Quasiperiodicity versus mixing instability in a kicked quantum system

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A two-level system subject to quasiperiodically modulated kicking is investigated under the aspects of ergodic theory. For weak kicking strength κ the correlation functions exhibit quasiperiodic behavior, whose character changes with increasing κ as recurrences become infrequent. A quantum instability with a transition to mixing behavior that was suggested for similar models does not arise. Analytic results are obtained for the case where two of three characteristic frequencies are incommensurate.

I. INTRODUCTION

The quantum analogs of classically chaotic systems, which are paraphrased by the term quantum chaos, are often found to lack chaotic behavior. The best-known example is the chaotic diffusion of angular momentum of the kicked rotator,[1] which is mimicked only for a finite time by the quantum system, before quantum interferences limit the growth of the mean-square displacement.[2] The temporal behavior is quasiperiodic and thus cannot be chaotic in the strict sense. A question of considerable interest is to what extent quantum mechanics can show any chaotic behavior at all, and whether there exists an instability in quantum mechanics causing a transition from nonmixing to mixing behavior. A system is called mixing if for $t \to \infty$ its correlation functions decay to zero (for all square-integrable variables of mean zero), i.e., the variables become statistically independent asymptotically. Mixing implies ergodicity. The above quasiperiodic behavior is not mixing.

This question is motivated by the following background. In periodically driven systems the time-evolution operator U is periodic in time, and according to the Floquet theorem wave packets evolve as superpositions of periodic functions multiplied by $\exp(i\omega_\nu t)$. In bounded systems the spectrum of eigenvalues ω_ν of the Floquet operator is discrete, and thus the correlation functions do not decay. However, if the system is subject to a quasiperiodic driving, the assumptions of the Floquet theorem do not hold, the time evolution need not be quasiperiodic, and correlation functions might decay. With increasing perturbations one might expect a transition from nonmixing to mixing behavior.

Quasiperiodic driving under these aspects was studied in a number of papers. Shepelyansky assumed a quasiperiodic modulation of the kicking strength for the kicked rotator and observed that the quantum limitations of diffusion mentioned above disappear, i.e., the mean-square displacement does not stop growing.[3] In the same model, Samuelides *et al.*[4] studied the spectral measure of the quasienergy spectrum and observed a transition from a pure point to a continuous spectrum. A continuous-time two-level system with periodic driving was subjected to a quasiperiodic modulation by Pomeau, Dorizzi, and Grammaticos.[5] Their results suggest a transition to an absolutely continuous spectrum and decaying correlations. This conclusion was questioned by later authors.[6] Sutherland studied a spin-$\frac{1}{2}$ system subject to a quasiperiodic Fibonacci sequence of kinks.[7] He concluded that there are autocorrelation functions decaying like a power law, weaker than a power law, and faster than a power law. The same model was used by Luck, Orland, and Smilansky,[8] who argued that there exists a singular continuous spectrum and that the temporal behavior is intermediate between quasiperiodic and chaotic. In a generalization to N-level systems by Graham,[9] a mixed discrete and continuous spectrum shows up as an additional possibility.

Comparing these conclusions we do not find a clear-cut answer to the question raised in the beginning. On one hand, part of this work was based on numerical results whose limitations and subtleties may cause misinterpretations and may have led to the partly contradicting conclusions. On the other hand, the rigorous results do not pertain to the decay of correlation functions, which is of major interest from the point of view of ergodic theory. The purpose of the present work is to find an unambiguous conclusion by considering a simple model which allows a partly analytical treatment and a simple and accurate numerical treatment. The model is a kicked two-state system with a quasiperiodic modulation of kicks. We shall concentrate on autocorrelation functions since they are more directly related to the question of mixing than the spectral properties. The simplicity of the numerical treatment also gives us some insight into possible misinterpretations related to incommensurate frequencies in numerical simulations.

II. DYNAMICS OF A KICKED TWO-LEVEL SYSTEM

As discussed above, let us assume a two-level system subject to a kicking perturbation that can be modulated quasiperiodically

$$H = H_0 + V(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) . \quad (1)$$

The unperturbed part with level spacing ϵ can always be written in terms of a Pauli spin operator

$$H_0=\epsilon\sigma_z \ . \tag{2}$$

Transitions can be caused by a transverse magnetic field of strength $B(t)$, which is allowed to be time dependent

$$V(t)=-\mu B(t)\sigma_x \ . \tag{3}$$

The temporal evolution need only be considered stepwise. Let $|\psi(n)\rangle$ denote the state right after the nth kick, then

$$|\psi(n+1)\rangle=e^{-iV(nT)T/\hbar}e^{-iH_0T/\hbar}|\psi(n)\rangle \ . \tag{4}$$

Here the time evolution operator U could be split into two parts, as usual in kicked systems. One part gives the evolution due to the kicks, the other part describes the evolution in between the kicks. If we represent the state of the system in a basis of unperturbed states $|\varphi_i\rangle$

$$|\psi(n)\rangle=\sum_{i=1}^{2} a_i(n)|\varphi_i\rangle \ , \tag{5}$$

where $|\varphi_1\rangle=|\sigma_z=-\frac{1}{2}\rangle$ and $|\varphi_2\rangle=|\sigma_z=\frac{1}{2}\rangle$, Eq. (4) becomes

$$a_i(n+1)=\sum_{j=1}^{2}\langle\varphi_i|e^{-iV(nT)T/\hbar}|\varphi_j\rangle e^{-i\epsilon_j T/\hbar}a_j(n) \ , \tag{6}$$

with $\epsilon_j=\pm\epsilon/2$. This is simply an iteration of 2×2 matrices, in a matrix notation

$$\mathbf{a}(n+1)=\underline{U}(n)\underline{U}_0\mathbf{a}(n) \ . \tag{7}$$

Here the unperturbed evolution between kicks is determined by the matrix

$$\underline{U}_0=\begin{bmatrix}\exp(-i\omega_0/2) & 0\\ 0 & \exp(i\omega_0/2)\end{bmatrix} , \tag{8}$$

with $\omega_0=2\epsilon T/\hbar$. Using as an abbreviation

$$k(n)=k_n=\frac{2\mu T}{\hbar}B(nT) \tag{9}$$

for the time-dependent kicking strength, the unitary operator in Eq. (6) can be written

$$e^{-iV(nT)T/\hbar}=e^{ik(n)\sigma_x/2} \ , \tag{10}$$

which can be simplified to

$$e^{-iV(nT)T/\hbar}=\underline{1}\cos\frac{k(n)}{2}+i\sigma_x\sin\frac{k(n)}{2} \ . \tag{11}$$

The matrix $\underline{U}(n)$ thus reads

$$\underline{U}(n)=\begin{bmatrix}\cos[k(n)/2] & i\sin[k(n)/2]\\ i\sin[k(n)/2] & \cos[k(n)/2]\end{bmatrix} . \tag{12}$$

Equation (7) depends on time merely through the kick number n. Lacking enough symbols, t will also be used henceforth to denote the kick number. [This use as an integer thus departs from the previous convention, e.g., in Eq. (1)]. The temporal evolution according to Eq. (7) consists in a repeated application of a diagonal matrix and a nondiagonal time-dependent matrix. It is more convenient to deal with a diagonal time-dependent matrix at the expense of nondiagonal elements for the time-independent matrix. This is achieved by a transformation to vectors $\mathbf{b}=\underline{X}\mathbf{a}$ in a σ_x basis, with

$$\underline{X}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} . \tag{13}$$

Equation (7) is replaced by

$$\mathbf{b}(t+1)=\underline{U}'(t)\underline{U}_0'\mathbf{b}(t) \ , \tag{14}$$

where

$$\underline{U}'(t)=\begin{bmatrix}e^{-ik(t)/2} & 0\\ 0 & e^{ik(t)/2}\end{bmatrix} , \tag{15}$$

$$\underline{U}_0'=\begin{bmatrix}\cos(\omega_0/2) & -i\sin(\omega_0/2)\\ -i\sin(\omega_0/2) & \cos(\omega_0/2)\end{bmatrix} . \tag{16}$$

For numerical and other reasons it is useful to carry out another transformation to obtain the real quantities A, B, and C (Bloch variables) defined as

$$A=|b_2|^2-|b_1|^2 \ , \tag{17a}$$

$$B=i(b_2b_1^*-b_1b_2^*) \ , \tag{17b}$$

$$C=b_2b_1^*+b_1b_2^* \ . \tag{17c}$$

A lengthy but straightforward calculation yields the equation of motion

$$\begin{bmatrix}A(t+1)\\ B(t+1)\\ C(t+1)\end{bmatrix}=\begin{bmatrix}\cos\omega_0 & \sin\omega_0\cos k_t & -\sin\omega_0\sin k_t\\ -\sin\omega_0 & \cos\omega_0\cos k_t & -\cos\omega_0\sin k_t\\ 0 & \sin k_t & \cos k_t\end{bmatrix}\begin{bmatrix}A(t)\\ B(t)\\ C(t)\end{bmatrix} , \tag{18}$$

where the abbreviation $k_t=k(t)=2\mu TB(tT)/\hbar$ was introduced. In particular, the kicking strength will be assumed to vary sinusoidally

$$k_t=\kappa\cos(\omega t) \tag{19}$$

with an amplitude κ and a frequency ω incommensurate with 2π. In some sense the parameter κ may instead be viewed as a frequency itself, since it appears as an argument of sinusoidal functions in Eq. (18).

The autocorrelation functions $C_1(t)$ and $C_2(t)$ for the variables $b_1(t)$ and $b_2(t)$ are defined as time averages, e.g.,

$$C_2(t)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}b_2^*(n)b_2(t+n) \ . \tag{20}$$

This quantity describes the extent to which the occupation of spin state two (in the σ_x basis) at time t is statistically dependent on its occupation at time zero. The fact that there are only two spin states and that the system must (partially) recur to a previously occupied state should not mislead to the conclusion that for this reason the correlation function be recurrent after arbitrarily long times. Instead, one can imagine that the flipping between the two states happens in a statistically independent way (for $t\to\infty$).

The notation of the correlation functions should not be confused with the Bloch variable $C(t)$ whose own correlation functions $C_A(t)$, $C_B(t)$, and $C_C(t)$ are useful for numerical computations, e.g.,

$$C_B(t)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}B(n)B(t+n)\ . \tag{21}$$

The Bloch variables fulfill the equation

$$A^2+B^2+C^2=1\ , \tag{22}$$

and consistently the left-hand side of Eq. (22) must be a dynamical invariant of Eq. (18). The Jacobian of this transformation is unity and the map is area preserving. A, B, and C thus are components of a real vector that is restricted to move on the unit sphere.

III. ANALYTIC RESULT IN A SPECIAL CASE

The above model has three characteristic frequencies, the kicking frequency 2π, the level spacing $\omega_0/2$, and the modulation frequency ω [Eq. (19)]. In the general case these frequencies may all be chosen incommensurate with each other. In a special case where only ω and the kicking frequency are incommensurate, the correlation functions can be calculated analytically as follows. As an example, let us assume ω_0 as an odd multiple of 2π.

The iteration according to Eqs. (14)–(16) yields

$$b_2(n)=e^{i\pi n}\exp\left[\frac{i}{2}\sum_{l=0}^{n-1}k_l\right]b_2(0)\ , \tag{23}$$

and the correlation function $C_2(t)$ can be expressed as

$$C_2(t)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{i\pi t}\exp\left[\frac{i}{2}\sum_{l=n}^{t+n-1}k_l\right]|b_2(0)|^2\ . \tag{24}$$

Assuming a sinusoidal modulation like in Eq. (19), i.e.,

$$k_l=\frac{\kappa}{2}(e^{i\omega l}+e^{-i\omega l})\ , \tag{25}$$

the sum over the phases k_l in Eq. (24) can be carried out

$$\begin{aligned}\sum_{l=n}^{n+t-1}k_l&=\frac{\kappa}{2}\left[e^{i\omega n}\sum_{l=0}^{t-1}e^{i\omega l}+\text{c.c.}\right]\\&=\frac{\kappa}{2}\left[e^{i\omega n}\frac{1-e^{i\omega t}}{1-e^{i\omega}}+\text{c.c.}\right]\ .\end{aligned} \tag{26}$$

This is further simplified to

$$\sum_{l=n}^{n+t-1}k_l=2\beta(t)\cos[\omega n+\gamma(t)]\ , \tag{27}$$

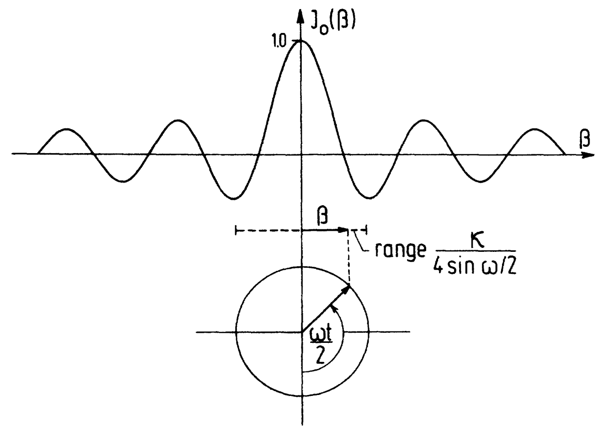

FIG. 1. Illustration of the quasiperiodicity of the correlation function $C_2(t)$ according to Eqs. (33) and (28).

where

$$\beta(t)=\frac{\kappa}{4\sin(\omega/2)}\sin\left[\frac{\omega t}{2}\right]\ , \tag{28}$$

$$\gamma(t)=\frac{\omega t}{2}-\frac{\omega}{2}\ , \tag{29}$$

Eq. (24) now becomes

$$C_2(t)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}(-1)^t\times e^{i\beta(t)\cos[\omega n+\gamma(t)]}|b_2(0)|^2\ . \tag{30}$$

Consider t as fixed and use the fact that the sequence of phases $\omega n+\gamma(\text{mod}2\pi)$ is ergodic and uniformly distributed on the circle for ω incommensurate with 2π (it can be generated e.g., by a homeomorphism of the circle), i.e.,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\delta(\varphi-\omega n-\gamma)d\varphi=\frac{1}{2\pi}d\varphi\ , \tag{31}$$

where $\omega n+\gamma$ is assumed mod2π. This leads to

$$C_2(t)=|b_2(0)|^2(-1)^t\frac{1}{2\pi}\int_0^{2\pi}d\varphi\, e^{i\beta(t)\cos\varphi}\ . \tag{32}$$

The integral yields the Bessel function J_0 and thus

$$C_2(t)=|b_2(0)|^2(-1)^tJ_0(\beta(t))\ . \tag{33}$$

It is instructive to analyze Eq. (33) in some detail. It is defined only for integer t and for $\omega/2\pi$ irrational is a quasiperiodic function based on the two frequencies π and $\omega/2$ according to Eq. (28). Thus it cannot decay to zero, but in the course of time must recur arbitrarily close to its initial value. Figure 1 illustrates the apparent randomness of the function and shows how the recurrences may be strongly suppressed: the argument β of the Bessel function varies in a range $\pm\kappa/[4\sin(\omega/2)]$. It can be viewed as the projection of a vector which revolves at an irrational frequency $\omega/2$. The phase $\omega t/2$ is ergodic and uniformly distributed as in Eq. (31). It thus recurs arbitrarily close to its initial value at $\omega t/2=0$.

When this happens, $J_0(\beta)$ and the autocorrelation function $C_2(t)$ recur close to their maxima.

When κ becomes very large, the range of β also becomes very large. The Bessel function decays asymptotically like $J_0(\beta) \sim \beta^{-1/2}$. The correlation function reaches large values only when β recurs sufficiently close to zero. For large κ, this may occur very infrequently and a possible suppression of recurrences of the correlation function Eq. (33) can thus be understood. More precisely, as the phases are distributed uniformly, the recurrences of the argument β follow the distribution

$$p(\beta) = \frac{1}{\pi}\left[\frac{\kappa^2}{16\sin^2(\omega/2)} - \beta^2\right]^{-1/2}, \qquad (34)$$

which diverges at the upper and lower end.

IV. NUMERICAL RESULTS

The simplicity of the equation of motion Eq. (18) allows an accurate determination of the autocorrelation functions for the general case of three incommensurate frequencies. The computations were done in double precision and the time averages in Eq. (21) were carried out over 4000 steps after their convergence had been tested by varying N.

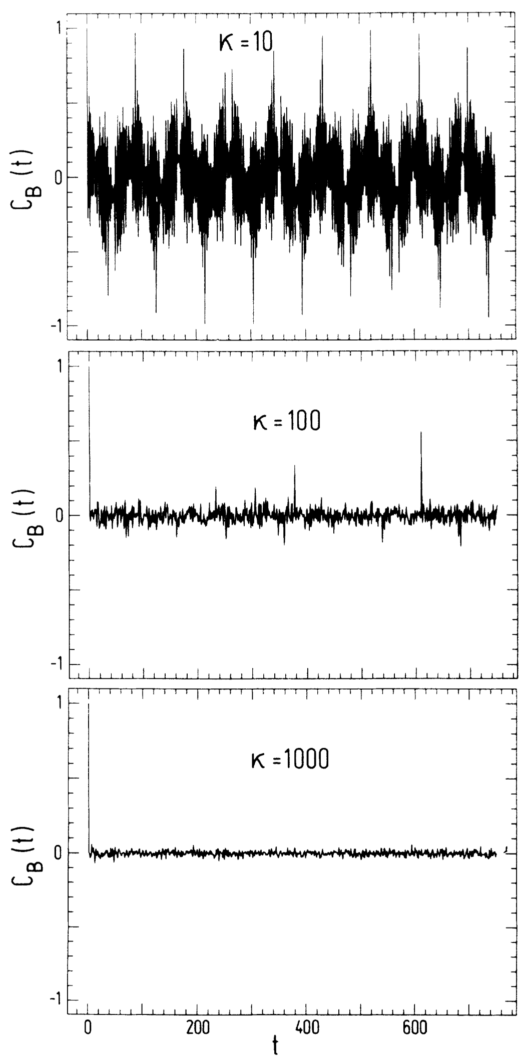

FIG. 2. Autocorrelation function of the Bloch variable B for increasing values of the kicking strength κ. The choice of frequencies is 2π for the kicking frequency, the golden mean for the modulation frequency ω, and $\omega_0/2 = \frac{1}{2}$ for the level spacing.

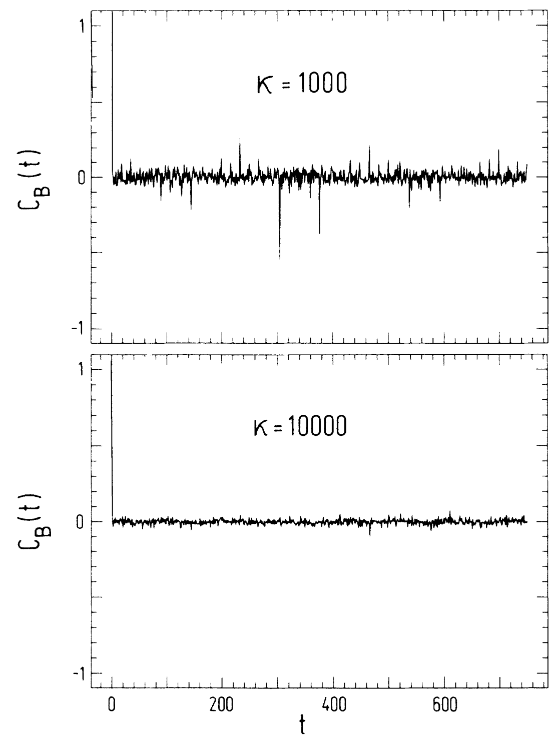

FIG. 3. Same as Fig. 2 apart from a commensurate choice $\omega_0/2 = 2\pi$ for the level spacing. The modulation and kicking frequencies are incommensurate as before.

For the figures shows here, the kicking frequency was 2π, the modulation frequency ω was chosen as the golden mean $\omega = (\sqrt{5}-1)/2$, and ω_0 as $\omega_0 = 1$ (except in Fig. 3). Figure 2 shows this case selecting the autocorrelation function of $B(t)$ for increasing amplitude κ. For $\kappa = 10$ and 100 the time dependence is obviously quasiperiodic, although it looks somewhat different in character. For $\kappa = 1000$ one finds a sharp initial drop followed by small values fluctuating around zero. Looking at these figures and interpreting the fluctuations as numerical errors, one might conclude, as was done in some of the previous work, that there is a transition from a quasiperiodic to a mixing behavior between $\kappa = 100$ and 1000. But now let us look at the case $\omega_0 = 4\pi$ (Fig. 3). Apart from a factor of 10 difference in κ this figure looks qualitatively the same as Figs. 2(b) and 2(c). In particular, with the same right as before, one might conclude that Fig. 3(b) pertains to mixing behavior. This, however, is wrong, as we know from the analytic argument of Sec. III that in the cases $\omega_0 = 2\pi n$ the correlation functions are genuinely quasiperiodic. Therefore it is not justified either to conclude for mixing in Fig. 2(c).

There are stronger arguments against mixing. Figure 4 shows a logarithmic version of Fig. 2(c). One notes that the correlation function reaches small values (e.g., 10^{-6})

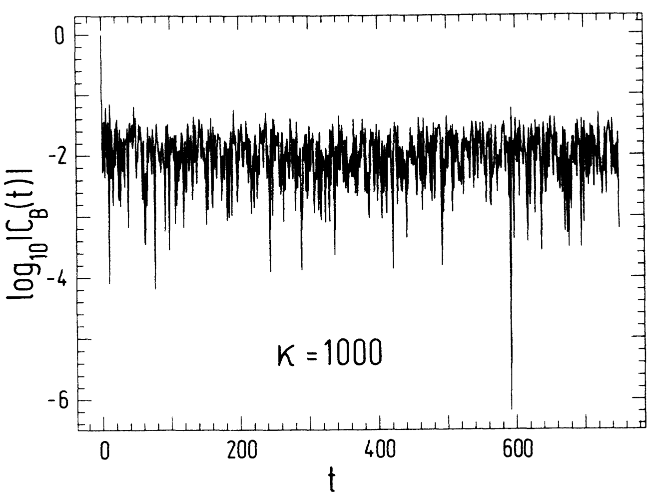

FIG. 4. Logarithm of the absolute value of the correlation function of Fig. 2(c).

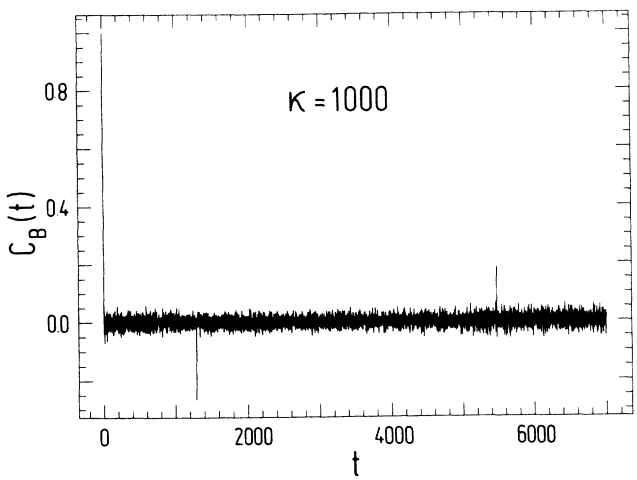

FIG. 5. Same as Fig. 2(c) in a larger window of time.

only exceptionally. After the initial drop it fluctuates around 10^{-2} without decaying further. This is similar to what one expects for the quasiperiodic behavior of Sec. III as illustrated in Fig. 1. Another similarity is the sharpness of the initial drop, which happens in a single time step. If the system had a mixing instability, one would typically expect a finite correlation time, which would gradually increase as the quasiperiodic regime is approached.

If it is true that increasing κ only makes recurrences of the correlation functions less frequent (e.g., by a mechanism as in Fig. 1), then one should be able to detect them again by increasing the simulation time. This was done in Fig. 5, which is otherwise identical to Fig. 2(c). Indeed, two large recurrences show up between $t=1000$ and 7000.

V. CONCLUSION

In this paper the nature of the correlation functions of a quasiperiodically kicked two-level system was analyzed. With increasing kicking strength κ the correlation functions change in aspect, but there is sufficient evidence that they remain quasiperiodic up to $\kappa=1000$. Increasing κ does not destroy correlations, but shifts them to (perhaps unobservably) long time scales. A possible mechanism might be similar to the one illustrated in Fig. 1, where the phase matching required for a recurrence may become more and more restrictive. An analogous phenomenon may show up in numerical spectral analysis. If the number of different frequency channels is sufficiently smaller than the number of δ peaks in a discrete spectrum, it will appear as if it was continuous.

Instead of analyzing spectral properties, the paper has concentrated only on correlation functions since they are most definitive from the point of view of ergodic theory. There is no mixing instability as κ is varied up to 1000. Of course, no conclusions can be drawn on different and more complicated systems. The present analysis, however, helps us to understand the subtleties that may affect the results of numerical simulations with incommensurate frequencies. These are presently of considerable interest, e.g., in relation with quantum localization and delocalization.[10] The presence or absence of correlations in a system may make a difference for the outcome of simulations. Numerically, however, a quasiperiodic system may become indistinguishable from a random system.

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[1]G. Casati, B. V. Chirikov, F. M. Izrailev, and J. Ford, in *Stochastic Behavior in Classical and Quantum Hamiltonian Systems*, Vol. 93, of *Lecture Notes in Physics*, edited by G. Casati and J. Ford (Springer, Berlin, 1979), p. 334; B. V. Chirikov, F. M. Izrailev, and D. L. Shepelyansky, Sov. Sci. Rev. Sect. C **2**, 209 (1981).
[2]T. Hogg and B. A. Huberman, Phys. Rev. Lett. **48**, 711 (1982); S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. Lett. **49**, 509 (1982).
[3]D. L. Shepelyansky, Physica **8D**, 208 (1983).
[4]M. Samuelides, R. Fleckinger, L. Touzillier, and J. Bellissard,

Europhys. Lett. **1**, 203 (1986).
[5]Y. Pomeau, B. Dorizzi, and G. Grammaticos, Phys. Rev. Lett. **56**, 681 (1986).
[6]R. Badii and P. F. Meier, Phys. Rev. Lett. **58**, 1045 (1987).
[7]B. Sutherland, Phys. Rev. Lett. **57**, 770 (1986).
[8]J. M. Luck, H. Orland, and U. Smilansky, J. Stat. Phys. **53**, 551 (1988).
[9]R. Graham, Europhys. Lett. **8**, 717 (1989).
[10]G. Casati, I. Guarneri, and D. L. Shepelyansky, Phys. Rev. Lett. **62**, 345 (1989); E. Doron and S. Fishman, *ibid.* **60**, 867 (1988).