Critical phenomena and phase transitions in optical bistability

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Critical phenomena in optical bistability (OB) are studied in detail. The theoretical analyses, mainly based on the steady-state solution of the "thermodynamics" Fokker-Planck equation [Fa Ou and Zixiong Qin, Opt. Commun. **65**, 455 (1988)] for absorptive OB, show that the critical phenomena in OB also can be brought into the framework of Landau's theory for the second-order phase transition, and the critical exponents obey the scaling laws.

I. INTRODUCTION

The phase transition in going from equilibrium to nonequilibrium is always a fascinating problem. It has been found that although the equilibrium phase transition may be very different in the mechanism of interaction, there are some common features. To study the analogy between the equilibrium and the far-from-equilibrium phase transition would just be for the purpose of finding the degree of similarity to a greater extent. In quantum optics, the atoms coupled with an optical (electromagnetic) field form an open system that exhibits phenomena analogous to phase transitions but far from thermodynamic equilibrium. The switching effect in optical bistability (OB) appears similar to a first-order phase transition,[1] and the threshold behavior of the laser is similar to a second-order phase transition.[2,3] The phenomena of first-order-like phase transition are shown also by the other dissipative systems in quantum optics. For instance, lasers with saturable absorbers, dye lasers, subharmonic and second-harmonic generators, and bidirectional ring lasers are such kinds of systems.[4,5] However, some authors,[4(a),4(b)] who are authoritative in the theory of OB, have stated that "the characteristic features of optical bistability are that it occurs in a purely passive system and that it never exhibits a second-order transition." The critical phenomenon studied in this paper is just the OB critical phenomenon that occurs in the passive cavity. As we know, the two-phase coexistence line of the first-order transition in the equilibrium system has a terminal point, the so-called critical point, and the critical point in the equilibrium transition is just the second-order transition (continuous transition) point. Although it is difficult to establish the concept of two-phase coexistence concerning OB, a similar critical point does exist. Let I_i, $I_\uparrow$, and $I_\downarrow$ represent the input light intensity and its up and down threshold, respectively; I_t the output (transmitted) light intensity of the system, and C the control parameter. On the I_i-C plane, one can always draw $I_\uparrow$-C and $I_\downarrow$-C curves from experimental or theoretical work [see Figs. 1(a) and 1(b)]. These two curves intersect only at one point and end at this point. From the figure we can see that this point is completely similar to the end point of the two-phase coexistence line in the equilibrium system.

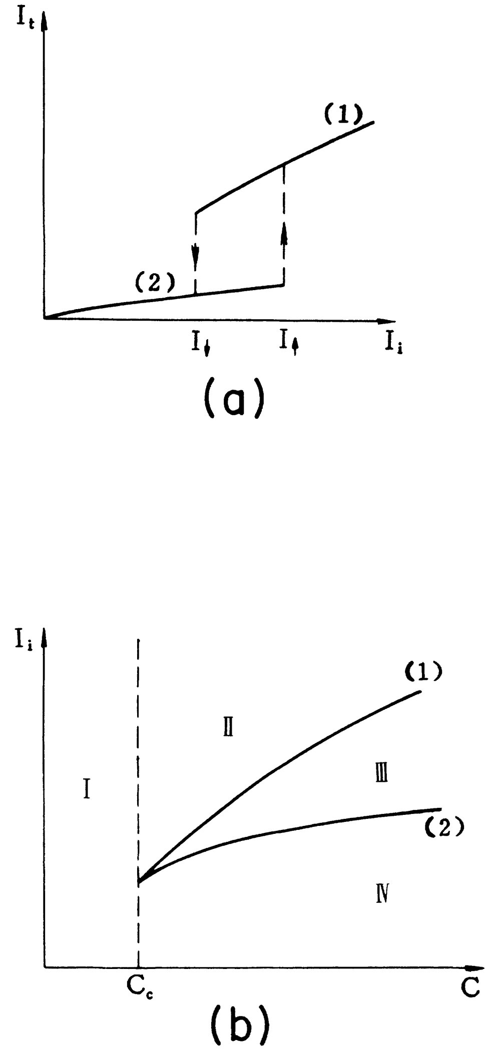

FIG. 1. (a) Transmitted intensity vs incident intensity for a bistable system: (1) high transmission ; (2) low transmission branch. (b) Phase diagram of the OB's: (1) $I_\uparrow$-C curve; (2) $I_\downarrow$-C curve. I. Single stable region; II. single stable region of the high branch. III. bistable region; IV. single stable region of the low branch.

Thus, as an analogy, it is also a critical point—the critical point of the OB system. Does this critical point of OB correspond to a continuous transition (second-order phase transition)? This paper is devoted to answering this question.

II. GENERALIZED THERMODYNAMIC POTENTIAL, ORDER PARAMETER AND EQUIVALENT EXTERNAL FIELD

The Fokker-Planck equation of the OB, or laser, is one kind of quantum-statistical theory of a nonequilibrium open system. From this equation the steady-state distribution $P_{st}(x)$ can be solved to have an exponential form:

$$P_{st}(x)=\mathcal{N}\exp[-\tilde{G}(x)/q] \;, \tag{1}$$

where $\mathcal{N}$ is the normalized factor and q is the fluctuation parameter; the variable x is the normalized transmitted field, while the function $\tilde{G}(x)$ is the so-called general thermodynamic potential which determines directly the properties of the steady-state statistical distribution of the system. In the case neglecting quantum fluctuation, the $\tilde{G}(x)$, due to pure absorptive OB (AOB), can be written in the following simple form:[6,7]

$$\tilde{G}(x)=\tfrac{1}{2}x^2-xy+C\ln(1+x^2) \;, \tag{2}$$

where y is the normalized incident field and C is the cooperative parameter (control parameter). If some internal factor of the system causes a small fluctuation of x to be δx, the corresponding change of $\tilde{G}(x)$ is $\Delta\tilde{G}(x)$ (y and C remain unchanged):

$$\Delta\tilde{G}=\frac{\partial\tilde{G}}{\partial x}\delta x+\frac{\partial^2 G}{\partial x^2}(\delta x)^2 \;; \tag{3}$$

$\partial\tilde{G}/\partial x=0$ is the condition that $\tilde{G}$ takes an extreme value and it also gives the macroscopic state equation

$$y=x\left[1+\frac{2C}{1+x^2}\right] , \tag{4}$$

while the condition $\partial^2 G/\partial x^2>0$ (i.e., $dy/dx>0$), under which the $\tilde{G}$ takes a minimum, is also the stability condition of the macroscopic steady state. Therefore, the $\tilde{G}(x)$ is equivalent to the Gibbs free energy in equilibrium thermodynamics. This problem has been discussed in detail in Ref. 6.

When $y=0$ in Eq. (4), which means no injected signal, we have

$$x\left[1+\frac{2C}{1+x^2}\right]=0 \;. \tag{5}$$

It can be taken as the laser state equation near threshold, while the parameter C in this equation plays a role of pumping and $C<0$ should be satisfied. Even though, this paper is mainly concerned with the OB critical phenomenon, which has not been carefully studied. However, the threshold behavior of the laser would be included naturally in our discussion.

It is easy to find the values of (x,y,C) in the critical point (x_c,y_c,C_c) by working out $dy/dx=0$ and $d^2y/dx^2=0$ according to Eq. (4):

$$(x_c,y_c,C_c)=(\sqrt{3},3\sqrt{3},4) \;. \tag{6}$$

For lasers, from Eq. (5), we have

$$(x_c,y_c,C_c)=(0,0,-\tfrac{1}{2}) \;. \tag{7}$$

The starting point for considering the OB critical behavior is the general thermodynamic potential $\tilde{G}(x)$, which reflects the statistical properties of the AOB steady state and is shown in Eq. (2). We expand $\tilde{G}(x)$ about the critical point $x=x_c$ and let

$$x=x_c+\eta, \quad \eta\ll x_c \;. \tag{8}$$

Then the Taylor series of $\tilde{G}$ may be expressed in terms of the deviation η:

$$\tilde{G}(\eta)=G_0+G_1\eta+G_2\eta^2+G_3\eta^3+G_4\eta^4 \;, \tag{9}$$

where η can be considered as an order parameter of the OB system. Later, we could see that η is analogous to the magnetization of the ferromagnetics, to the wave function of the electron pairs in superconductor, and to the difference between the gas density and liquid density at the gas-liquid coexistence state (strictly speaking, analogous to the deviation of the gas or liquid density from its critical value). In order to conveniently compare with Landau's theory, the series of $\tilde{G}$ has been truncated to the fifth power of η. By using Eq. (2) it is easy to obtain the expansion coefficients G_n, $n=0,1,2,3,4$, as follows:

$$G_0=\tfrac{1}{2}x_c^2-x_cy+C_c(1-t)\ln(1+x_c^2), \quad t=1-\frac{C}{C_c} \tag{9a}$$

(for lasers, $x_c=0$, then $G_0=0$);

$$G_1=-y+x_c\left[1+\frac{2C}{1+x_c^2}\right]\equiv-\tilde{H} \tag{9b}$$

(for lasers, $y=0$, $x_c=0$, then $\tilde{H}=0$);

$$G_2=\frac{t}{2} \tag{9c}$$

(for lasers, the same);

$$G_3=\frac{C}{3!}\frac{4x_c^3-12x_c}{(1+x_c^2)^3}=0 \tag{9d}$$

(for AOB, $x_c=\sqrt{3}$; for lasers, $x_c=0$);

$$G_4=b(1-t)$$

[for AOB, $b=1/C_c^2=(\frac{1}{4})^2$; for lasers, $b=C_c^2=\frac{1}{4}$]. It is worth noting that the quantity $\tilde{H}$, defined in Eq. (9b), may be called the "equivalent external field," because it is fairly similar to the external magnetic field in ferromagnetics. Let the incident field y operate in a special value $y_{\rm op}$, which means

$$y=y_{\rm op}=x_c\left[1+\frac{2C}{1+x_c^2}\right] \;; \tag{10}$$

the corresponding equivalent external field $\tilde{H}$ will be zero, i.e.,

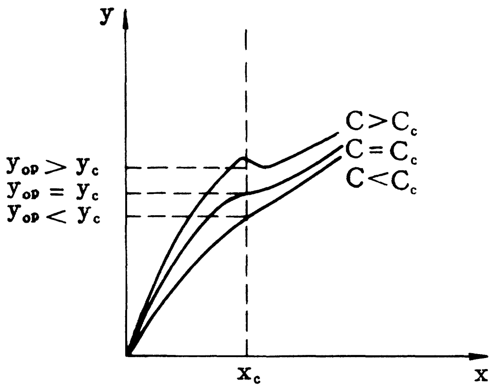

FIG. 2. The x-y curves for different values of the parameter C in the vicinity of the critical point.

$$\tilde{H}=0 \ . \tag{11}$$

Such a $y_{\rm op}$ corresponds to the values of y with the same $x=x_c(\eta=0)$ in the equal C curves (see Fig. 2). Inserting Eqs. (9a)–(9e) into Eq. (9), we have the expression of the general thermodynamic potential of AOB near the critical point:

$$\tilde{G}(\eta)=\tfrac{1}{2}x_c^2-x_c y+C_c(1-t)\ln(1+x_c^2) \\ -\tilde{H}\eta+\frac{t}{2}\eta^2+b(1-t)\eta^4 \ . \tag{12}$$

In fact, the obtaining of this equation has brought the OB critical phenomena of passive cavity ($C>0$) into the Landau's second-order phase transition theory.

III. THE STEADY-STATE EQUATION NEAR THE CRITICAL POINT

From $\partial\tilde{G}/\partial\eta=0$, we obtain the steady-state equation near the critical point:

$$\tilde{H}=t\eta+4b(1-t)\eta^3 \ . \tag{13}$$

If we set the incident field y to be the $y_{\rm op}$ shown in Eq. (10), which means $\tilde{H}=0$, we have

$$\eta[t+4b(1-t)\eta^2]=0 \ . \tag{14}$$

The solutions of this equation are as follows:

$$\eta=0, \quad \eta=\eta_\pm=\pm\frac{1}{2\sqrt{b}}(-t)^{1/2} \ . \tag{15}$$

The above results represent the bifurcation of one stable mode into two stable modes in OB critical phenomena: When $t>0$ ($C<C_c$), only the solution $\eta=0$ is stable, since $\partial^2\tilde{G}/\partial\eta^2|_{\eta=0}=t>0$. As soon as $t<0$ ($C>C_c$), the solution $\eta=0$ becomes unstable; since $\partial^2\tilde{G}/\partial\eta^2|_{\eta=\eta_\pm}=-2t>0$, the two solutions $\eta=\eta_\pm$ are stable. These three solutions meet at the critical point $t=0$ with $\eta=0$, which is critically unstable. $\eta=\eta_+>0$ corresponds to the high transmission branch; $\eta=\eta<0$, the low branch. Because the equivalent external field $\tilde{H}=0$ in the given case, η_+ and η_- are completely similar to two opposite spontaneous magnetizations in ferromagnetics. In summary,

$t>0$ ($C<C_c$), $\eta=0$ stable, no OB ;

$t=0$ ($C=C_c$), $\eta=0$ critically unstable ;

$$t<0 \ (C>C_c), \quad \eta=\eta_\pm=\pm(-t)^\beta/2\sqrt{b} \ , \\ \beta=\tfrac{1}{2} \ \text{OB occurs} \ . \tag{16}$$

Now, let us consider the case of $\tilde{H}=y-x_c[1+2C/(1+x_c^2)]\neq 0$ ($x_c=\sqrt{3}$). For a critical value $C_c=4$, from Eq (13), we have

$$\eta=(\tilde{H}/4b)^{1/\delta}, \quad \delta=3 \ . \tag{17}$$

This result indicates that, although C takes the critical value $C_c=4$, the incident field y does not equal the corresponding operating value, i.e. $y\neq y_{\rm op}=y_c=3\sqrt{3}$. Consequently, it is obvious that $x\neq x_c$, i.e., $\eta\neq 0$. In this case the η is equivalent to the magnetization of ferromagnetics under the external magnetic field at the critical temperature. From Eq. (13) "the equivalent susceptibility" $\tilde{X}$ in the vicinity of critical point of AOB can be defined as follows:

$$\tilde{X}\equiv\frac{\partial\eta}{\partial\tilde{H}}=\left[\frac{\partial\tilde{H}}{\partial\eta}\right]^{-1}=|t|^{-\nu}, \quad \nu=1 \ . \tag{18}$$

We continue to discuss the solutions of the order parameter under the condition the equivalent field $\tilde{H}\neq 0$. For this purpose, Eq. (13) is rewritten in the following typical form of third-order algebraic equation:

$$\eta^3+t'\eta+\tilde{H}'=0 \ , \tag{19}$$

where

$$t'=\frac{t}{4b}, \quad \tilde{H}'=-\frac{\tilde{H}}{4b}=\frac{y-y_{\rm op}}{4b} \ . \tag{20}$$

The discriminant of the solution of Eq. (19) is

$$\Delta=(\tilde{H}'/2)^2+(t'/3)^3 \ . \tag{21}$$

When $\Delta>0$, the η has only one real root, which contains two situations: (i) C is smaller than the critical value C_c, i.e., $t'>0$, this is absolutely the single stable case, no matter what the incident field y is taken as; (ii) $t'<0$, but C is slightly above C_c for the given y, deviating from $y_{\rm op}(\tilde{H}\neq 0)$ the single real root of η means a single stable state in the high branch or in the low branch, i.e., beyond the bistable region. When $\Delta<0$, the η has three unequal real roots, which means that not only is C above $C_c(t'<0)$ but also the y has been biased into the bistable region. The above discussion shows that above the critical point, $\tilde{H}=0$ ensures that the incident y holds always in the bistable region. Taking $\tilde{H}=0$ would make clearer the demonstration of the characteristic features of OB critical phenomena. So far, the OB critical features, revealed by us, are completely similar to ferromagnetics, while the critical phenomena of ferromagnetics are a prototypical second-order transition.

IV. SECOND-ORDER TRANSITION AND SCALING RULE

In order to further demonstrate that the OB critical phenomena are in accord with the general definition of a second-order transition (Ehrenfest definition), we consider the formal entropy $\tilde{S}$ introduced in Ref. 6. $\tilde{S}$ is just the first-order derivation of $\tilde{G}$, i.e., $\tilde{S}=\partial\tilde{G}/\partial C$. First, we consider the case of the generalized thermodynamic potential $\tilde{G}$ with $\tilde{H}\neq 0$. Referring to Eqs. (12), (9a), and (9b), we have

$$\tilde{S}=\left[\frac{\partial \tilde{G}}{\partial C}\right]_{\eta}=\ln(1+x_c^2)-\frac{\partial \tilde{H}}{\partial C}\eta-\frac{\eta^2}{2C_c} ,$$

where

$$-\frac{\partial \tilde{H}}{\partial C}=\frac{2x_c}{1+x_c^2} \quad (\eta^4 \text{ being neglected}) . \tag{22}$$

The above equation is just the approximate series expansion of $\tilde{S}$, formulated in Ref. 6 as follows:

$$\tilde{S}=\ln(1+x^2)=\ln[1+(x_c+\eta)^2] . \tag{23}$$

The advantage of the formal entropy $\tilde{S}$ formulated as Eq.(22) is that it includes the terms of the first power of η. If we switch up or down the OB's, i.e., altering the sign of η, then the formal entropy $\tilde{S}$ is discontinuously changed. Evidently, this shows the character of first-order transition. However, in order to demonstrate the second-order transition of OB we would prefer to use the $\tilde{G}(\eta)$ with $\tilde{H}=0$ to derive the formal entropy $\tilde{S}$. In this case the result is

$$\tilde{S}=\frac{\partial \tilde{G}}{\partial C}=\ln(1+x_c^2)-\frac{\eta^2}{2C_c} . \tag{24}$$

Substituting the solution of η, expressed by Eqs. (16) with $\tilde{H}=0$, into the above equation, we have

$$\begin{aligned} &\tilde{S}_0=\ln(1+x_c^2), \quad t>0 \quad (C<C_c) \\ &\tilde{S}=\ln(1+x_c^2)+\frac{t}{8b}, \quad t<0 \quad (C>C_c) . \end{aligned} \tag{25}$$

Therefore, if the system passes through the critical point, the change of formal entropy $\tilde{S}$ is continuous. Because the cooperative parameter C is equivalent to the temperature in equilibrium thermodynamics, the formal specific heat $\tilde{c}$ can be defined by the following fashion:[3,6]

$$\tilde{c}=C\frac{\partial \tilde{S}}{\partial C}=-\frac{C}{C_c}\frac{\partial \tilde{S}}{\partial t} . \tag{26}$$

According to Eq. (25) we have

$$\tilde{c}=\begin{cases} -\dfrac{C}{C_c}\dfrac{\partial \tilde{S}}{\partial t}=0, & t>0 \\ -\dfrac{C}{C_c}\dfrac{\partial \tilde{S}}{\partial t}=-\dfrac{C}{8bC_c^2}, & t<0 . \end{cases} \tag{27a}$$

Thus, when the system passes through the critical point, the formal specific heat $\tilde{c}$—the second-order derivation of $\tilde{G}$—would discontinuously change. This situation clearly indicates that the OB critical phenomena are of a second-order phase transition.

At the OB critical point formal specific heat $\tilde{c}$ has a definite jump [see Eq. (27a)], that is,

$$\begin{aligned} \tilde{c}(t\to 0^+)-\tilde{c}(t\to 0^-) &= \lim_{C\to C_c}\frac{C}{8bC_c^2} \\ &= \lim_{C\to C_c}\frac{C}{2C_c} \quad (\text{for AOB}) \\ &= \tfrac{1}{2} . \end{aligned} \tag{27b}$$

The singularity of the "specific heat" $\tilde{c}$ may be expressed as follows:

$$\tilde{c}\sim|t|^{-\alpha}+\text{nonsingular parts} . \tag{28}$$

Combining the critical exponents ($\beta=\frac{1}{2}$, $\delta=3$, $\nu=1$, and $\alpha=0$) in Eqs. (16), (17), (18), and (28) we find that they are just the same as that of the ferromagnetics, and obey the following scaling rules:[8]

$$\alpha+2\beta+\nu=2, \quad \alpha+\beta(\delta+1)=2 . \tag{29}$$

V. CONCLUSIONS

Summarizing the above discussion on OB critical phenomena, we would like to make the following conclusions.

(i) The optical bistability is not only an example of the first-order transition, but also plays the role of a prototype of the second-order transition.

(ii) The OB critical phenomena can be brought into the framework of Landau phase transition theory, at least for the case of AOB.

(iii) The relations among the critical exponents of AOB obey the scaling rules, as shown in Eqs. (29).

(iv) Although the purpose of the present paper is to study the OB critical phenomena, if comparing it with the work of Degiorgio and Scully (analogy between laser and ferromagnetics[2]) and with the work of Graham and Haken (analogy between laser and superconductor[3]), it is easy to find that the discussion of this paper has included the laser threshold behavior with injected signal[2] ($y\neq 0$, $\tilde{H}\neq 0$) and/or without injected signal ($y=0$, $\tilde{H}=0$, since for lasers, $x_c=0$). Thus this paper would be a development and synthesis of the related works of above authors.

(v) In principle, the discussion on OB critical behavior in this paper is based on the statistical theory, because the starting point is the generalized thermodynamic potential $\tilde{G}$ which represents the statistical properties of AOB. Certainly the works of other authors mentioned above are the same.

(vi) The generalized potential $\tilde{G}$, related to our discussion, is a particular form shown as Eq. (2) which satisfies the principle of detailed balance.[6] Therefore, in the studied case, the foundation of an analogy between the equilibrium and the far-from-equilibrium phase transition would still be the detailed balance.

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[1]R. Bonifacio and L. A. Lugiato, in *Dissipative Systems in Quantum Optics,* edited by R. Bonifacio (Springer-Verlag, Berlin, 1982), pp. 61 and 64.

[2](a) V. Degiorgio and M. O. Scully, Physics Rev. A **2**, 1170 (1970); (b) M. Sargent III and M. O. Scully, in *Laser Handbook,* edited by F. T. Arrechi and E. O. Schulz-Dubois (North-Holland, Amsterdam, 1972), Vol. I.

[3]R. Graham and H. Haken, Z. Phys. **237**, 31 (1970); H. Haken, Ref. 2(b).

[4](a) L. A. Lugiato, in *Progress in Optics XXI,* edited by E. Wolf (Elsevier Science, New York, 1984), pp. 71 and 188; (b) R. Bonifacio and L. A. Lugiata (Ref. 1), p. 84.

[5]C. R. Willis, in *Optical Bistability,* edited by C. M. Bowden *et al.* (Plenum, New York, 1981), p. 431.

[6]Fa Ou and Zixiong Qin, Opt. Commun. **65**, 455 (1988).

[7]R. Bonifacio, M. Gronchi, and L. A. Lugiato, Phys. Rev. A **18**, 2266 (1978).

[8]S. K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, New York, 1976).