Lamb shift: Additional recoil corrections and estimates of radiative proton effects

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Some new corrections to the hydrogen Lamb shift from multiphoton exchange graphs have been calculated and found to be 1 kHz. Radiative corrections to the proton line have also been considered for diagrams containing two exchanged Coulomb interactions and have been found to be of the order of $Z^2\alpha(Z\alpha)^5m^3/M^2$ and therefore negligible.

During the past few years recoil corrections to the Lamb shift of hydrogen reduced the theoretical value to 1057.866 MHz. Corrections presented arose from several sources.[1]

(i) Terms of order $\alpha(Z\alpha)^5m^2/M$ originating from lowest-order electron-radiative corrections with two photons exchanged to the proton. These are called radiative recoil terms.

(ii) Terms of order $(Z\alpha)^6m^2/M$ (nonradiative recoil) which arise from double- and triple-Coulomb interactions, single-transverse interactions, and "seagull" interactions.

It was mentioned that additional terms could be significant to the accuracy required. The present Brief Report provides the results of a calculation of double-transverse–single-Coulomb terms, discussion of insignificant radiative effects on the proton line, and some comments on diagrams which must still be evaluated. The current comparison of theory and experiment is briefly reviewed.

The order $1/M$ Lamb shift contributions can be contained in the set of diagrams shown in Fig. 1. The "dot" on the proton propagator signifies the removal of the positive-energy proton pole contribution. This subtraction is necessary since such terms are already contained in lower-order calculations involving only the seagull graph. When the diagrams of Fig. 1 are combined to obtain a recoil contribution, it is found that the important terms reside only in Figs. 1(c) and 1(f), that is, those graphs in which the Coulomb interaction on the electron side occurs between the transverse photons. The resulting proton expression simplifies to

$$-i\frac{(Ze)^3}{M}g_{ij}(-2\pi i)\delta(q_{3_0}) ,$$

where q_3 is the Coulomb line four-momentum. To construct the entire graph, electron, photon, and Coulomb propagators are needed and the electron line Dirac algebra must be simplified. For evaluation it suffices to use zero three-momentum on the external lines. Consequently, the energy shift occurs in S states and is proportional to the product of the square of the coordinate-space wave function and the two-loop Feynman graphs discussed earlier.

We have done the two-loop integral numerically. The δ function above reduces the integral to seven dimensions and further reduction to six is accomplished by doing a contour integration using Cauchy's theorem. The residual six-dimensional integral is actually only three dimensional, involving an angle θ between three momenta $\mathbf{p}_1$ and $\mathbf{p}_2$, as well as integration on $|\mathbf{p}_1|$ and $|\mathbf{p}_2|$. The integration is carried out using the Monte Carlo integration program VEGAS. This results in a correction

$$[(2.4\pm0.003)/\pi](m^2/M)[(Z\alpha)^6/n^3] ,$$

which, for $n=2$ states of hydrogen, produces an additional Lamb shift of 1 kHz. Inclusion of this term changes the Lamb shift to 1057.867 MHz.

Bhatt and Grotch calculated radiative recoil when the radiative corrections were on the electron line.[2] Are there any corresponding corrections with radiative effects on the proton line? The lowest-order terms of impor-

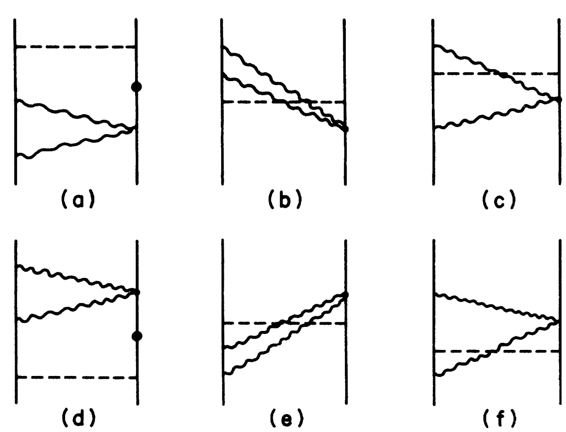

FIG. 1. Recoil contributions due to double-transverse–single-Coulomb exchange. The dot on proton propagators signifies the removal of the positive-energy proton pole contribution. The right-hand line is the proton line.

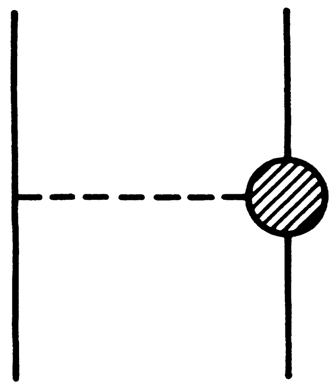

FIG. 2. Lowest-order radiative effects on the proton line. The right-hand line is the proton line.

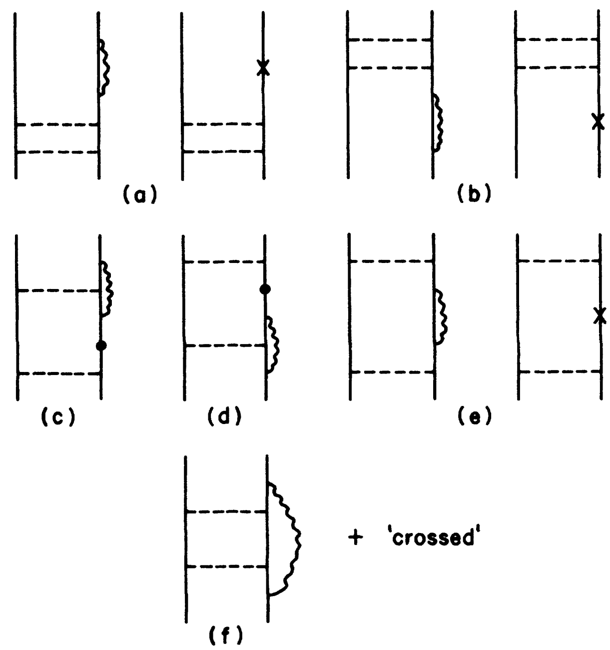

FIG. 3. Higher-order radiative effects on the proton line. The cross signifies the mass counterterm subtraction. "Crossed" describes the other possible time ordering of the exchanged Coulombs. The right-hand line is the proton line.

tance can be depicted in Fig. 2, which contains vertex corrections described by appropriate form factors. A first-principles calculation of this diagram is not possible since it involves unknown strong interactions. Nevertheless, the effect of this graph has been previously considered, and is known to produce a finite proton extension and consequently to alter the Coulomb interaction at shorter distances. This perturbation results in a proton radius correction of 145 kHz for $r_p = 0.862$ fm.

In this Brief Report, we also report on the result of the investigation of additional radiative corrections to the proton line and their contribution to the Lamb shift. In particular, we studied radiative corrections to the two Coulomb exchange diagrams (ladder and crossed), as in Fig. 3. Any contributions smaller than 1 kHz are considered negligible. For example, any corrections of the order of $Z^3\alpha(Z\alpha)^5 m^3/M^2$ are extremely small and can be neglected.

To carry out this analysis, it is convenient to treat the radiative photon in the Coulomb gauge. The conclusions are, however, gauge independent. It will be seen that the contribution of each of the diagrams is of order $Z^2\alpha(Z\alpha)^5 m^3/M^2$. To see this we first examine the external self-energy diagrams [Figs. 3(a) and 3(b)]. It is easily established that these do not contribute, since in this approach[3] the heavy particle (in this case a proton) is on its positive-energy mass shell. Consequently, after mass renormalization there is no contribution since the external self-energy is exactly canceled by the mass counterterm.

In the remaining diagrams it is convenient, for the purpose of this analysis, to decompose each of the proton propagators into a positive- and negative-energy part, i.e.,

$$S_F(p) = \left[\frac{\Lambda^+(p)}{p_0 - E(p) + i\epsilon} + \frac{\Lambda^-(p)}{p_0 + E(p) - i\epsilon} \right] \gamma_0 \, . \quad (1)$$

Thus, for each of the above diagrams, the three proton propagators result in eight terms $(2\times2\times2)$. This decomposition has the advantage that we can focus our attention on the terms containing only the positive-energy projection operators since the negative-energy projection operators, for small proton momentum, go as p^2/M^2. Thus, the expectation value of an operator between two Coulomb states in which one of the factors is the negative-energy projection operator is already decreased by a factor of $(m/M)^2$ relative to the expectation value without the presence of the negative-energy projection operator.

With the foregoing in mind, the vertex diagrams Figs. 3(c) and 3(d) can be shown to be of order $Z^2\alpha(Z\alpha)^5(m^3/M^2)$. This is shown explicitly by first subtracting the positive-energy poles of the propagator which are shown by a dot on some of the propagators in Fig. 3. These indicate that this contribution is to be removed since it has already been included in the one-Coulomb vertex calculation. When this is done, a potentially large, residual, nonrecoil term is found to be negligible as a result of the renormalization constraint that requires $\bar{u}(p)\Lambda^R_\mu(p,p)u(p) = 0$.

The treatments of the internal self-energy and the spanning diagrams [Figs. 3(e) and 3(f), respectively] are similar, and our discussion will be applicable to both. We first discuss the internal self-energy since it is the simpler of the two and then point out how it differs from the calculation of the spanning diagram.

In the contribution of the internal self-energy diagram, the structure in the numerator precludes this diagram from contributing terms larger than $Z^2\alpha(Z\alpha)^5(m/M)^2 m$. This can be seen from the factor

$$\frac{\gamma^\mu \Lambda^+(p-k)\gamma_\mu}{k^2} \xrightarrow[\text{Coulomb gauge}]{} - \left[\frac{\gamma^0 \Lambda^+(p-k)\gamma_0}{\mathbf{k}^2} + \frac{\gamma_{\rm tr}\cdot\Lambda^+(p-k)\gamma_{\rm tr}}{k^2} \right] . \quad (2)$$

For the first term, there is no contribution since the radiative photon (being in the Coulomb gauge) cannot contribute to the k_0 integration (k being the photon momentum) and there is only one factor remaining which depends on k_0. Thus, the contour of integration in the k_0 plane can be chosen to exclude the k_0 pole. The second term of Eq. (2) (that is, the transverse part) can be reduced to the form

$$-\frac{1}{k^2}\left[2\Lambda^-(p-k)+\frac{\boldsymbol{\alpha}\cdot(\mathbf{p}-\mathbf{k})}{E(p-k)}\right] . \tag{3}$$

Again, the negative-energy projection operator, as well as the second term in (3), will lead, at most, to a quadratic recoil correction. As it stands, the second term of (3) is of order m/M but another factor of this order comes about because of the "odd" nature of the $\boldsymbol{\alpha}$ operators which couple large to small components of the wave function, the latter of which are of relative order m/M with respect to the upper components. If one had considered, in place of (2), the contribution arising from

$$\frac{\gamma^\mu\Lambda^-(p-k)\gamma_\mu}{k^2} , \tag{4}$$

the transverse terms here lead to a positive-energy projection operator when the γ_μ is commuted through the negative projection operator. However, it is found through explicit calculation, after performing the k_0 and the p_0 integrations (where p is the loop momentum) that the negative-energy poles depress the integral by a factor of $1/M^2$.

The same discussion applies directly to the spanning diagram. The only difference is that now there are three projection operators sandwiched between the gamma matrices. However, it is readily seen that argument goes through unchanged. This discussion is the same for both the ladder and the crossed diagrams. Thus, all of the above contributions are of order $Z^2\alpha(Z\alpha)^5(m/M)^2m$.

Radiative corrections on the proton side already contribute 145 kHz to the Lamb shift but these are entirely due to the single-Coulomb exchange modified by a proton form factor which originates from strong-interaction effects. Thus, the proton size correction is, in fact, a radiative effect produced primarily by mesonic effects. It is logical to determine whether the strong-interaction effects can also correct the two-Coulomb interaction significantly and thereby contribute to the Lamb shift. To investigate this, the radiative photon in our previous considerations may be replaced by a pseudoscalar particle (i.e., a pion). On the external lines there is no contribution as the mass counter terms again cancel the external self-energy contribution. For the vertex, any contribution would be a small correction to the recoil effect of order $(Z\alpha)^5m^2/M$. That is, this correction would be multiplied by a factor of the ratio of the proton radius to the Bohr radius. Since the γ_μ's in Eq. (2) are now replaced by γ^5's, an equation similar to that of Eq. (3) is obtained without the second term. Thus, the analysis is similar to that of the transverse photon in the above [without the presence of the second term in Eq. (3)]. In this case, it is found for both the internal electron self-energy and the spanning diagram that the contribution is less than $Z^2\alpha_s(Z\alpha)^5(m/M)^2m$.

It must be pointed out that if any of these additional corrections were not negligible, we would be at a loss as to how to evaluate them. For one thing, we really cannot calculate strong-interaction contributions with any confidence, but more important, we cannot extract and isolate these terms since their presence certainly shows up in some manner in the measured proton radius.

We have now calculated a further mass correction to the Lamb shift and obtained the result 1057.867 MHz as quoted earlier. This number is based on the proton radius of Ref. 4. If an older value[5] is used this will be reduced by 18 kHz and produce a result closer to the experimental values 1057.845(9) (Ref. 6) and 1057.851(2) MHz.[7] Unfortunately, we cannot yet draw any definitive conclusions since there are still possible Lamb shift contributions which are significant. Specifically, it is still necessary to evaluate the rather complicated two-loop nonrecoil binding corrections of order $m\alpha^2(Z\alpha)^5$.[8] Research on this problem is currently being carried out.

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[1]G. W. Erickson and H. Grotch, Phys. Rev. Lett. **60**, 2611 (1988); **63**, 1326 (1989).

[2]G. C. Bhatt and H. Grotch, Ann. Phys. (NY) **178**, 1 (1987).

[3]H. Grotch and D. R. Yennie, Rev. Mod. Phys. **41**, 350 (1969).

[4]G. G. Simon, Ch. Schmitt, F. Borkowski, and V. H. Walther, Nucl. Phys. A **333**, 381 (1980).

[5]D. J. Drickey and L. N. Hand, Phys. Rev. Lett. **9**, 521 (1962); L. N. Hand, D. G. Miller, and R. Wilson, Rev. Mod. Phys. **35**, 335 (1963).

[6]S. R. Lundeen and F. M. Pipkin, Phys. Rev. Lett. **46**, 232 (1981); Metrol. **22**, 9 (1986).

[7]V. G. Pal'Chikov, Yu. L. Sokolov, and V. P. Yakovlev, Lett. J. Tech. Phys. **38**, 347 (1983).

[8]H. Grotch, Phys. Scr. **T21**, 86 (1988).