terms of classical mechanics.

We shall describe elementary excitations of a medium as a set of harmonic vibrations whose eigenfrequencies ω_k make up a continuous (or quasicontinuous) spectrum. Their Hamiltonian is of the form

$$H_m = \tfrac{1}{2}\sum_k (p_k^2 + \omega_k^2 q_k^2) . \tag{A1}$$

Evolution of the coordinate q_k of the kth mode in the absence of coupling is given by the expression

$$q_k^{(0)}(\tau) = A_k \cos(\omega_k \tau + \phi_k) . \tag{A2}$$

In thermal equilibrium the distribution of the vibrations is Gibbsian, and thus the phases ϕ_k are uniformly distributed in the interval $[0,2\pi]$, while the distribution of the amplitudes is of the form

$$w(\ldots, A_k^2, \ldots) = \prod_k (\omega_k^2/2T)\exp(-\omega_k^2 A_k^2/2T) , \tag{A3}$$

where T is temperature in energy units.

We assume that the Hamiltonian of interaction between the singled-out nonlinear oscillator under consideration and the thermostat contains the term

$$H_i = \tfrac{1}{2}q^2 \sum_{k,k'} V_{kk'} q_k q_{k'} \tag{A4}$$

parallel to the terms causing linear friction in Eq. (2). The interaction (A4) is supposed weak, while the characteristic width ω_{th} of the frequency spectrum of the thermostat is supposed sufficiently large. In addition the doubled oscillator frequency $2\omega_0$ is assumed to lie out of the range of combined frequencies $|\omega_k \pm \omega_{k'}|$.

The interaction (A4) causes an addition

$$-\frac{\partial H_i}{\partial q} = -q(\tau)[f_1(\tau)+f_2(\tau)] ,$$
$$f_1(\tau) = \sum_k V_{kk} q_k^2(\tau), \quad f_2(\tau) = \sum_{\substack{k,k'\\ k\neq k'}} V_{kk'} q_k(\tau) q_{k'}(\tau) \tag{A5}$$

to the rhs of Eq. (2) for the oscillator coordinate $q(\tau)$. It gives rise also to the forces $-\partial H_i/\partial q_k$ in the equations of motion for the coordinates q_k of the bath modes.

To lowest order in H_i we can replace $q_k(\tau)$ in $f_1(\tau)$ and $f_2(\tau)$ by $q_k^{(0)}(\tau)$ [Eq. (A2)]. We denote the results by $f_1^{(0)}(\tau)$ and $f_2^{(0)}(\tau)$, respectively. Equation (A3) yields

$$\langle f_1^{(0)}(\tau)\rangle = 2\omega_0 P_1, \quad P_1 = \frac{T}{2\omega_0} \sum_k V_{kk} \omega_k^{-2} ,$$
$$\langle f_2^{(0)}(\tau)\rangle = 0 . \tag{A6}$$

When

$$P_1 \ll \omega_0 \tag{A7}$$

the quantity P_1, from Eqs. (2), (A5), and (A6), obviously equals the shift of the oscillator eigenfrequency ω_0. The inequality (A7) is one of the criteria of smallness of the interaction (A4).

For a large number N of eigenmodes (degrees of freedom) of the medium, $N \gg 1$ (the statistical limit), $\langle f_1(\tau)\rangle$ may be shown to equal $\langle f_1^{(0)}(\tau)\rangle$. Moreover,

$$\langle f_1(\tau_1) \cdots f_1(\tau_m)\rangle = (2\omega_0 P_1)^m$$

for arbitrary m [this is a consequence of the coupling parameters $V_{kk'}$ in (A4) being proportional to N^{-1}]. Therefore $f_1(\tau)$ in Eq. (A5) may be replaced by $2\omega_0 P_1$ and thus the respective term in (A5) comes simply to the renormalization of ω_0.

According to (A2)

$$\langle f_2^{(0)}(\tau) f_2^{(0)}(\tau')\rangle = T^2 \sum_{\substack{k,k'\\ k\neq k'}} V_{kk'}^2 \omega_k^{-2} \omega_{k'}^{-2} \{\cos[(\omega_k - \omega_{k'})(\tau-\tau')] - \cos[(\omega_k+\omega_{k'})(\tau-\tau')]\} \equiv K\tilde{\delta}(\tau-\tau') ,$$
$$K = 2\pi T^2 \sum_{\substack{k,k'\\ k\neq k'}} V_{kk'}^2 \omega_k^{-2}\omega_{k'}^{-2} \delta(\omega_k - \omega_{k'}) . \tag{A8}$$

The function $\tilde{\delta}(\tau)$ introduced here is of the order of ω_{th} for the time $|\tau| \lesssim \omega_{\text{th}}^{-1}$, while it is small and fast oscillating for $|\tau| \gg \omega_{\text{th}}^{-1}$. In addition, it follows from the definition of $\tilde{\delta}(\tau)$ that

$$\int_{-\tau_1}^{\tau_2} \tilde{\delta}(\tau)d\tau = 1, \quad \tau_1, \tau_2 \gg \omega_{\text{th}}^{-1} .$$

Therefore on a time scale exceeding substantially ω_{th}^{-1} the function $\tilde{\delta}(\tau)$ behaves as a δ function. We suppose a time $\sim\omega_{\text{th}}^{-1}$ to be small as compared with those times within which the amplitude (proportional to $x^{1/2}$) and the slow part of the phase (θ) of the oscillator are varied [see Eq. (4)].

$$\omega_{\text{th}} \gg \Gamma, |\delta\omega|, P_1 , \tag{A9}$$

so we substitute $\delta(\tau)$ for $\tilde{\delta}(\tau)$ into Eq. (A8) bearing in mind the equations for x,θ "coarsened" over times $\sim\omega_{\text{th}}^{-1}$ (cf. Ref. 5).

The random process $f_2^{(0)}(\tau)$ is not strictly Gaussian: The higher-order correlators do not decouple exactly into combinations of the pair correlators (A8). The additions are, however, fast oscillating and localized within narrow time intervals, e.g., the products of the pair correlators involved in the correlator

$$\langle f_2^{0}(\tau_1) f_2^{(0)}(\tau_2) f_2^{(0)}(\tau_3) f_2^{(0)}(\tau_4)\rangle$$

are comparitively large, provided the instants $\tau_1, \tau_2, \tau_3, \tau_4$ are close pairwise (e.g., $|\tau_1-\tau_3| \sim |\tau_2-\tau_4| \sim \omega_{\text{th}}^{-1}$), but an interval between the pairs may be arbitrary. The non-Gaussian correction to this correlator is substantial within a narrow range $|\tau_1-\tau_2| \sim |\tau_1-\tau_3| \sim |\tau_1-\tau_4| \sim \omega_{\text{th}}^{-1}$. On coarsening over the times greater than ω_{th}^{-1} the non-Gaussian corrections may be neglected.

Therefore, on a coarse-grained time scale the function $f_2^{(0)}(\tau)$ is Gaussian and δ correlated, i.e., it represents white noise. The noise intensity equals K. Since $K \sim \omega_0^2 P_1^2/\omega_{\text{th}}$, this noise is weak, the mean-square departure of the oscillator phase θ per the period $2\pi\omega_0^{-1}$ is small: $\langle(\Delta\theta)^2\rangle = 2\pi K/\omega_0^3 \ll 1$. This enables us to go to the slow variables x,θ [Eq. (4)] in Eq. (2) with (A5) having been added to its right-hand side. The resulting equations for x,θ coincide with (5), provided we replace

$$\delta\omega \to \delta\omega - P_1, \quad \frac{d\phi_F}{dt} \to \frac{d\phi_F}{dt} - (2\omega_F|\delta\omega|)^{-1} f_2^{(0)} .$$

The latter substitution is equivalent to the following renormalization of the phase-diffusion coefficient in Eq. (9):

$$D \to D + (8\omega_F^2|\delta\omega|)^{-1}K . \tag{A10}$$

The phase diffusion described by Eqs. (5), (9), and (A10) contributes to the "transverse" relaxation of the oscillator. The quantity $K/8\omega_0^2$ equals the high-temperature value of the so-called modulational or Raman damping Γ_m of the time correlation functions of the oscillator which has been obtained in quantum theory by essentially different methods.[5,12] The necessary condition for the bistability in a resonance field to occur is obvious from Eqs. (10) and (A10) to be of the form

$$\Gamma_m = K/8\omega_0^2 \ll \Gamma \ . \tag{A11}$$

By making use of the approach given in Ref. 5 the terms in $f_2(\tau)$ of higher order in H_i (A4) may be shown to renormalize the oscillator nonlinearity parameter γ in Eq. (2) (when $2\omega_0$ lies out of the range of combined frequencies $|\omega_k \pm \omega_{k'}|$). The role of other types of interaction with a thermostat was discussed in Ref. 5 and references cited therein.

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