PHYSICAL REVIEW A VOLUME 41, NUMBER 5 1 MARCH 1990

Sub-Poissonian laser

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(Received 6 July 1989)

By intermittently and periodically pumping N atoms and observing time-scale separation for pumping, emission, and photon leakage from the cavity a laser with up to 50%, sub-Poissonian photon statistics can be realized. We present a stroboscopic period-to-period description of such a laser and clarify how pumping and spontaneous emission can be eliminated and cavity damping be minimized as sources of photon noise.

I. INTRODUCTION

The classical fiction of a monochromatic plane wave with fixed amplitude and phase finds it closest possible approximation in the output of single-mode lasers operated far above threshold. The linewidth falls off in inverse proportion to the intensity and on a time scale smaller than the inverse linewidth the laser field appears to be in a coherent state, as is manifest, e.g., in Poissonian photon statistics. According to Heisenberg's uncertainty principle, the uncertainty product for the electric and the magnetic field cannot be reduced below the limit attained for a coherent state.

Noise suppression below the coherent-state limit is possible for a certain observable, but only at the expense of increasing the noise displayed by other quantities. For instance, efforts are under way to achieve squeezed-state radiation from lasers. On the other hand, one may try to stiffen the photon number distribution to sub-Poissonian width with the ultimate goal of producing a photon number eigenstate. In a previous paper,[1] which we shall refer to as I, we have reviewed the literature devoted to such efforts and presented a particular scheme for photon noise reduction.

The present paper is devoted to a more detailed study of a simple model of a sub-Poissonian laser. Our noise reduction scheme, due in essence to Golubev and Sokolov,[2] involves intermittent periodic rather than continuous pumping and separation of time scales for pumping, emission, and photon leakage from the cavity. By making the pumping the fastest of the three processes, practically instantaneous lifting of *all* atoms to the upper working level occurs at the beginning of each cycle of the laser operation. We thus eliminate the pump as a noise source. Photon leakage through the nonideal mirror is required to be the slowest process for two reasons. First, the escape of light from the cavity is an intrinsically random process on the energy scale on which individual photons can be detected and the ensuing photon noise is the smaller the higher the quality of the cavity. Second, small cavity damping implies strong intracavity fields and thus rapid Rabi oscillations between the two atomic working levels; stimulated emission is thereby favored over spontaneous emission and a further quieting of photon noise results. For a complete suppression of spontaneous emission as a source of photon noise we require fast depletion of the lower working level of the atoms by incoherent return to the ground level. When the time-scale requirements just discussed are respected in full the photon noise becomes minimal in a sense: as was already shown in I, the spectral variance of the output intensity reaches a vanishing limit at zero frequency.

The cyclic operation of the laser in consideration invites a stroboscopic cycle-to-cycle description. Within each cycle pumping, emission, and leakage take place separately and successively, each phase allowing for rigorous description. Even the stroboscopic description of the sequence of cycles can be given exactly in the limit of ideal time-scale separation. In fact, the theory of our periodically pumped laser with separated time scales is considerably simpler than that of the conventional laser.

Our last scheme is in some aspects reminiscent of the so-called single-atom maser.[3,4] The lattice device also involves effectively instantaneous pumping by the repeated injection of excited atoms in the maser cavity. Our scheme may indeed be looked up as a generalization of the single-atom maser to the case of periodic "injection" of N excited atoms. With $N \gg 1$ and good time-scale separation our scheme, even though originally proposed for Rydberg transitions,[2] might even work for optical frequencies.

Section II of the present paper describes our model in detail. In Sec. III we give the rigorous treatment of the extreme case of time-scale separation. In Sec. IV we investigate the effect of a nonideal pump with pump efficiency $p < 1$. Section V is devoted to corrections due to spontaneous emission and in Sec. VI we even allow for strong perturbations of the low-noise regime by effective competition of incoherent atomic transitions with the coherent emission of laser photons. In Sec. VII we illustrate our theoretical investigations with some numerical results. Finally, two appendixes contain the derivation of rate equations governing the competition of incoherent and coherent processes.

II. MODEL

We consider N identical active atoms in a resonator one mode of which is brought to self-sustained oscillations. For simplicity we assume exact resonance between the mode frequency ω and two working levels of each atom, 1 and 2, i.e., $E_2-E_1=\hbar\omega$. Two more levels will be of interest, the ground level 0 with $E_0<E_1$, and an auxiliary level 3 with $E_3>E_2$ (see Fig. 1). A periodic train of pump pulses repeatedly brings the atoms from the ground level 0 to the upper working level 2, via the auxiliary level 3. While the pump is off the atoms return to the ground level, ideally via the lower working level 1 such that with the transition $2\rightarrow 1$ each atom deposits one photon in the lasing atom. Under stationary conditions the whole system will display periodic modulations in time of the expected number of photons $\langle n \rangle$ in the (nonideal) resonator as well as of the expected number of atoms $\langle N_i \rangle$ in level i (see Fig. 2).

In order to achieve low noise in the photon number we follow Golubev and Sokolov[2] and require a separation of time scales for the various processes taking place during one cycle. Besides the period T and the duration T_0 of the pump pulse there are four relevant times to be controlled. During a time τ_{pump} the atoms undergo the sequence of transitions $0\rightarrow 3\rightarrow 2$; the subsequent coherent emission of photons into the lasing mode and the atomic decay $2\rightarrow 1\rightarrow 0$ is characterized by a time τ_{emission}; leakage through the nonideal mirror limits the lifetime $1/\kappa$ of photons in the resonator; finally, incoherent atomic relaxations $2\rightarrow 0$ (nonradiative or due to spontaneous emission into nonresonant modes) or $2\rightarrow 1$ (again nonradiative or due to spontaneous emission into resonant but nonlasing "sideway" modes) have a time scale τ_{spont}. The limit we have in mind is

$$\tau_{\text{emission}} \ll \tau_{\text{spont}} \tag{2.1}$$

and

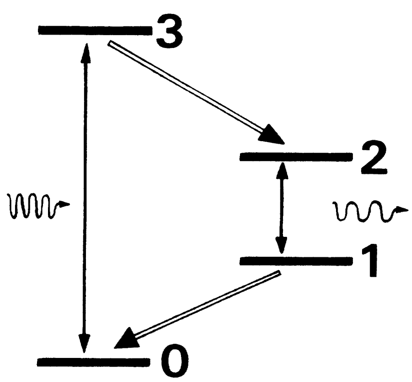

FIG. 1. Level scheme for the active atoms.

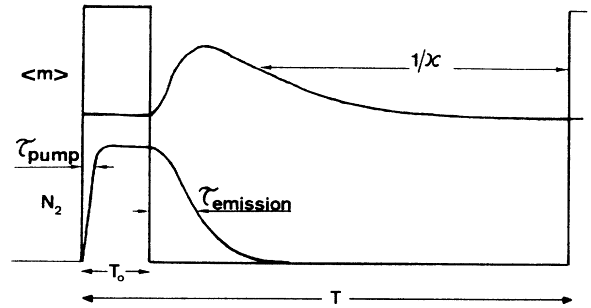

FIG. 2. Schematic time dependence of the mean photon number and the number of atoms in the upper working level.

$$\tau_{\text{pump}} \ll T_0 \ll \tau_{\text{emission}} \ll T < \kappa^{-1} . \tag{2.2}$$

Violation of (2.1) would not only imply reduced efficiency of the laser but also, due to the intrinsically stochastic nature of spontaneous-emission events, increased photon number fluctuations. On the other hand, (2.2) secures that the population of the upper working level by the pump, the coherent emission of photons, and photon leakage take place separately and successively.

It is easy to see, without any calculation, that the latter separation of time scales eliminates the pump as a noise source for the photon number in the laser mode. Due to $\tau_{\text{pump}} \ll T_0$ *all* N atoms assemble in the upper working level before the atom-field interaction begins to deplete that population. Next by $\tau_{\text{emission}} \ll T$ we make sure that all atoms have returned to the ground level before the beginning of the next cycle. Inasmuch as (2.1) precludes incoherent "channels" for the atoms to return to the ground level, every atom arriving in that level is certain to have left behind one photon in the laser mode. Up to this point during each cycle *precisely* N photons have been added to the laser mode. Only during the remaining phase of each cycle, when *on the average* N photons leak out of the resonator (in a stationary regime), can and do uncertainties arise for the photon number since the leakage of a photon in an intrinsically quantum mechanical, i.e., random, event. Even the role of this latter damping as a noise source is minimized by $T \ll 1/\kappa$, as we shall show in Sec. III.

It may be well to comment in some detail on each of the three phases of one cycle, pumping, coherent emission, and leakage. For the sake of concreteness we may imagine the pump pulse as coupling to the atomic transition $0\leftrightarrow 3$ according to the Hamiltonian

$$H_{03} = \sum_{\mu=1}^{N} \hbar\Omega_{30}(S_{03}^{\mu}+S_{30}^{\mu}) . \tag{2.3}$$

Here Ω_{30} is a Rabi frequency proportional to the electric field of the pump. For simplicity, we take the pump pulse to be rectangularly shaped in time such that Ω_{30} is constant for $0\le\tau\le T_0$. The operator S_{03}^{μ} lowers the μth atom from level 3 to level 0 while $S_{30}^{\mu}=(S_{03}^{\mu})^{\dagger}$ is the corresponding raising operator.[5] The population of level 2 may be achieved incoherently with a rate w_{32}. The whole pumping process is then described by the following master equation[6] for the atomic density operator $A(\tau)$,

$$\dot{A}(\tau) = -(i/\hbar)[H_{30}, A(\tau)] + \Lambda_{32}A(\tau) ,$$
$$\Lambda_{32}A(\tau) = (w_{32}/2)\sum_{\mu=1}^{N}\{[S_{23}^{\mu}, A(\tau)S_{32}^{\mu}] + [S_{23}^{\mu}A(\tau), S_{32}^{\mu}]\} , \quad (2.4)$$

where S_{32}^{μ} and $S_{23}^{\mu} = (S_{32}^{\mu})^{\dagger}$ are the raising and lowering operators for the atomic transition $2 \rightarrow 3$. The initial condition for the pump is

$$A(0) = \prod_{\mu=1}^{N} |0\rangle_{\mu\,\mu}\langle 0| . \quad (2.5)$$

In the interesting limit of fast drainage $3 \rightarrow 2$ the characteristic time of the process reads

$$1/\tau_{\text{pump}} = 4\Omega_{30}^{2}/w_{32}, \quad \Omega_{30} \ll w_{32} , \quad (2.6)$$

and for times $\tau_{\text{pump}} \ll \tau < T_0$ the stationary solution of the master equation (2.4),

$$\overline{A} = \prod_{\mu=1}^{N} |2\rangle_{\mu\,\mu}\langle 2| , \quad (2.7)$$

is approached. Due to (2.1) and (2.2) the state of the laser mode does not change during the pumping process. If the full density operator just before the pump is

$$W(0) = A(0)\rho(0) , \quad (2.8)$$

with $\rho(0)$ referring to the laser mode, then right after the pump we have

$$\overline{W} = \overline{A}\rho(0) . \quad (2.9)$$

The coherent-emission phase begins with (2.9) as the initial condition. We now have to deal with the interaction between the atoms and the field mode,

$$H_{12} = \hbar g \sum_{\mu=1}^{N} (S_{21}^{\mu} b + S_{12}^{\mu} b^{\dagger}) . \quad (2.10)$$

Here g is an elementary Rabi frequency referring to a fictitious electric field corresponding to a single laser photon. In the Hamiltonian (2.10) we encounter the photon annihilation and creation operators b and $b^{\dagger}$ whose commutator is $[b, b^{\dagger}] = 1$. Inasmuch as we want every atom to contribute one photon to the laser mode we must provide for an incoherent drainage of atomic population from level 1 back to the ground level with a rate w_{10} by far exceeding the rates w_{21} and w_{20} of incoherent decays $2 \rightarrow 1$ and $2 \rightarrow 0$, respectively. The full description of all competing processes involved is given, in the interaction picture, by the master equation

$$\dot{W}(\tau) = -(i/\hbar)[H_{12}, W(\tau)] + (\Lambda_{10} + \Lambda_{21} + \Lambda_{20})W(\tau) , \quad (2.11)$$

where the generators of incoherent decay Λ_{10}, Λ_{21}, and Λ_{20} are defined analogously to (2.4), with the appropriate rates and raising and lowering operators for each transition. We should note that in all damping generators only transitions downward in energy are accounted for.

Starting from the initial condition (2.9) the density operator $W(\tau)$ solving the master equation (2.11) approaches the stationary form

$$\overline{\overline{W}} = \overline{\overline{\rho}} \prod_{\mu=1}^{N} |0\rangle_{\mu\,\mu}\langle 0| , \quad (2.12)$$

with a reduced density operator $\overline{\overline{\rho}}$ for the laser mode depending on the frequencies w_{10}, w_{20}, w_{21}, and g as well as, in general, on the initial operator $\rho(0)$. We shall discuss $\overline{\overline{\rho}}$ and the time scale on which it is approached in the sections to follow. The simplest situation, to be studied in the next section, arises when w_{21} and w_{20} are completely negligible, i.e., when the limit (2.1) is taken to the extreme. In that case $\overline{\overline{\rho}}$ differs from $\rho(0)$ only by the addition of precisely N photons. When, moreover, the leakage $1 \rightarrow 0$ is fast compared to the Rabi oscillations $2 \rightarrow 1$ the relevant time scale is analogous to (2.6),

$$1/\tau_{\text{emission}} = 4g^{2}n/w_{10}, \quad w_{10} \gg g\sqrt{n} , \quad (2.13)$$

where n is a typical photon number for the laser mode, i.e., $g\sqrt{n}$ a typical Rabi frequency.

The concluding phase of each cycle, photon leakage through the nonideal mirror, begins with (2.12) as an initial condition. Actually, since the atomic ground state is not at all affected by the field damping we may simply consider the evolution of the reduced density operator $\rho(\tau)$ away from $\overline{\overline{\rho}}$ towards the final form $\rho(T)$. The appropriate master equation is[6]

$$\dot{\rho}(\tau) = \kappa\{[b, \rho(\tau)b^{\dagger}] + [b\rho(\tau), b^{\dagger}]\} \equiv \Lambda\rho(\tau) . \quad (2.14)$$

We shall discuss the exact explicit form of the solution $\rho(\tau) = e^{\Lambda\tau}\overline{\overline{\rho}}$ in the next section.

The evolution of the field density operator ρ over one whole cycle may be written as[7] $\rho(T) = DE\rho(0)$ where $D = \exp(\Lambda T)$ and E is the map defined by $\overline{\overline{\rho}} = E\rho(0)$. Obviously, D describes the photon damping and E the combined effect of coherent and incoherent emission. We shall, from this point on, focus our attention on the state of the system at one instant per cycle, the moment just before the pump pulse is switched on. The stroboscopic description of the cycle-to-cycle evolution is given by the dissipative quantum map

$$\rho(t+1) = DE\rho(t) \quad \text{or} \quad \rho(t) = (DE)^{t}\rho(0) , \quad t = 0, 1, 2, \ldots , \quad (2.15)$$

where the integer t counts the number of cycles passed.

III. SUPPRESSION OF PUMP NOISE AND SPONTANEOUS EMISSION

We here consider the ideal case for which nonradiative and spontaneous-emission transitions $2 \rightarrow 0$ and $2 \rightarrow 1$ are so weak as to make w_{20} and w_{21} entirely negligible against $g\sqrt{n}$ and w_{10}. In that limit the stroboscopic map (2.15) becomes so simple that it can, in fact, be solved rigorously. Since neither the emission nor the damping distinguishes any phase of the field amplitude, we most conveniently use the Fock representation for ρ, which is defined by $b|n\rangle=\sqrt{n}\,|n-1\rangle$, $n=0,1,2,\ldots$. If $\rho(t)$ is diagonal, $\rho_{mn}(t)=\delta_{mn}\rho_m(t)$, so will be $\rho(t+1)$ whereupon the map (2.16) takes the form

$$\rho_m(t+1)=\sum_{n,k} D_{mk}E_{kn}\rho_n(t)\ . \tag{3.1}$$

The matrix E_{kn} follows from (2.11) and (2.12) with Λ_{20} and Λ_{21} dropped as

$$E_{kn}=\delta_{k,n+N}\ . \tag{3.2}$$

Indeed, since no atom can arrive in the ground level without having left one photon in the laser mode, the conditional probability of having k photons in the state (2.12) after having had n photons in the state (2.9) is unity for $k=n+N$ and zero otherwise. Incidentally, the corresponding stationary solution of (2.12),

$$\bar{\bar{\rho}}_n=\begin{cases}\rho_{n-N}(0), & \text{for } n\geq N\\ 0, & \text{for } 0\leq n<N\end{cases} \tag{3.3}$$

depends on the initial state (2.9). This memory is due to a conservation law obeyed by the master equation (2.11) with Λ_{20} and Λ_{21} dropped,

$$b^{\dagger}b+\sum_{\mu=1}^{N} S_{22}^{\mu}=\text{const},\quad \text{for } w_{20}=w_{21}=0\ . \tag{3.4}$$

This conservation law once more explains why the photon number draws no uncertainty from either the pump or the emission since the number of atoms in level 2 is sharp both after the pump [$N_2=N$ according to (2.9)] and after the emission [$N_2=0$, see (2.12)].

The element D_{mk} of the damping matrix in (3.1) obviously is the conditional probability of finding m photons at the end of the cycle provided there were k after the emission phase. By writing out the master equation (2.14) as a set of differential equations for the probabilities ρ_n one immediately verifies the well-known binomial distribution, with $d=e^{-2\kappa T}$,

$$D_{mk}=\begin{cases}\begin{bmatrix}k\\ m\end{bmatrix} d^{m}(1-d)^{k-m}, & k\geq m\\ 0, & k<m\ .\end{cases} \tag{3.5}$$

Clearly, D_{mk} must vanish for $k<m$ since the master equation (2.14) describes photon leakage out of the cavity.[8]

By inserting the matrices E and D from (3.2) and (3.5) into the map (3.1) and taking moments we easily find the stationary mean and variance of the photon number as

$$\begin{aligned}\langle m\rangle_\infty&=\frac{Nd}{1-d}\ ,\\ \sigma_\infty^2&=\langle m^2\rangle_\infty-(\langle m\rangle_\infty)^2=\frac{\langle m\rangle_\infty}{1+d}\ .\end{aligned} \tag{3.6}$$

Sub-Poissonian behavior is manifest since $0\leq d\leq 1$. Actually, in the limit $\kappa T\ll 1$, which is part of (2.2), we have $d\to 1$ and $1-d\to 2\kappa T$, thus

$$\langle m\rangle_\infty\to N/2\kappa T\gg N\gg 1,\quad \sigma_\infty^2\to\langle m\rangle_\infty/2\ , \tag{3.7}$$

i.e., a variance 50% below the Poissonian level. It is to be noted that this ideal behavior results from a complete suppression of the pump and of spontaneous emission as noise sources for the number of photons; even the only remaining noise source, photon leakage, is minimized in its effect in the limit (3.7).

Higher-order moments $\langle m^{\nu}\rangle_\infty$ can be calculated similarly. To within corrections of relative order $1/\langle m\rangle_\infty$ they can all be obtained as the moments of a Gaussian distribution with mean and variance given by (3.7). Needless to say, to the accuracy mentioned the photon number can be looked upon as continuous when moments of that Gaussian are to be evaluated.

The time dependence of the νth moment $\langle m^{\nu}\rangle_t$ is readily accessible as well, since the map (3.1), (3.2), and (3.5) couples this moment only to lower-order ones. Especially, the mean and the variance evolve as

$$\begin{aligned}&\langle m\rangle_t-\langle m\rangle_\infty=d^t(\langle m\rangle_0-\langle m\rangle_\infty)\ ,\\ &\sigma_t^2-\sigma_\infty^2=d^{2t}(\sigma_0^2-\sigma_\infty^2)+(d^t-d^{2t})(\langle m\rangle_0-\langle m\rangle_\infty)\ .\end{aligned} \tag{3.8}$$

This approach to equilibrium takes place on the time scale $1/\kappa$ as must be the case, all other time scales being negligibly small by comparison.

Finally, in order to characterize the intensity spectrum of the emitted radiation we must evaluate the normally second-order correlation function

$$\begin{aligned}g(t)&=\langle b^{\dagger}(0)b^{\dagger}(t)b(t)b(0)\rangle\\ &=\sum_{m,n} m\,[(DE)^t]_{mn}(n+1)\rho_{n+1}(\infty)\ ,\end{aligned} \tag{3.9}$$

where $(DE)^t$ is the tth power of the matrix DE from (3.1). By using the Gaussian nature of $\rho_n(\infty)$ we had shown in I that[9] to within corrections of order $1/\langle m\rangle_\infty$,

$$\begin{aligned}g(t)=&\ \langle m(t)m(0)\rangle(1+\langle m\rangle_\infty\sigma_\infty^{-2})\\ &-\langle m(t)m(0)^2\rangle\sigma_\infty^{-2}+\langle m\rangle_\infty\ ,\end{aligned} \tag{3.10}$$

where

$$\langle m(t)m(0)^{\nu}\rangle=\sum_{m,n} m\,[(DE)^t]_{mn}n^{\nu}\rho_n(\infty)\ . \tag{3.11}$$

The latter correlation functions evolve, according to (3.1), as[10]

$$\langle m(t+1)m(0)^{\mu}\rangle=d\langle m(t)m(0)^{\mu}\rangle+Nd\langle m^{\mu}\rangle_\infty\ ,$$

i.e.,

$$\begin{aligned}&\langle m(t)m(0)\rangle=d^t\sigma_\infty^2+\langle m\rangle_\infty^2\ ,\\ &\langle m(t)m(0)^2\rangle=d^t2\sigma_\infty^2\langle m\rangle_\infty+\langle m\rangle_\infty(\sigma_\infty^2+\langle m\rangle_\infty^2)\ .\end{aligned} \tag{3.12}$$

Putting together (3.10)–(3.12) we arrive at

$$g(t)=\langle m\rangle_\infty^2-(\langle m\rangle_\infty-\sigma_\infty^2)d^t\ , \tag{3.13}$$

a result formally identical with Eq. (3.20) of I where the photon number was modeled so as to undergo a Gaussian Markov process with a continuous time. As was shown in Sec. III of I this result implies a spectacular noise reduction in the laser output: the spectral variance[11]

$\langle \delta i^2 \rangle_\omega$ displays a Lorentzian dip below the shot noise limit $\kappa \langle m \rangle_\infty$,

$$\langle \delta i^2 \rangle_\omega = \kappa \langle m \rangle_\infty \left[1 - 2(1 - \sigma_\infty^2 / \langle m \rangle_\infty) \frac{\kappa^2}{\kappa^2 + \omega^2} \right] . \tag{3.14}$$

The dip is centered around zero frequency and there the spectral variance reaches the ideal value $\langle \delta i^2 \rangle_{\omega=0} = 0$ in the limit (3.7).

IV. PUMP FLUCTUATIONS

We now propose to consider the effect of a nonideal pump.[12] To that end we assume each of the N atoms to be brought to level 2 with probability p, independently of the fate of all other atoms, before the pump pulse is switched off. The probability of getting any subset of l atoms excited and the remaining $N-l$ ones left in the ground level is then given by the binomial distribution $\binom{N}{l} p^l (1-p)^{N-l}$ and this also is the probability of getting l photons added to the laser mode during the emission phase of any cycle. The photon number balance (3.1) is therefore modified to

$$\rho_m(t+1) = \sum_{k=\max(m,N)}^{\infty} \begin{bmatrix} k \\ m \end{bmatrix} d^m (1-d)^{k-m} \times \sum_{l=0}^{\min(k,N)} \begin{bmatrix} N \\ l \end{bmatrix} p^l (1-p)^{N-l} \times \rho_{k-l}(t) . \tag{4.1}$$

This cycle-to-cycle recursion relation for the photon number distribution can again be solved rigorously.

As might be expected, the stationary mean photon number is now diminished by the factor p to

$$\langle m \rangle_\infty = \frac{Npd}{1-d} . \tag{4.2}$$

The stationary variance, on the other hand, changes as

$$\sigma_\infty^2 = \langle m \rangle_\infty [1 + d(1-p)]/(1+d) . \tag{4.3}$$

By reducing the pump efficiency p from unity to zero the ratio $\sigma_\infty^2 / \langle m \rangle_\infty$ is increased from the "ideal" value $\frac{1}{2}$ to the Poissonian value 1. For fixed pump efficiency, on the other hand, the ratio $\sigma_\infty^2 / \langle m \rangle_\infty$ falls monotonously towards $1 - p/2$ when the damping parameter d grows to the lossless-cavity limit $d = 1$.

Similarly simple are the changes brought about for the time dependence of moments and correlation functions. Especially, the normally ordered second-order correlation function $g(t)$ and the spectral variance $\langle \delta i^2 \rangle_\omega$ are still given by (3.13) and (3.14) with $\langle m \rangle_\infty$ and σ_∞^2 as in (4.2) and (4.3). The minimal spectral variance no longer vanishes but takes the finite value

$$\langle \delta i^2 \rangle_{\omega=0} = \kappa \langle m \rangle_\infty [1 - 2dp/(1+d)] \approx \kappa \langle m \rangle_\infty (1-p) . \tag{4.4}$$

In brief, while sub-Poissonian effects are still present as long as $p > 0$, an inefficient pump not only deteriorates the laser output but also the noise reduction. The at least partial preservation of sub-Poissonian effects is, of course, due to the fact that the inefficient pump considered does not violate the time-scale separation (2.1,2).

The foregoing considerations apply not only to a nonideal pump efficiency but also to a situation where the number of active atoms itself undergoes cycle-to-cycle fluctuations. The latter situations typically arises in gas and dye lasers. To appreciate the equivalence of the two cases one must realize that the binomial distribution for having l out of N atoms pumped is essentially a Gaussian, provided $\langle l \rangle = Np \gg 1$. On the other hand, in the limit of a large mean the probability distribution for the number of atoms in a cell filled with a gas or a liquid is, by the central limit theorem, a Gaussian as well. By simply equating the means $\langle l \rangle = Np$ and the variances $\langle (\Delta l)^2 \rangle = Np(1-p)$, one may translate back and forth between the two situations.

V. INCOHERENT TRANSITION WITHIN THE LASING LEVEL PAIR

The most important spontaneous-emission processes capable of generating laser noise are the ones accompanying atomic transition $2 \rightarrow 1$. Such events are controlled by the same dipole-matrix element as the coherent interaction H_{21}; they even tend to be favored over the coherent interaction inasmuch as there are, at least for transition frequencies in the optical range, many modes available for accepting spontaneously emitted quanta. In order to secure predominance of the coherent interaction we must require the rate w_{21} to be smaller than both the typical Rabi frequency $g\sqrt{\langle m \rangle_\infty}$ and the rate w_{10}. The same condition must be obeyed by the rate w_{20} of incoherent transitions $2 \rightarrow 0$, which altogether shortcut level 1. We shall actually require

$$w_{20}, w_{21}, g\sqrt{\langle m \rangle_\infty} \ll w_{10} , \tag{5.1}$$

thereby both implementing (2.1) and establishing a certain adiabatic limit for the atomic relaxation $2 \rightarrow 1 \rightarrow 0$ and the accompanying coherent emission of photons. Indeed, the right-hand inequality in (5.1) means that no atom can complete a full Rabi cycle $2 \rightarrow 1 \rightarrow 2$ since every atom temporarily visiting level 1 is immediately and irreversibly sucked into the ground level. It follows that all observables can be classified as either fast or slow depending on whether or not their time rates of change have a contribution proportional to w_{10}. A few units of time $1/w_{10}$ into the emission phase the fast variables have settled in rigid adiabatic-equilibrium relations with the slow ones and from then on the whole system relaxes "slowly" towards the final equilibrium state (2.12) of the emission phase.

While for $w_{20} = w_{21} = 0$ the field part $\bar{\bar{\rho}}$ of the final state (2.12) was accessible without effort in Sec. III, it is not a trivial matter to calculate the influence of finite rates w_{20} and w_{21} on $\bar{\bar{\rho}}$. We must study the master equation (2.11). The adiabatic limit (5.1) actually facilitates that study since it allows the adiabatic elimination of all fast observables. As we show in Appendix A the master equation

(2.11) and the limit (5.1) imply the following rate equation for the probability $p(m,N_2,\tau)$ of having m photons in the laser mode and N_2 atoms in level 2 at time τ:

$$\dot{p}(m,N_2,\tau)=w[(N_2+1)p(m,N_2+1,\tau)-N_2p(m,N_2,\tau)] +\gamma[m(N_2+1)p(m-1,N_2+1,\tau) -(m+1)N_2p(m,N_2,\tau)]\ , \quad (5.2)$$

where $\gamma=4g^2/w_{10}$ and $w=w_{20}+w_{21}$. The terms $\sim w$ constitute the simplest Markovian process for atoms incoherently exiting from level 2. Similarly intuitive are the terms $\sim\gamma$ in (5.2): formally arising from H_{21} and Λ_{10} they actually constitute the simplest Markov process for atoms generating one laser photon for each transition $2\rightarrow1$ such that the conservation law $N_2+m=\text{const}$ holds [see also (3.4); the latter conservation law is, of course, broken by the incoherent processes involving the rate w]. Indeed, the process described by (5.2) must be Markovian since it follows from the higher-dimensional Markovian process (2.11) by adiabatic elimination of fast variables. The rate constant $\gamma=4g^2/w_{10}$ is most simply calculated by using Fermi's golden rule for the interaction H_{12} and taking the final state in the transition $2\rightarrow1$ as of width w_{10}; in Appendix A γ will result from a systematic adiabatic elimination procedure.

Assuming all N atoms excited and m_0 photons present at $\tau=0$, we are facing the initial probability

$$p(m,N_2,0)=\delta_{m,m_0}\delta_{N,N_2}\ . \quad (5.3)$$

In the mN_2 plane all points with integer coordinates lying on the triangle (see Fig. 3)

$$m_0\le m\le m_0+N-N_2\ ,\qquad 0\le N_2\le N\ , \quad (5.4)$$

become accessible for $0<\tau<\infty$ while for $\tau\rightarrow\infty$ a stationary solution,

$$p(m,N_2,\infty)=E_{m,m_0}\delta_{N_2,0}\ ,$$

$$\text{with } E_{m,m_0}\neq0 \text{ only for } m_0\le m\le m_0+N\ , \quad (5.5)$$

is approached. Due to the initial condition (5.3) E_{m,m_0} is the conditional probability for finding m photons at the end of the emission phase after having had am_0 before.

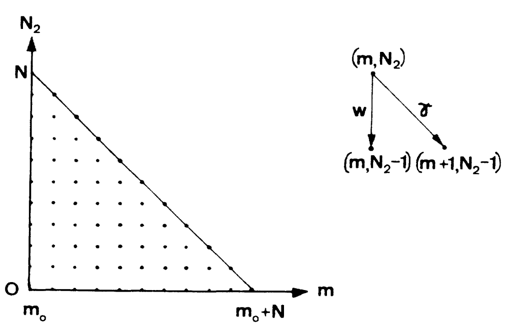

FIG. 3. Discrete state space corresponding to the rate equation (5.2) and the initial condition (5.3).

We achieve some insight into the competition between coherent and incoherent transitions by taking m_0 and N sufficiently large for a deterministic approximation to (5.2) and (5.3) to make sense. We may then extract moment equations from (5.2) and factorize as $\langle mN_2\rangle=\langle m\rangle\langle N_2\rangle$ to obtain the classical equations of motion

$$\langle\dot{N}_2\rangle=-w\langle N_2\rangle-\gamma\langle m\rangle\langle N_2\rangle\ ,\qquad \langle\dot{m}\rangle=+\gamma\langle m\rangle\langle N_2\rangle\ . \quad (5.6)$$

These admit the first integral

$$\langle m(\tau)\rangle+\langle N_2(\tau)\rangle-m_0-N =-(w/\gamma)\ln(\langle m(\tau)\rangle/m_0)\ , \quad (5.7)$$

which generalizes the conservation law (3.4). Moreover, for $\tau\rightarrow\infty$ Eqs. (5.6) yield $\langle N_2(\infty)\rangle=0$ as well as the transcendental equation for the mean photon number

$$\overline{\overline{m}}=m_0+N-(w/\gamma)\ln(\overline{\overline{m}}/m_0)\ . \quad (5.8)$$

An interesting consequence to be drawn from this equation is that the ratio w/γ need not be small compared to unity for the gain reduction per cycle to be small compared to the maximum gain N; rather, we only need $w/\gamma\ll N/\ln(1+N/m_0)$, which in the interesting case $m_0\gg N$, means $w\ll\gamma m_0$. To finally ascertain the self-consistency of all approximations we may check, by linearizing (5.6) around the stationary solution, that equilibrium is approached on the time sale $\tau_{\text{emission}}=1/(w+\gamma\overline{\overline{m}})$. Again, by $w\ll\gamma\overline{\overline{m}}$ we secure predominance of stimulated emission over incoherent decay. It will become clear below that the latter condition is a self-consistency condition for the deterministic behavior (5.8) to be faithful to the exact one.

It may be well to note a further property of the process (5.2,3) which is manifest both in (5.5) and the deterministic result (5.8): the field part E_{mm_0} of the stationary solution, in contrast to the atomic part, still depends on the initial state through the parameter m_0. Even though the conservation law (3.4) is now broken and the conditional probability E_{mm_0} therefore broader than δ_{m,m_0+N}, there is no photon sink accounted for in (5.2) which could cause complete loss of memory of the initial number of photons.

The rate equation (5.2) allows for a rigorous solution obeying the initial condition (5.3). This is all the more interesting as Kirchhoff's famous construction[13] fails due to the unidirectionality of the probability flow through the triangle of Fig. 3: N_2 being forbidden to increase, probability only flows "downwards"; starting from the upper left corner $(m,N_2)=(m_0,N)$, it spreads over the triangle and eventually gets stuck on the $N+1$ points $(m,0)$ with $m_0\le m\le m_0+N$.

For the explicit solution of the rate equation we conveniently employ the Laplace transform $\tilde{p}(m,N_2,\sigma)=\int_0^\infty d\tau\, p(m,N_2,t)e^{-\sigma\tau}$ and rewrite (5.2) as

$$\tilde{p}(m,N_2,\sigma)=\frac{1}{\sigma+N_2[(m+1)\gamma+w]}\times[\gamma m(N_2+1)\tilde{p}(m-1,N_2+1,\sigma)+w(N_2+1)\tilde{p}(m,N_2+1,\sigma)] . \quad (5.9)$$

This assumes that the point (m,N_2) lies on neither of the two boundaries for m. If $m=m_0$, the term $\sim\gamma$ in the last set of square brackets must be dropped since these points cannot be populated by coherent transitions. Correspondingly, points with $m=m_0+N-N_2$ cannot be reached through incoherent transitions and therefore the term $\sim w$ in the last set of square brackets does not arise. Finally, for the initially populated point (m_0,N) the last term in the square brackets must be replaced with unity,

$$\tilde{p}(m_0,N,\sigma)=\frac{1}{\sigma+N[(m_0+1)\gamma+w]} , \quad (5.10)$$

that contribution arising from the initial condition (5.3); indeed, the initial point draws no probability from any other point but only feeds its two "lower" neighbors.

Starting with (5.10) and repeatedly invoking (5.9) we may construct $\tilde{p}(m,N_2,\sigma)$ iteratively, lowering N_2 in unit steps from the initial value N to the current one. The stationary probabilities finally follow with the help of the limit theorem

$$p(m,N_2,\infty)=\lim_{\sigma\to 0}\sigma\tilde{p}(m,N_2,\sigma) . \quad (5.11)$$

From (5.11) and (5.9) we immediately conclude that $p(m,N_2,\infty)$ vanishes unless $N_2=0$, a result already anticipated on physical grounds in (5.5) above.

Most easily evaluated is the probability for the minimal photon number m_0,

$$\tilde{p}(m_0,0,\sigma)=\frac{w}{\sigma}\frac{2w}{\sigma+[(m_0+1)\gamma+w]}\times\frac{3w}{\sigma+2[(m_0+1)\gamma+w]}\cdots\times\frac{Nw}{\sigma+(N-1)[(m_0+1)\gamma+w]}\times\frac{1}{\sigma+N[(m_0+1)\gamma+w]} . \quad (5.12)$$

This is a product of $N+1$ factors, one (the right most) stemming from the initial point [see (5.10)] and then one for each of the N successive steps down the N_2 axis, $m=m_0$. Note that all of these steps are incoherent. The limit theorem (5.11) now yields the stationary probability

$$E_{m_0,m_0}=[\epsilon f_1(\epsilon)]^N , \quad (5.13)$$

with

$$f_\nu(\epsilon)=(m_0+\nu+\epsilon)^{-1}, \quad \epsilon=w/\gamma . \quad (5.14)$$

Similarly simple is the probability of the maximal number of photons m_0+N,

$$E_{m_0+N,m_0}=\frac{(m_0+N)!}{m_0!}\prod_{\nu=1}^{N}f_\nu . \quad (5.15)$$

Since only a sequence of N coherent transitions can result in the addition of N photons, this latter probability remains nonzero and, in fact, approaches unity in the limit $w\to 0$, which was considered in Secs. III and IV.

The only complication arising for the other points $(m_0+N-\nu,0)$ with $\nu\neq 0,N$ is the number of paths on the triangle of Fig. 3 connecting these points to the initial point (m_0,N). Probability can flow through all of these $\binom{N}{m-m_0}=\binom{N}{N-\nu}$ paths and therefore each path contributes additively to $E_{m_0+N-\nu,m_0}$; each contribution is similar in structure to (5.13) and (5.14) except that ν steps along the path are now of the incoherent type $(\sim w)$ while the remaining $N-\nu$ ones are coherent $(\sim\gamma)$. The paths differ only in the ordering of coherent and incoherent steps. There is no difficulty in sorting out the bookkeeping as

$$E_{m_0+N-\nu,m_0}=\frac{(m_0+N-\nu)!}{m_0!}\epsilon^\nu\left[\prod_{\lambda=1}^{N-\nu}f_\lambda\right]\times\sum_{1\le\mu_1\le\mu_2\le\cdots\le\mu_\nu\le N+1-\nu}f_{\mu_1}f_{\mu_2}\cdots f_{\mu_\nu} . \quad (5.16)$$

The various paths contributing to E_{mm_0} are labeled by the ν fold index $\mu_1,\mu_2,\ldots,\mu_\nu$, which specifies which ν out of N steps are of the incoherent type.

As it stands the above E_{m,m_0} is rigorous for arbitrary integer values of m_0 and N. Moreover, this E_{m,m_0} is quite tractable as an ingredient in the stroboscopic laser dynamics $\rho(t+1)=DE\rho(t)$, see (3.1). We think it is quite appropriate, though, to take advantage of some substantial simplifications arising in the limit $1\ll N\ll m_0$, $\epsilon\ll m_0$ in which an interesting laser would be operated. To that end we must realize that to within corrections of relative order $1/m_0$ or ϵ/m_0, the equalities $\mu_i=\mu_j$ in the ν fold sum may at will be excluded or admitted. We may therefore write

$$\sum_{1\le\mu_1\le\mu_2\le\cdots\le\mu_\nu\le N+1-\nu}f_{\mu_1}\cdots f_{\mu_\nu}\approx\sum_{\cdots<\cdots}\cdots=\frac{1}{\nu!}\sum_{\cdots\neq\cdots}\cdots\approx\frac{1}{\nu!}\left[\sum_{\mu=1}^{N+1-\nu}f_\mu\right]^\nu$$
$$=\frac{1}{\nu!}[\psi(m_0+N+2-\nu+\epsilon)-\psi(m_0+1+\epsilon)]^\nu$$
$$\approx\frac{1}{\nu!}\left[\ln\frac{m_0+N+2-\nu+\epsilon}{m_0+1+\epsilon}\right]^\nu , \quad (5.17)$$

where the digamma function ψ and its asymptotic behavior at large argument are incurred. By finally expressing the remaining product $(\prod_\lambda f_\lambda)$ in (5.16) in terms of the Γ functions, we arrive at the compact expression

$$E_{mm_0}=\frac{\Gamma(m+1)\Gamma(m_0+1+\epsilon)}{\Gamma(m_0+1)\Gamma(m+1+\epsilon)\Gamma(m_0+N-m+1)}\times\left[\frac{w}{\gamma}\ln\frac{m+2+\epsilon}{m_0+1+\epsilon}\right]^{m_0+N-m}, \tag{5.18}$$

which for the sake of consistency should be read with the Γ functions approximated by Stirling's formula. It is quite satisfactory to check that this asymptotic probability E_{mm_0} is maximal, up to corrections of relative weight $1/m_0$, ϵ/m_0, and N/m_0 at $m=\overline{\overline{m}}$ as given by the deterministic result (5.8).

At this point it is an easy matter to quantitatively study the laser noise generated by the incoherent transition $2\to 1$. Through numerically iterating the map (3.1) with E from (5.18) we have verified that the stationary photon number variance σ_∞^2 remains close to 50% sub-Poissonian as long as $w \ll \gamma\langle m\rangle_\infty$, see Sec. VII. An even more convincing demonstration of this result follows from an asymptotic treatment of the cycle-to-cycle evolution of the mean and the variance of the photon number. Towards that goal we assume $\rho_m(t)$ to be a Gaussian with mean $\langle m\rangle_t$ and variance σ_t^2. The conditional probability E_{mm_0} may, for $m,m_0\gg 1,\epsilon$, also be approximated by a Gaussian function of m, with its peak at $\overline{\overline{m}}(m_0)$ given by (5.8) and variance as

$$\frac{1}{\overline{\overline{\sigma}}^{\,2}}=-\left[\frac{\partial^2\ln E_{mm_0}}{\partial m^2}\right]_{m=\overline{\overline{m}}}=\epsilon\ln\left[\frac{\overline{\overline{m}}(m_0)}{m_0}\right] \tag{5.19}$$

if $1,\epsilon\ll m_0,\overline{\overline{m}}$. The expression (5.19) is valid to the same degree of accuracy as (5.8). The photon number distribution after the emisson phase $[E\rho(t)]_m=\sum_{m_0}E_{mm_0}\rho_{m_0}(t)$ can then, without incurring an error beyond the ones already accepted in (5.8) and (5.19), be written as a convolution of two Gaussians, $\rho_{m_0}(t)$ being one and E_{mm_0} with $\ln[\overline{\overline{m}}(m_0)/m_0]$ replaced by $\ln[\overline{\overline{m}}(\langle m\rangle_t)/m_t]$ the other. That convolution is a Gaussian as well, the mean and the variance being the sums of the corresponding quantities for the two convolution partners. At the end of the emission phase we are thus confronting

$$\begin{aligned}\overline{\overline{m}}&=\langle m\rangle_t+N-\epsilon\ln(\overline{\overline{m}}/\langle m\rangle_t)\ ,\\ \overline{\overline{\sigma}}^{\,2}&=\sigma_t^2+\epsilon\ln(\overline{\overline{m}}/\langle m\rangle_t)\ .\end{aligned} \tag{5.20}$$

A complete set of cycle-to-cycle recursion relations is finally obtained by extracting from (3.8) the effect of the damping phase,

$$\begin{aligned}\langle m\rangle_{t+1}&=d\overline{\overline{m}}\ ,\\ \sigma_{t+1}^2&=d^2\overline{\overline{\sigma}}^{\,2}+d(1-d)\overline{\overline{m}}\ .\end{aligned} \tag{5.21}$$

Surprisingly, the stationary solutions $\langle m\rangle_\infty$ and σ_∞^2 of (5.20) and (5.21) can be given in closed form since from (5.21) we have $\ln(\overline{\overline{m}}/\langle m\rangle_\infty)=-\ln d$. A little algebra then yields

$$\begin{aligned}\langle m\rangle_\infty&=\frac{d(N+\epsilon\ln d)}{1-d}\ ,\\ \sigma_\infty^2&=\langle m\rangle_\infty/(1+d)-\epsilon d^2\ln d/(1-d^2)\ ,\end{aligned} \tag{5.22}$$

and the reduction of the sub-Poissonian effect by incoherent transitions is manifest from the ratio

$$\frac{\sigma_\infty^2}{\langle m\rangle_\infty}=\frac{1}{1+d}\left[1-\frac{\epsilon d\ln d}{N+\epsilon\ln d}\right]. \tag{5.23}$$

It is to be noted that these results are based on the limit (5.1), i.e., $\epsilon=w/\gamma\ll(\langle m\rangle_\infty)$. However, for sufficiently weak damping $d\to 1$, we have $N\ll(\langle m\rangle_\infty)$ and ϵ need not be small compared to N; we may even admit $\epsilon\ln d$ as comparable in magnitude to the number of atoms in which case the degradation of sub-Poissonian effects would be quite noticeable.

VI. ALL ATOMIC TRANSITIONS IN COMPETITION

A situation revealing qualitatively new aspects arises when we drop the requirement that w_{10} be larger than $g\sqrt{\langle m\rangle_\infty}$, w_{20}, and w_{21}. Especially, allowing the same magnitude for w_{21} and w_{20} as for w_{10} amounts to a much more serious assault on the photon number stiffness then the one launched in Secs. IV and V: indeed, the coherent-emission channel then no longer offers the fastest return of the atoms to their ground level.

Effective competition of all decay channels requires numerical analysis. For the sake of a reasonable balance of numerical effort and insight we slightly modify the model by including in the generators Λ_{ij} the effect of phase destroying processes. Such processes do not at all affect the rates w_{ij} for the change of the populations of the atomic levels i and j; they increase the decay constants for the atomic polarizations S_{ij}^μ with $i\neq j$ from $\frac{1}{2}w_{ij}$ to $\frac{1}{2}(w_{ij}+\eta_{ij})$. The corresponding increments of the generators read[6]

$$\delta\Lambda_{ij}A=-\tfrac{1}{2}\eta_{ij}\sum_{\mu=1}^{N}(S_{ii}^\mu AS_{jj}^\mu+S_{jj}^\mu AS_{ii}^\mu)\ , \tag{6.1}$$

with $ij=21,20,10$.

A new adiabatic limit now becomes accessible in which the effective dimensionality of the emission phase is lowered considerably. In fact, the master equation (2.11) with the $\delta\Lambda_{ij}$ included may be replaced by a rate equation for the probability $p(m,N_1,N_2,\tau)$ of finding, at time τ during the emission phase, m photons, N_1 and N_2 atoms in levels 1 and 2, respectively, provided we require at least

$$w_{21}\ll\eta_{21}\ , \tag{6.2}$$

or, to facilitate the derivation, the corresponding limits for the other two transitions $2\to 0$ and $1\to 0$ as well. The rate equation, to be derived in Appendix B, reads

$$\begin{aligned}\dot{p}(m,N_1,N_2) = {} & \gamma[m(N_2+1)p(m-1,N_1-1,N_2+1)-(m+1)N_2p(m,N_1,N_2)] \\ & +\gamma[(m+1)(N_1+1)p(m+1,N_1+1,N_2-1)-mN_1p(m,N_1,N_2)] \\ & +w_{10}[(N_1+1)p(m,N_1+1),N_2)-N_1p(m,N_1,N_2)] \\ & +w_{20}[(N_2+1)p(m,N_1,N_2+1)-N_2p(m,N_1,N_2)] \\ & +w_{21}[(N_2+1)p(m,N_1-1,N_2+1)-N_2p(m,N_1,N_2)] \ ,\end{aligned} \tag{6.3}$$

where now $\gamma=4g^2/\eta_{21}$. In contrast to Sec. V, no restriction is placed on the relative sizes of the rates γ and w_{ij}. As a consequence, absorption terms $\sim\gamma$ are now present.

When at this point the limit (5.1) is imposed the population of level 1 may be adiabatically eliminated $[p(m,N_1,N_2)\rightarrow\delta_{N_1,0}p(m,N_2)]$ whereupon (5.2) is recovered. On the other hand, as the incoherent side channels are allowed to complete more and more effectively with the coherent emission we must expect rapid deterioration of the laser output as well as a transition from sub-Poissonian to super-Poissonian fluctuations. These expectations are indeed borne out in the numerical results to be presented in the next section.

VII. NUMERICAL RESULTS

We here propose to illustrate the above considerations with a few numbers and graphs. First we would like to demonstrate how spontaneous emission, according to the incoherent transitions $2\rightarrow0$ and $2\rightarrow1$, gets suppressed as the number of photons in the lasing mode grows. To that end we assume the vacuum as the initial state of the mode, disregard the field damping $(d=1)$, and consider $\rho_m(t)=(E^t)_{m,0}$ with the conditional probability E_{mm_0} given in (5.13), (5.15), and (5.16). Figure 4 shows $\rho_m(t)$ for $t=5$, 10, 15, and 20 with $N=100$ and $\epsilon=100$. For the same values of d, ϵ, and N, Fig. 5 displays the time dependence of the mean and the variance of the photon number. Once the number of photons has become large the mean increases linearly with t while the variance tends to a constant. This is expected since for large $\langle m\rangle_t$ almost all atoms deexcite by stimulated emission, thus increasing the number of photons *precisely* by N in each cycle and leaving the variance unchanged. At early times, on the other hand, while $\langle m\rangle_t$ is still small, spontaneous emission dominates and the increase of $\langle m\rangle_t$ with the number t of cycles is sublinear. It is quite interesting to see the fluctuations go sub-Poissonian for sufficiently large times.

For Fig. 6 we have allowed damping with $d=0.99$, keeping $\epsilon=100$ and $N=100$. We display $\langle m\rangle_t$ and σ_t^2 as derived from $\rho(t)=(DE)^t\rho(0)$ with $\rho(0)$ corresponding to the vacuum and E from Sec. V. During the first few cycles the variance is larger than the mean due to spontaneous emission, but is ultimately reduced below the mean, resulting in sub-Poissonian statistics. The stationary values reached are $\langle m\rangle_\infty=9800$ and $\sigma_\infty^2=5024$ while the asymptotic predictions (5.22) are, respectively, 9800 and 4974. Such excellent accuracy of the asymptotic result is not surprising since $\epsilon/\langle m\rangle_\infty\approx0.01\ll1$.

Figure 7 refers to the same situation as Fig. 6 except that the damping is larger, $d=0.9$, and the number of atoms smaller, $N=20$. The steady-state photon number is now much smaller, $\langle m\rangle_\infty\approx85.2$. In fact, $\epsilon/\langle m\rangle_\infty\approx1.2$ such that predominance of stimulated over spontaneous emission is not reached. The fluctuations must therefore be super-Poissonian. Even though the condition of validity of the asymptotic results (5.22) is

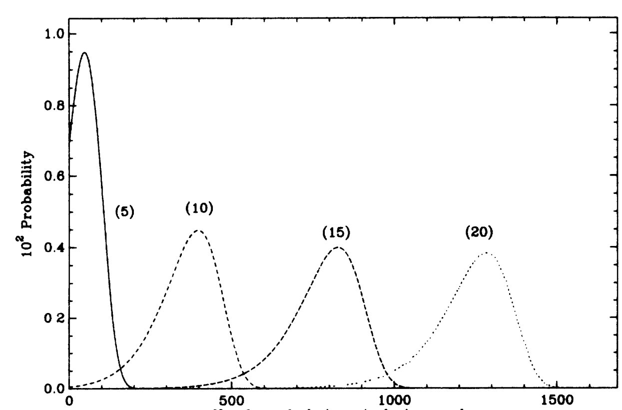

FIG. 4. Snapshots of the photon number distribution in a lossless cavity $(d=1)$ for $N=100$ and $\epsilon=100$, taken after 5, 10, 15, and 20 cycles.

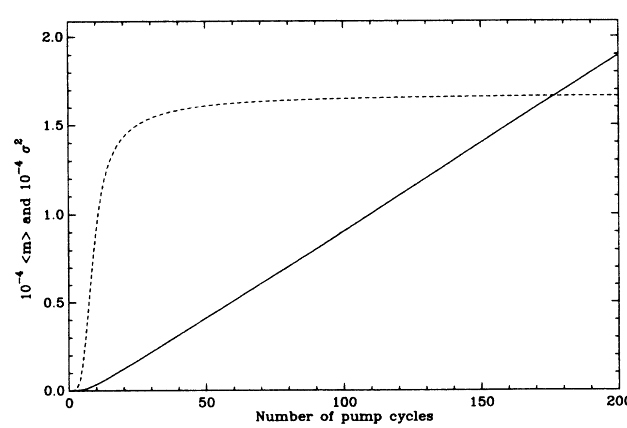

FIG. 5. Time dependence of the mean and the variance of the photon number for $N=100$, $\epsilon=100$, and $d=1$.

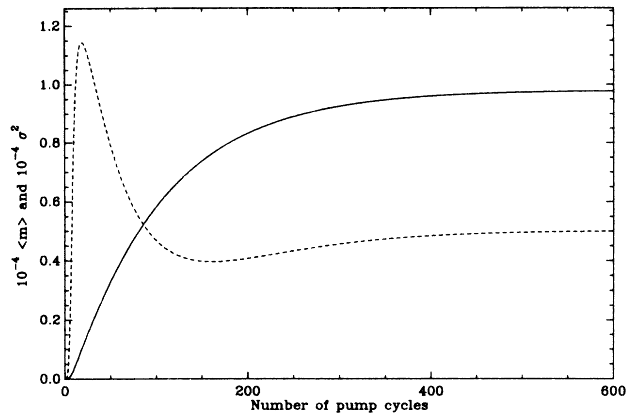

FIG. 6. Time dependence of the mean and the variance of the photon number for $N=100$, $\epsilon=100$, and $d=0.99$.

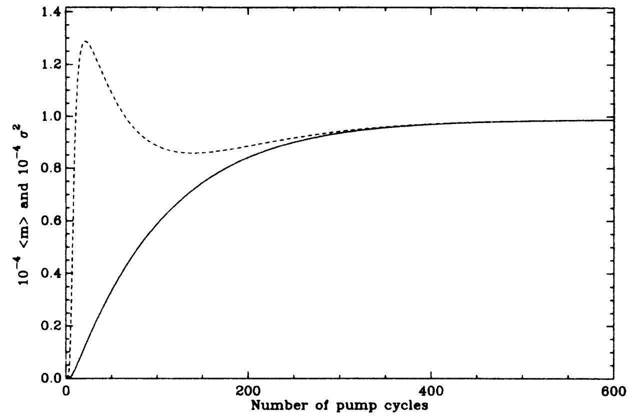

FIG. 9. Same as Fig. 8 except that $p=0.01$ and $Np=100$.

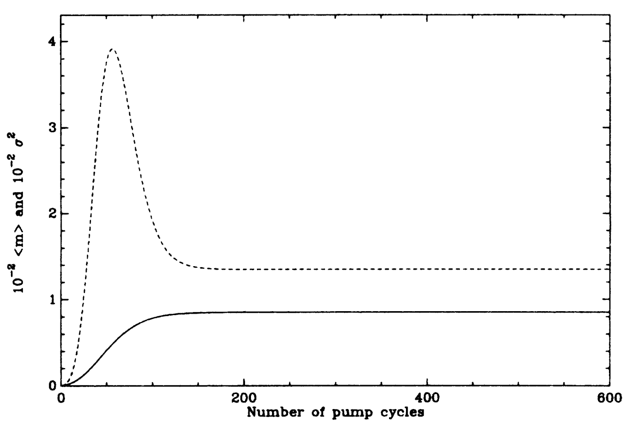

FIG. 7. Time dependence of the mean and the variance of the photon number for $N=20$, $\epsilon=100$, and $d=0.9$.

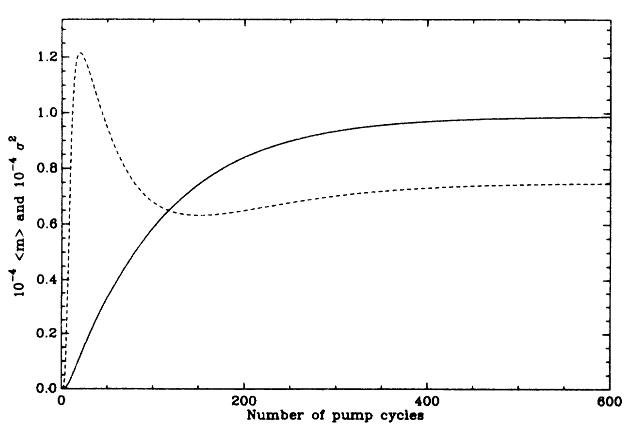

FIG. 8. Time dependence of the mean and the variance of the photon number for a nonideal pump $p=0.5$, $Np=100$, $\epsilon=100$, and $d=0.99$.

not violated $[\epsilon/\langle m \rangle_\infty$ not small] the mean $\langle m \rangle_\infty$ is still amazingly close to that prediction (error smaller than 1 ppt); however, the variance $\sigma^2_\infty \approx 133$ does reveal the inapplicability of the asymptotic prediction, which is about 30% too small.

In Figs. 8 and 9 we explore the effects of a nonideal pump, allowing for an efficiency $p<1$. In Fig. 8, $p=0.5$ and in Fig. 9, $p=0.01$, but in both cases the mean number of excited atoms is kept at $Np=100$. To facilitate the comparison with the ideal-pump situation in Fig. 6, the cavity damping is set to $d=0.99$. The evolution of the mean $\langle m \rangle_t$ is essentially the same in Figs, 6, 8, and 9, but the steady-state variances are quite different. From Eq. (4.2) we expect a steady-state mean of 9900 while Eq. (4.3) predicts steady-state variances of 7438 and 9851 for Figs. 8 and 9, respectively. These predictions agree well with the results of simulation (mean 9900 and variances, respectively, 7511 and 9900); actually, such agreement can be improved to excellent by putting together (4.2)

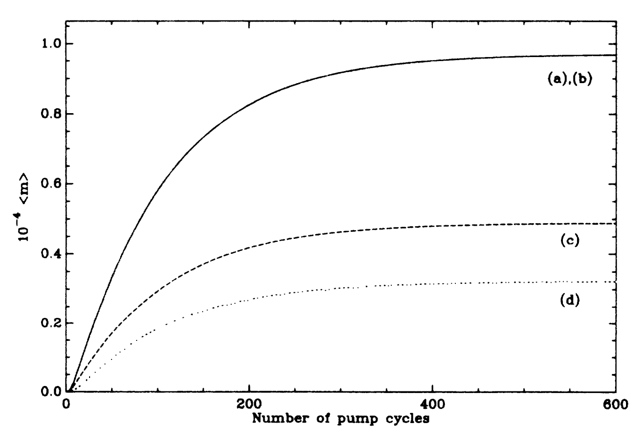

FIG. 10. Time dependence of the mean photon number for all incoherent transitions on $N=100$, $\epsilon=100$, and $d=0.99$. (a) $w_{10}=100w_{20}$, $w_{21}=0$; (b) $w_{10}=100w_{21}$, $w_{20}=0$; (c) $w_{10}=w_{20}$, $w_{21}=0$; (d) $w_{10}=w_{21}+w_{20}+2w_{20}$.

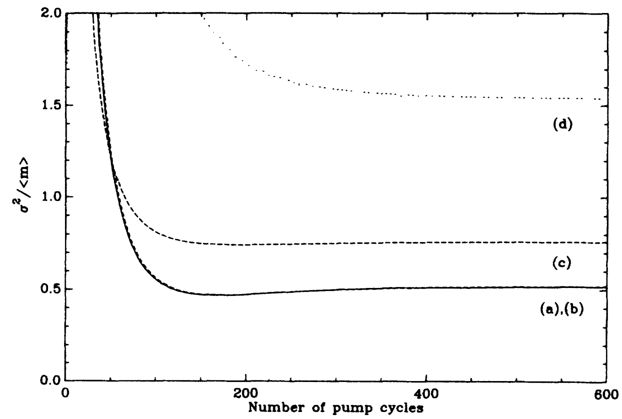

FIG. 11. Time dependence of the ratio variance to mean of the photon number, all parameters as in Fig. 10.

and (4.3) with (5.22) as was seen above in Fig. 6. The low pump efficiency used for Fig. 9 approximates a Poissonian pump, and as expected in this case, the photon number also exhibits near-Poissonian statistics.

We finally turn to the rate equation (6.3) which allows for effective competition of all atomic transitions. Figures 10 and 11 show the evolution of $\langle m \rangle_t$ and the ratio $\sigma_t^2/\langle m \rangle_t$ for a quiet pump ($p=1$), small damping ($d=0.99$), $\epsilon = w/\gamma = (w_{21}+w_{20})/\gamma = 100$, and $N=100$. The figures are based on approximating $p(m,N_1,N_2)$ by a trivariate Gaussian as described in paper I.

In curves (a) and (b), $w_{10}=100w$ such that the decay $1\rightarrow 0$ is very fast. Curve (a) is for $w_{21}=0$, i.e., $w=w_{20}$ whereas (b) has $w_{20}=0$, i.e., $w=w_{21}$. The two curves (a) and (b), being hardly distinguishable in the plots, we find confirmed the analysis of Sec. VI, which is paramount to the assumption that the lower working level is depleted, $N_1=0$; in this case noise suppression can be very effective.

For curves (c) and (d), on the other hand, $w_{10}=w=w_{21}+w_{20}$ such that the lower working level is not depleted during most of the emission phase. In curve (c), $w_{21}=0$, i.e., $w=w_{20}$, whereas in curve (d) $w_{21}=w_{20}=w/2$. We see that the degree of noise suppression and the steady-state photon number strongly depend on w_{21} for fixed $w=w_{21}+w_{20}$. As w_{21} is increased from zero, the incoherent transitions into the lower working level reduce the inversion N_2-N_1 and thus the gain of the laser and the steady-state photon number; the relative variance grows since the atoms may now make multiple transitions between the working levels and not every downward transition $2\rightarrow 1$ produces a photon in the lasing mode. When w_{21} is further increased the steady-state photon number $\langle m \rangle_\infty$ falls drastically.

Note. After finishing this work, we learned that J. Bergon, L. Davidovich, M. Orszag, C. Benkert, M. Hillary, and M. O. Scully have independently done similar investigations reaching partially identical conclusions.[4] We are grateful to M. O. Scully for informing us about that work.

ACKNOWLEDGMENTS

We have enjoyed fruitful discussions with M. J. Collet, R. Grobe, T. A. B. Kennedy, R. Levien, D. von der Linde, and M. Wilkens. This work was partially supported by the New Zealand University Grants Committee, the University of Auckland Foundation, the Sonderforschungsbereich Unordnung und grosse Fluktuationen der Deutschen Forschungsgemeinschaft and the U.S. Office of Naval Research.

APPENDIX A

In order to derive the rate equation (5.2) from the operator master equation (2.11) we must first employ a c-number representative for the density operator. A convenient choice is

$$W(\tau) \rightarrow P_{mn}(N_1,N_2,S_{01},S_{02},S_{12},\tau) , \qquad \text{(A1)}$$

which is a matrix with respect to the field mode, the indices m and n referring to the Fock representation, $b|m\rangle = \sqrt{m}\,|m-1\rangle$. The remaining variables are c-number associates of the operators used above. The real quantities N_1 and N_2 count the atoms in levels 1 and 2 while the polarization variables are complex; all of these atomic variables are meant as collective ones, i.e., are associated with the sums of the corresponding single-atom operators over all N atoms, e.g.,

$$S_{01} \leftrightarrow \sum_{\mu=1}^{N} \hat{S}_{01}^{\mu}, \quad S_{01}^{*} \leftrightarrow \sum_{\mu=1}^{N} \hat{S}_{10}^{\mu} , \qquad \text{(A2)}$$

where for the sake of clarity a caret is used to distinguish operators. With respect to the atomic variables P is a quasiprobability whose moments equal expectation values of "normally ordered" products of observables as[14]

$$\langle \hat{S}_{01}\hat{S}_{02}\hat{S}_{12}\hat{N}_1\hat{N}_2\hat{S}_{21}\hat{S}_{20}\hat{S}_{10} \rangle = \sum_m \int dN_1 \int dN_2 \int d^2S_{01} \int d^2S_{02} \int d^2S_{12} P_{mm} |S_{01}|^2 |S_{02}|^2 |S_{12}|^2 N_1 N_2 . \qquad \text{(A3)}$$

The latter identity also holds with arbitrary integral exponents on the operators and their c-number representatives.

The c-number variables chosen form a complete set in the sense that the master equation (2.11) for $W(\tau)$ can be translated[14] into an equivalent equation of motion for the quasiprobability P

$$\dot{P}_{mn} = \sum_{m'n'} L_{mm'nn'} P_{m'n'} + (\Lambda_{20}+\Lambda_{21}+\Lambda_{10}) P_{mn} . \qquad \text{(A4)}$$

The coherent-interaction piece in (2.11), $LW = -(i/\hbar)[H_{12},W]$, here gives rise to the tetrad $L_{mm'nn'}$ whose elements are differential operators with respect to the atomic variables,

$$L_{mm'nn'}=g\sqrt{m}\,\delta_{m',m-1}\delta_{n'n}[e^{-\partial_1+\partial_2}S_{12}+\partial_{12}^*(N_1-N_2)+\partial_{02}^*S_{01}^*-\partial_{12}^{*2}S_{12}^*]$$
$$-g\sqrt{m+1}\,\delta_{m',m+1}\delta_{n'n}(S_{12}^*+\partial_{01}^*S_{02}^*)+\text{H.c.}\ , \qquad (A5)$$

where H.c. implies $m\leftrightarrow n$ and $m'\leftrightarrow n'$ as well as complex conjugation for all polarization variables, $S_{ij}\to S_{ij}^*$, $\partial_{ij}\equiv\partial/\partial S_{ij}\to\partial_{ij}^*$, and the population variables N_i and $\partial_i\equiv\partial/\partial N_i$ remaining unchanged. The exponentials $e^{\pm\partial_i}$ appearing here and below signal that the population variables N_i take on integer values only.

The generators Λ_{20}, Λ_{22}, and Λ_{10} do not directly involve the lasing mode and are thus diagonal with respect to the photon number indices

$$\Lambda_{20}=\tfrac{1}{2}w_{20}\{[1+\partial_{01}\partial_{01}^*e^{-\partial_1}+(-\partial_{02}+\partial_{01}\partial_{12})(-\partial_{02}^*+\partial_{01}^*\partial_{12}^*)e^{-\partial_2}]$$
$$\times[\partial_{02}\partial_{02}^*(N-N_1-N_2)+\partial_{12}\partial_{12}^*e^{\partial_1}N_1+e^{\partial_2}N_2$$
$$+2\partial_{02}\partial_{12}^*e^{\partial_1}(S_{01}+\partial_{12}S_{02})+2\partial_{02}e^{\partial_2}S_{02}+2\partial_{12}e^{\partial_2}S_{12}]-N_2-(\partial_{02}S_{02}+\partial_{12}S_{12})+\text{c.c.}\}\ , \qquad (A6)$$

$$\Lambda_{21}=\tfrac{1}{2}w_{21}\{(e^{-\partial_1}+\partial_{12}\partial_{12}^*e^{-\partial_2})$$
$$\times[\partial_{02}\partial_{02}^*(N-N_1-N_2)+\partial_{12}\partial_{12}^*e^{\partial_1}N_1+e^{\partial_2}N_2$$
$$+2\partial_{02}\partial_{12}^*e^{\partial_1}(S_{01}+\partial_{12}S_{02})+2\partial_{02}e^{\partial_2}S_{02}+2\partial_{12}e^{\partial_2}S_{12}]-N_2-(\partial_{02}S_{02}+\partial_{12}S_{12})+\text{c.c.}\}\ ,$$

$$\Lambda_{10}=\tfrac{1}{2}w_{10}\{[1+\partial_{01}\partial_{01}^*e^{-\partial_1}+(-\partial_{02}+\partial_{01}\partial_{12})(-\partial_{02}^*+\partial_{01}^*\partial_{12}^*)e^{-\partial_2}]$$
$$\times[\partial_{01}\partial_{01}^*(N-N_1-N_2)+e^{\partial_1}N_1+2\partial_{01}e^{\partial_1}(S_{01}+\partial_{12}S_{02})]-N_1-\partial_{01}S_{01}+\partial_{12}S_{12}+\text{c.c.}\}\ .$$

Due to the normal order of the operators $\hat{S}_{ij}^{\mu}$ worked into the definition of the quasiprobability P the above c-number versions of the generators Λ_{ij} do *not* follow from one another by mere index juggling. It may also be well to note that the second and higher derivatives with respect to the polarizations secure consistency with the microscopic constraints $\hat{S}_{ij}^{\mu}\hat{S}_{jk}^{\mu}=\hat{S}_{ik}^{\mu}$; this is how the collective c-number variables remember that they refer to a collection of three-level atoms. Finally, the conservation law $\hat{S}_{00}^{\mu}+\hat{S}_{11}^{\mu}+\hat{S}_{22}^{\mu}=1$ is the only one of the microscopic constraints on the observables of a three-level atom with a simple macroscopic consequence $N_0+N_1+N_2=N$; this is why the collective population of the ground level need not be considered as an independent variable and why neither N_0 nor ∂_0 show up in (A6).

The most important property of the quasiprobability P is that it easily lends itself to an implementation of the adiabatic limit (5.1) to which we now proceed. All variables assigned nonzero time rates of change for their means by Λ_{10} are fast; these are obviously N_1, S_{01}, and S_{12}. The remaining variables N_2 and S_{02}, as well as the photon number, are slow. On the time scale $1/w_{10}$ the fast variables relax into a certain adiabatic equilibrium contingent on the current values of the slow variables; they play no independent dynamical role any more during the subsequent slow relaxation towards a final stationary state. To achieve adiabatic elimination we must construct the evolution equation $\dot{p}=lp$ for the reduced quasiprobability of the slow variables[15]

$$p_{mn}(N_2,S_{02},\tau)=\int_{\text{fast}}P_{mn}(N_1,N_2,S_{02},S_{01},S_{12},\tau)\ ,$$
$$\int_{\text{fast}}=\int dN_1d^2S_{01}d^2S_{12}\ . \qquad (A7)$$

The generator l of the slow evolution can be calculated as a power series in the small time-scale ratios implied by (5.1).

To keep the promise of deriving (5.2) we must eventually go two more steps. First, the generator l, a tetrad $l_{mm'nn'}$ like the coherent interaction piece (A5), will turn out not to couple the diagonal elements p_{mm} with the offdiagonal ones p_{mn} with $m\neq n$. Second, to leading order in the time-scale ratio the polarization S_{02} and the population N_2 will be seen to evolve separately such that we can, without additional approximations, simply integrate out S_{02} and thus find (5.2) to hold for

$$p(m,N_2,\tau)\equiv\int d^2S_{02}\,p_{mm}(N_2,S_{02},\tau)\ . \qquad (A8)$$

It must be stressed that this latter elimination of S_{02} and the off-diagonal elements p_{mn} with $m\neq n$ is not an adiabatic elimination; it rather arises from the fact that the lowest-order evolution of $p_{mn}(N_2,S_{02},\tau)$ separates rigorously into several subdynamics.

For the first, adiabatic elimination we rewrite (A8) in the form

$$p(\tau)=\int_{\text{fast}}P(\tau)=\int_{\text{fast}}e^{(L+\Lambda_{20}+\Lambda_{21}+\Lambda_{10})\tau}P(0)\ , \qquad (A9)$$

and assume, for the sake of technical convenience, the special initial condition

$$P(0)=Rp(0),\quad \int_{\text{fast}}R=1,\quad \Lambda_{10}R=0\ , \qquad (A10)$$

with a reference distribution R which is normalized and stationary with respect to Λ_{10}; evidently, R is associated with the atomic ground state. Putting together (A9) and (A10) we find the quasiprobability $p(\tau)$ related to its initial form by the time evolution operator

$$U(\tau)=\int_{\text{fast}} e^{(L+\Lambda_{20}+\Lambda_{21}+\Lambda_{10})\tau}R \ , \tag{A11}$$

and thus the time rate of change $\dot{p}(\tau)$ to $p(\tau)$ by $\dot{p}(\tau)=\dot{U}(\tau)U^{-1}(\tau)p(\tau)$. The formal generator of infinitesimal time evolution $\dot{U}(\tau)U(\tau)^{-1}$ is time dependent, but relaxes towards a stationary form l on the fast time scale $1/w_{10}$,

$$l=\lim_{\tau\gg 1/w_{10}} \dot{U}(\tau)U(\tau)^{-1} \ . \tag{A12}$$

It may be well to point out that a different choice of the reference distribution, not obeying $\Lambda_{10}R=0$, would yield a different operator $\dot{U}(\tau)U(\tau)^{-1}$ at finite times $\tau\lesssim 1/w_{10}$, but the same asymptotic reduced generator (A12).

By expanding $U(\tau)$ in powers of $(L+\Lambda_{20}+\Lambda_{21})$, $U(\tau)=U^{(0)}(\tau)+U^{(1)}(\tau)+U^{(2)}(\tau)+\ldots$, we perturbatively construct the generator l. This calculation follows standard lines and requires only a few comments. In addition to the properties of R noted in (A10) we shall need the identity $\int_{\text{fast}}\Lambda_{10}(\cdots)=0$ as well as $\int_{\text{fast}}\partial_{\text{fast}}(\cdots)=0$ where ∂_{fast} may denote the derivative with respect to any of the fast variables S_{01}, S_{12}, and N_1. We then immediately have $U^{(0)}(\tau)=1$, $U^{(1)}(\tau)=\dot{U}^{(1)}(\infty)\tau$, and thus $l^{(0)}=0$, $l^{(1)}=\dot{U}^{(1)}(\infty)=\int_{\text{fast}}(L+\Lambda_{20}+\Lambda_{21})R$. Moreover, the reference distribution R being related to the atomic ground state it assigns vanishing means to the fast variables

$$\int_{\text{fast}}N_1R=\int_{\text{fast}}S_{12}R=\int_{\text{fast}}S_{01}R=0 \ .$$

By inspection of (A5) and (A6) we at once conclude $\int_{\text{fast}}LR=0$ and

$$\begin{aligned} l^{(1)}=\tfrac{1}{2}\{&[w_{21}+w_{20}(1+\partial_{02}\partial_{02}^{*}e^{-\partial_2})]\\ &\times[\partial_{02}\partial_{02}^{*}(N-N_2)+e^{\partial_2}N_2+\partial_{02}e^{\partial_2}\partial_{02}]+\text{c.c.}\}\\ &-\tfrac{1}{2}(w_{21}+w_{20})(N_2+\partial_{02}S_{02}+\text{c.c.}) \ . \end{aligned} \tag{A13}$$

At this point a result anticipated above becomes accessible: by simply integrating out the polarization S_{02}, we obtain the first-order contribution to the generator for the motion of $p(m,N_2,\tau)$ as

$$\tilde{l}^{(1)}=(w_{21}+w_{20})(e^{\partial_2}-1)N_2 \ . \tag{A14}$$

This indeed gives the incoherent-transition terms in the rate equation (5.2).

The coherent-transition terms $(\sim\gamma)$ in (5.2) arise in second order. Given that there are three original slow processes $(\sim w_{20},w_{21},g)$ one might expect contributions to $l^{(2)}$ from all six bilinear combinations of these small rates. Actually, due to the properties of R already mentioned, only g^2 arises. As the single new ingredient in the second-order calculation we meet the time-dependent mean

$$\int_{\text{fast}}S_{01}e^{\Lambda_{10}\tau}(\cdots)=e^{-w_{10}\tau/2}\int_{\text{fast}}S_{01}(\cdots) \ .$$

By immediately integrating out S_{02} we arrive at

$$\begin{aligned}\tilde{l}^{(2)}&=\int_0^{\infty}d\tau\int d^2S_{02}\int_{\text{fast}}Le^{\Lambda_{10}\tau}LR\\ &=\frac{2}{w_{10}}\int d^2S_{02}\int_{\text{fast}}L^2R \ .\end{aligned} \tag{A15}$$

We finally insert L into (A5) and find the tetrad, with $\gamma=4g^2/w_{10}$,

$$\begin{aligned}\tilde{l}^{(2)}_{mm'nn'}=\gamma[&\sqrt{mn}\,e^{\partial_2}\delta_{m',m-1}\delta_{n',n-1}\\ &-\tfrac{1}{2}(m+n+2)\delta_{m'm}\delta_{n'n}] \ .\end{aligned} \tag{A16}$$

Obviously now, if $p_{nm}(N_2,t)$ is diagonal in the photon number indices, so is $\dot{p}_{mn}(N_2,t)$ and that time rate of change is given by the matrix $\tilde{l}^{(2)}_{mm'nn'}$, which, in fact, yields the coherent-interaction terms in (5.2).

APPENDIX B

The derivation of (6.3) proceeds quite similarly to the one of (5.2) just presented. If

$$\eta_{21}\gg w_{21},w_{20},w_{10},\eta_{20},\eta_{10} \ , \tag{B1}$$

only the polarization S_{12} is fast and can be eliminated adiabatically. However, the other two polarizations S_{02} and S_{01} behave similarly as S_{02} in Appendix A and can be integrated out in the end.

To proceed with the calculation, the increments $\delta\Lambda_{ij}$ from (6.1) must be translated into c-number form

$$\begin{aligned}\delta\Lambda_{10}&=\tfrac{1}{2}\eta_{10}\{[\partial_{01}e^{-\partial_1}+\partial_{12}^{*}(-\partial_{02}+\partial_{01}\partial_{12})e^{-\partial_2}][\partial_{01}^{*}(N-N_1-N_2)+e^{\partial_1}(S_{01}+\partial_{12}S_{02})]+\text{c.c.}\} \ ,\\ \delta\Lambda_{20}&=\tfrac{1}{2}\eta_{20}\{[\partial_{02}-\partial_{01}\partial_{12})[S_{02}+e^{-\partial_2+\partial_1}\partial_{12}^{*}(S_{01}+\partial_{12}S_{02})+e^{-\partial_2}\partial_{02}^{*}(N-N_1-N_2)]+\text{c.c.}\} \ ,\\ \delta\Lambda_{21}&=\tfrac{1}{2}\eta_{21}\{\partial_{12}[\partial_{02}^{*}\partial_{01}e^{-\partial_2}(N-N_1-N_2)+\partial_{02}^{*}e^{-\partial_2+\partial_1}(S_{01}^{*}+\partial_{12}^{*}S_{02}^{*})\\ &\quad+\partial_{12}^{*}\partial_{01}e^{-\partial_2-\partial_1}(S_{01}+\partial_{12}S_{02})+\partial_{12}^{*}e^{-\partial_2+\partial_1}N_1+\partial_{01}S_{02}+S_{12}]+\text{c.c.}\} \ .\end{aligned} \tag{B2}$$

Due to (B1) $\delta\Lambda_{21}$ now generates the fast part of the evolution. The adiabatic reduction integration is $\int_{\text{fast}}\equiv\int d^2S_{21}$ and we obviously have $\int_{\text{fast}}\delta\Lambda_{21}=0$.

It is again convenient to adopt an initial condition analogous to (A10) with a normalized reference distribution R which is stationary with respect to $\delta\Lambda_{21}$. Formally, $R=\lim_{t\to\infty}e^{\delta\Lambda_{21}\tau R_0}$ with R_0 arbitrary save for $\int_{\text{fast}}R_0=1$. We need not construct R in full since only its moments

$$\langle S_{12}\rangle_R = \lim_{\tau\to\infty}\int_{\text{fast}} S_{12}e^{\delta\Lambda_{21}\tau}R_0 = 0\ ,$$
$$\langle (S_{12})^2\rangle_R = 0\ , \quad \text{(B3)}$$
$$\langle |S_{12}|^2\rangle_R = e^{-\partial_2+\partial_1}N_1$$

will occur; these are easily found by considering their time rates of change before taking the limit of $t\to\infty$. The third of Eqs. (B3) requires special comment. One might expect R and its moments to depend "parametrically" on the slow variables; after all, by its very definition R invites the interpretation as an adiabatic equilibrium distribution of S_{12} contingent on the "parameters" in $\delta\Lambda_{21}$ such as $S_{02}, N_1, \ldots$. But $\delta\Lambda_{21}$ also contains the "parameters" $\partial_1, \partial_2, \ldots$ and therefore the moments of R may and do involve these operators as well.

These explanations given, the remainder of the argument is the same as in Appendix A. The incoherent-transition terms in (6.3) arise in first and the coherent-transition terms in second order of the expansion in powers of the slow generators.

Needless to say, if all three phase destruction rates η_{ij} are much larger than the w_{ij} and $g\sqrt{\langle m\rangle_\infty}$, all polarizations may be eliminated adiabatically. The rate equation (6.3) comes out unchanged.

[1]F. Haake, S. M. Tan, and D. F. Walls, Phys. Rev. A **40**, 7121 (1989).

[2]Yu. M. Golubev and I. V. Sokolov, Zh. Eksp. Teor. Fiz. **xx**, xxx (19xx) [Sov. Phys.—JETP **60**, 234 (1984)].

[3]P. Filipowicz, J. Javanainen, and P. Meystre, Opt. Commun. **58**, 327 (1986); Phys. Rev. A **34**, 3077 (1986).

[4]J. Bergou, L. Davidovich, M. Orszag, C. Benkert, M. Hillery, and M. O. Scully, Opt. Commun. **72**, 82 (1989).

[5]The atomic operators S_{ij}^μ are defined by $S_{ij}^\mu S_{kl}^\mu = \delta_{jk}S_{il}^\mu$ and $[S_{ij}^\mu, S_{kl}^\nu]=0$ for $\mu\neq\nu$. Latin subscripts label the levels 0,1,2,3 while greek subscripts label atoms $\mu=1,2,\ldots N$.

[6](a) H. Haken, in *Licht und Materie,* Vol. XXV/2c of *Handbuch der Physik,* edited by L. Genzel (Springer, Berlin, 1970); (b) F. Haake, in *Quantum Statistics in Optics and Solid State Physics,* Vol. 66 of *Springer Tracts in Modern Physics,* edited by Editor (Publisher, City, 1973).

[7]Due to (2.2) the duration of the damping phase may be identified with the duration of a cycle.

[8]At sufficiently low frequencies and for sufficiently high temperatures outside the resonator, the master equation (2.14) would have to be modified so as to allow for thermal photons entering the resonator.

[9]The verbal description of the derivation in I assumes a continuously evolving time; actually, there is no change in the argument for discrete t.

[10]In deriving the third from the first of Eqs. (3.12) we again admit errors of order $1/\langle m\rangle_\infty$.

[11]There is some freedom in defining the Fourier transform of a stroboscopic signal; the interesting limit $w\langle\kappa \ll 1/T$, however, its unique.

[12]The considerations to follow are similar in spirit to the ones of Ref. 4.

[13]There are no directed maximal trees with nonzero contributions; see H. Haken, *Synergetics: An Introduction* (Springer, Berlin, 1978).

[14]See in *Licht und Materie* [Ref. 6(a)], Chap. IX.

[15]F. Haake, Z. Phys. B **48**, 31 (1982).