Exact solution in the semiclassical Jaynes-Cummings model without the rotating-wave approximation

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(Received 17 July 1989)

Following the lines sketched by Kujawski [Phys. Rev. A **37**, 1386 (1988)] for the semiclassical version of the Jaynes-Cummings model without the rotating-wave approximation, we have obtained an alternative analytic periodic solution given in terms of elliptic functions. We discuss domains of values of the coupling constant and the detuning for which known yet exact periodic solutions exist.

One of the simplest theoretical models used in quantum optics is the Jaynes-Cummings (JC) model.[1] The model consists of a two-state atom interacting with a single model of the electromagnetic field in a resonant cavity. A system of N two-state atoms interacting with a single-mode field was studied by Tavis and Cummings.[2] Both of these models, the JC model and the generalized one, are exactly solvable within the rotating-wave approximation (RWA) for the field treated classically or quantum mechanically. (In both cases solvability is assured by the existence of an additional integral of motion which reduces the problems to a diagonalization of *finite* matrices. Since the order of these matrices depends on N the final goal can be achieved only numerically in the case of a large number of atoms.) The chaotic behavior in the semiclassical, non-RWA Jaynes-Cummings model was noticed for the first time by Belobrov, Zaslavskii, and Tartakovskii.[3] They showed that the chaos was a consequence of inclusion of terms normally neglected in the RWA. Similar conclusions were drawn independently by Milonni, Ackerhalt, and Galbraith,[4] although they studied a slightly different model than Belobrov, Zaslavskii, and Tartakovskii.

The two-level atom model has its counterpart in condensed-matter physics, which makes it interesting from a different physical point of view. Studying the Jahn-Teller effect,[5] Judd has discovered a class of exact isolated solutions (eigenstates) of the model.[6] The most complete and simple description of these solutions, also for the optical applications, has been given by Reik, Nusser, and Amarante Ribeiro[7] in terms of the Neumann series expansion for eigenvectors in the Bargmann representation for boson operators. A method of obtaining exact isolated solutions for the class of quantum optical systems (two-level and multilevel) without the RWA has been presented by Kuś and Lewenstein.[8] On the other hand, an example of an exact analytic solution in the semiclassical Jaynes-Cummings model has been discovered by Kujawski.[9] In this solution the electromagnetic field is described by the elliptic cosine function.

The aim of our Brief Report is to present another example of an exact periodic solution in the semiclassical Jaynes-Cummings model without the RWA. Each of the periodic solutions expressed in terms of the elliptic Jacobian functions is valid only for a definite domain of values of the coupling constant and the detuning.

The Hamiltonian for a quantum system of N two-level atoms interacting with a single-mode field in a resonant cavity (for the Belobrov, Zaslavskii, and Tartakovskii model) is[10,11]

$$H=\tfrac{1}{2}\hbar\omega_0 S_z+\hbar\omega(a^\dagger a+\tfrac{1}{2})+\hbar\lambda S_x(a+a^\dagger)\ , \tag{1}$$

in which λ is the coupling constant, $S_z=\sum_{j=1}^{N}\sigma_{zj}$, $S_x=\sum_{j=1}^{N}\sigma_{xj}$, and σ_{lj} is the lth Cartesian component for the jth two-level system, and in which $a^\dagger$ and a are photon creation and annihilation operators.

One can easily obtain the Heisenberg operator equations and next the semiclassical model equations:

$$\dot S_1=-S_2\ , \tag{2a}$$

$$\dot S_2=S_1+S_3E\ , \tag{2b}$$

$$\dot S_3=-S_2E\ , \tag{2c}$$

$$\ddot E+\mu^2E=\alpha S_1\ , \tag{3}$$

where the dimensionless parameter $\mu=\omega/\omega_0$, the coupling constant $\alpha=N\mu(2\lambda/\omega_0)^2$, and a dot denotes a time derivative. Time is scaled with the atomic transition frequency ω_0 and therefore dimensionless. S_1,S_2,S_3 are components of the Bloch vector and represent atomic polarization and inversion, whereas the electric field $E=-2(\lambda/\omega_0)A$.

As was noticed before, the two-state atom model has its counterpart in solid-state physics. One example is the dynamical Jahn-Teller and pseudo-Jahn-Teller effect in which vibrational modes interact with electronic levels and a second, simpler one is an electron hopping between two sites and interacting with one vibrational mode.[12] The Hamiltonian of this last system makes its appearance in optics, where it describes a two-level atom interacting with one linearly polarized radiation mode. In contradiction to the solid-state analog, the interaction constant is much smaller in the optical case. There are also no objections as to being far away from the resonance, otherwise than in the optical case. The conclusion is that giving up strictly optical applications of the semiclassical two-level atom model and searching for other ones, for instance, in condensed-matter physics, one can obtain a larger allowed range of values of the dimensionless parameter μ and the coupling constant α.

The system of Eqs. (2) and (3) possesses two conserved quantities, length of the Bloch vector and energy,

$$S_1^2+S_2^2+S_3^2=1 , \tag{4}$$

$$W=\alpha S_3-\alpha S_1 E+\tfrac{1}{2}\mu^2 E^2+\tfrac{1}{2}\dot{E}^2 . \tag{5}$$

For Eqs. (2) and (3) and considering the conservation law (4), the new solution has been found:

$$E=E_0 \mathrm{dn}(\Omega t, m) , \tag{6}$$

$$m=2\left[1+\frac{1}{\chi}(1-\sqrt{1+\chi})\right] , \tag{7}$$

$$\Omega^2=\frac{\chi(\mu^2-\frac{1}{3})}{4(\sqrt{1+\chi}-1)}, \quad E_0^2=16\Omega^2 , \tag{8}$$

and

$$\chi=-(\mu^2-\tfrac{1}{3})^{-2}\{\tfrac{4}{3}[\alpha^2-4(\mu^2-\tfrac{1}{9})^3]^{1/2}+(\mu^2-\tfrac{1}{9})(\mu^2-\tfrac{17}{9})\} .$$

The inversion S_3 and components of the Bloch vector describing polarization in terms of the electric field E have been obtained in the form

$$S_1=\frac{1}{\alpha}[\tfrac{3}{2}(\mu^2-\tfrac{1}{9})E-\tfrac{1}{8}E^3], \quad S_2=-\dot{S}_1 , \tag{9}$$

$$S_3=\frac{1}{\alpha}\{-\tfrac{3}{2}(\mu^2-\tfrac{1}{9})^2-[\alpha^2-4(\mu^2-\tfrac{1}{9})^3]^{1/2}+\tfrac{3}{4}(\mu^2-\tfrac{1}{9})E^2-\tfrac{3}{32}E^4\} . \tag{10}$$

In this way each component of the Bloch vector is a combination of the elliptic Jacobian functions, therefore also a periodic function.

The second conservation law (5) must be kept; this means energy is allowed to take only specific values:

$$W=-\tfrac{5}{3}[\alpha^2-4(\mu^2-\tfrac{1}{9})^3]^{1/2}-2(\mu^2-\tfrac{1}{9})(\mu^2-\tfrac{5}{9}) . \tag{11}$$

The main point of the consideration is whether one can find such parameters μ and α for which the solution (6) is valid. From the fact that W, Ω, and E_0 are real and $0 \le m \le 1$,[13] domains of values of the coupling constant α and the dimensionless parameter μ have been found:

$$\mu^2\in(\tfrac{1}{3},\tfrac{17}{9}) , \tag{12}$$

$$\alpha^2\in(\tfrac{9}{16}(\mu^2-\tfrac{1}{9})^2(\mu^2+\tfrac{5}{3})^2, 4\mu^4(\mu^2-\tfrac{1}{12})] . \tag{13}$$

One can see that the minimum value of α is $\alpha>\frac{1}{3}$; then it is the case of condensed-matter physics or solid-state physics rather than the optical one.

It is easy to verify that every elliptic Jacobian function is the solution of Eqs. (2) and (3) with the conservation law (4). For each of them one can find specific values of energy (5) for which a solution is valid. Still, it is not possible, for each of these functions, to obtain domains of values of μ and α for which a solution does exist. An example is the solution for the electric field in the form $E=E_0\mathrm{sn}(\Omega t, m)$. One can find expressions for m, Ω^2, E_0^2, and W, but domains of values of μ and α are empty.

The essential observation is that only even elliptic Jacobian functions are the solutions for the problem (See Ref. 13). The question is whether it is an accident or rather the specific property of the set of Eqs. (2)–(4) to have some inner symmetry.

As for the solution, where the electric field has the form

$$E=E_0\mathrm{cn}(\tilde{\Omega}t, m) , \tag{14}$$

$$\tilde{\Omega}^2=\pm\tfrac{1}{2}(\mu^2-\tfrac{1}{3})\sqrt{1+\chi} , \tag{15}$$

$$m=\frac{1}{2}+\frac{\mu^2-\frac{1}{3}}{4\tilde{\Omega}^2}, \quad E_0^2=16m\tilde{\Omega}^2=4(\mu^2-\tfrac{1}{3})+8\tilde{\Omega}^2 , \tag{16}$$

and

$$\chi=(\mu^2-\tfrac{1}{3})^{-2}\{\pm\tfrac{4}{3}[\alpha^2-4(\mu^2-\tfrac{1}{9})^3]^{1/2}-(\mu^2-\tfrac{1}{9})(\mu^2-\tfrac{17}{9})\} , \tag{17}$$

there are two possibilities as to how to choose the sign and each sign one should consider separately. The same problem takes place with expressions for S_3 and W:

$$S_3=\frac{1}{\alpha}\{-\tfrac{3}{2}(\mu^2-\tfrac{1}{9})^2\pm[\alpha^2-4(\mu^2-\tfrac{1}{9})^3]^{1/2}+\tfrac{3}{4}(\mu^2-\tfrac{1}{9})E^2-\tfrac{3}{32}E^4\} , \tag{18}$$

$$W=\pm\tfrac{5}{3}[\alpha^2-4(\mu^2-\tfrac{1}{9})^3]^{1/2}-2(\mu^2-\tfrac{1}{9})(\mu^2-\tfrac{5}{9}) . \tag{19}$$

The domains of values of μ and α for which the solution (13) is valid has been found:

$$\alpha^2\ge\tfrac{9}{16}(\mu^2-\tfrac{1}{9})^2(\mu^2+\tfrac{5}{3})^2 , \tag{20}$$

$$\alpha^2\in(4(\mu^2-\tfrac{1}{9})^3, \tfrac{9}{16}(\mu^2-\tfrac{1}{9})^2(\mu^2+\tfrac{5}{3})^2] , \tag{21}$$

$$\mu^2\in[\tfrac{1}{3},\tfrac{17}{9}) , \tag{22}$$

$$\mu^2\in[\tfrac{17}{9},+\infty) , \tag{23}$$

$$\mu^2\in[0,\tfrac{1}{9}] , \tag{24}$$

$$\mu^2\in(\tfrac{1}{9},\tfrac{1}{3}) . \tag{25}$$

The clue as to how to choose the sign in individual expressions is the following: (1) for + in the expression (15) and for − in (17)–(19) there are conditions (22) and (21); (2) for + in the expression (15) and for + in (17)–(19) there are conditions (23) and (20); (3) for − in the expression (15) and for + in (17)–(19) conditions (24),(20); (4) for − in the expression (15) and for − in (17)–(19) conditions (25),(21). The range of values of parameters μ and α is very large and one has to choose the parameters which fit to a given physical reality.

Since the model exhibits a chaotic behavior for the large set of parameters involved the knowledge of particular exact solutions could be interesting for investigations concerning transition from nonchaotic to chaotic motion.

The authors acknowledge the support of Polish Government Project No. CPBP 0107.

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