Classical dynamics and ground-state phase transitions of a model SU(1,1) Hamiltonian

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We discuss the classical dynamics of a model SU(1,1) Hamiltonian system that may have some application in many-boson systems. A ground-state phase transition is shown to exist in this model, and the energy surface is shown to undergo a qualitative change at this transition.

I. INTRODUCTION

In this paper we study the phase-space flow of the classical dynamics for certain model Hamiltonian systems associated with the dynamical group SU(1,1). We shall refer to these systems as SU(1,1) Hamiltonian systems in the sense that the Hamiltonians may be composed of the generators of SU(1,1) [or one of the locally isomorphic forms: SO(2,1), Sp(2*R*), and Sl(2*R*)].[1] In the work presented here we discuss, among other things, ground-state phase transitions and the classical motion of such systems in a mean-field approximation where the classical phase-space picture arises from the projection of the dynamics onto the coherent states (CS) associated with SU(1,1).[2] In previous work,[3] a path-integral form of the propagator over the SU(1,1) CS as been developed for SU(1,1) systems. In the continuous limit of the propagator the associated classical dynamics may be obtained and the phase space of this motion has the form of the Lobachevsky plane[3] (see also Ref. 2). It has also been shown that SU(1,1) Hamiltonians, which are linear in the SU(1,1) generators,[4] are coherence preserving under time evolution for an arbitrary initial SU(1,1) CS. This turns out to be an important class of systems which have applications in quantum optics since they model various parametric amplifiers[5,6] associated with the production of nonclassical states (i.e., squeezed and photon antibunched) of the quantized electromagnetic field. Also such linear systems have been studied in regard to associated time-dependent invariants,[7] as models of damped harmonic oscillators,[8] and as systems whose path integrals can be calculated exactly.[9]

In this paper we study systems which are not coherence preserving and are special cases of a more general anharmonic oscillator. Such Hamiltonians have applications in quantum mechanics if, for instance, one is interested in the time evolution of a wave packet that may be squeezed.[10] Again from quantum optics, some four-photon systems[11] can be realized as SU(1,1) Hamiltonians.[6,12] Also, previously a phase-integral quantization rule has been developed in the context of the large-N approximation and has successively been applied to various even-powered anharmonic oscillators.[13] In this paper, we study an SU(1,1) Hamiltonian which is at most quadratic in the generators.

The work we undertake in this paper may be regarded as the noncompact analog of the work done in the dynamics and ground-state phase transitions for SU(2) Hamiltonians[14] associated with the Lipkin-Meshkov-Glick[15] (LMG) model from nuclear physics. As far as we are aware, no such studies have been carried out for SU(1,1) systems. Aside from the anharmonic oscillator problem previously mentioned in connection with the four-photon systems, other many-body problems are relevant to SU(1,1). (For ordinary single-particle systems, see Ref. 1.) Other examples where SU(1,1) is an exact dynamical group are a quasispin formulation for many-boson systems in a spherical field (such as the nuclear many-surface-phonon state),[16] coupled anharmonic oscillators,[17] superfluid helium,[18] N interacting particles with a quadratic pair potential,[19] and spin waves in localized-spin models.[20] This does not exhaust the possibilities, but rather points out that the models we consider here may have a great deal of relevance if the systems are extended to include various self-interactions and interactions with external systems. For example, previously the existence of ground-state phase transitions in the fully quantized degenerate parametric amplifier has been discussed.[21] This system fundamentally consists of two interacting harmonic oscillators, one of which is taken to be in an ordinary coherent state and the other in an SU(1,1) CS. The compact analog of this system is the Dicke model,[22] where the electromagnetic field is quantized. Phase transitions for that system have been discussed by Gilmore.[23]

Finally, we mention one other area of concern for which our studies may have some relevance. This is related to the breaking of dynamical "symmetry," quantum nonintegrability, and the connection between classical and quantum chaos.[24,25] "Symmetry" in this case actually means "coherence preserving." Elsewhere we shall consider this type of phenomenon in connection with the system studied in Ref. 21.

The plan of the paper is as follows. In Sec. II we discuss the SU(1,1) model system, the SU(1,1) CS, and the associated classical phase space. The equations of motion for the model system are derived. In Sec. III we solve these equations numerically to obtain the energy surfaces. Ground-state phase transitions and their relation to the fixed points on the energy surfaces are discussed. The qualitative change in the surface due to the phase transition is also discussed. In Sec. IV, we conclude the paper with some brief remarks.

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II. THE MODEL

This section relies rather heavily on work presented elsewhere.[2-4] Only the briefest review will be presented here.

The Lie algebra of SU(1,1) consists of three generators $K_0, K_\pm$ satisfying the commutation relations

$$[K_0,K_\pm]=\pm K_\pm,\quad [K_-,K_+]=2K_0\ , \tag{2.1}$$

and the Casimir invariant

$$C=K_0^2-\tfrac{1}{2}(K_+K_-+K_-K_+)\ . \tag{2.2}$$

We consider only the positive discrete representation $\mathcal{D}^\dagger(k)$ whose basis states $\{|m,k\rangle\}$ diagonalize the operator K_0, i.e., $K_0|m,k\rangle=(m+k)|m,k\rangle$, $m=0,1,2,\ldots$, $C|m,k\rangle=k(k-1)|m,k\rangle$, and $k>0$. The coherent states for these representations are given as[2]

$$|\xi,k\rangle=S(z)|0,k\rangle\ , \tag{2.3}$$

where

$$S(z)=\exp(zK_+-z^*K_-) \tag{2.4}$$

and where $z=-(\theta/2)e^{-i\phi}$ and $\xi=-\tanh(\theta/2)e^{-i\phi}$. θ and ϕ are group parameters with ranges $(-\infty,\infty)$ and $[0,2\pi]$, respectively.

Assuming the Hamiltonian of a system to be composed of SU(1,1) generators, i.e., $H=H(K_0,K_\pm)$, then from the continuum limit of the path-integral expression for the propagator calculated over SU(1,1) CS,[3] we obtain the classical equations of motion for ξ and ξ^* as

$$\dot\xi=\{\xi,\mathcal{H}\},\dot\xi^*=\{\xi^*,\mathcal{H}\}\ , \tag{2.5}$$

where $\mathcal{H}=\langle\xi,k|H|\xi,k\rangle$ and $\{\ ,\ \}$ defines a generalized Poisson bracket

$$\{A,B\}=\frac{(1-|\xi|^2)^2}{2ik}\left[\frac{\partial A}{\partial\xi}\frac{\partial B}{\partial\xi^*}-\frac{\partial A}{\partial\xi^*}\frac{\partial B}{\partial\xi}\right] \tag{2.6}$$

or, in terms of the parameters θ and ϕ,

$$\{A,B\}=\frac{1}{k,\sinh\theta}\left[\frac{\partial A}{\partial\phi}\frac{\partial B}{\partial\theta}-\frac{\partial A}{\partial\theta}\frac{\partial B}{\partial\phi}\right]\ . \tag{2.7}$$

The equations of motion for θ and ϕ are then

$$\dot\theta=-(k\sinh\theta)^{-1}\frac{\partial\mathcal{H}}{\partial\phi}\ , \tag{2.8a}$$

$$\dot\phi=(k\sinh\theta)^{-1}\frac{\partial\mathcal{H}}{\partial\theta}\ . \tag{2.8b}$$

Actually, Eq. (2.8a) can be written as

$$\frac{d}{dt}\cosh\theta=-\frac{1}{k}\frac{\partial\mathcal{H}}{\partial\phi} \tag{2.9}$$

which implies that $\cosh\theta$ is essentially a momentum-type variable. In fact, as will become clear, it plays the role of an action variable conjugate to the angle variable ϕ. However, it is convenient to define the momentum variable as $p=\cosh\theta-1$ so that $p\geq 0$. Setting $\phi=q$, Eqs. (2.8) may be written as

$$\dot p=-\frac{1}{k}\frac{\partial\mathcal{H}}{\partial q}\ , \tag{2.10a}$$

$$\dot q=\frac{1}{k}\frac{\partial\mathcal{H}}{\partial p}\ . \tag{2.10b}$$

The model SU(1,1) Hamiltonian system we consider is

$$H=2\omega K_0+\tfrac{1}{2}\gamma(K_+K_-+K_-K_+)+\tfrac{1}{2}\lambda(K_+^2+K_-^2) \tag{2.11}$$

which, in fact, is the SU(1,1) analog of the LMG model for SU(2).[15] Actually, all the salient features of the dynamics show up for the model where $\gamma=0$ so we consider only this case such that

$$H=2\omega K_0+\tfrac{1}{2}\lambda(K_+^2+K_-^2)\ . \tag{2.12}$$

We shall briefly describe the effects of the added terms later.

In calculating the classical dynamics, we make the mean-field approximation that

$$\begin{aligned}\mathcal{H}&=\langle H(K_0,K_\pm)\rangle\\ &\approx H(\langle K_0\rangle,\langle K_\pm\rangle)\ .\end{aligned} \tag{2.13}$$

Since

$$\begin{aligned}&\langle\xi,k|K_0|\xi,k\rangle=k\cosh\theta\ ,\\ &\langle\xi,k|K_\pm|\xi,k\rangle=-k\sinh\theta\, e^{\pm i\phi}\ ,\end{aligned} \tag{2.14}$$

then

$$\mathcal{H}=2\omega k\cosh\theta+\lambda k^2\sinh^2\theta\cos(2\phi)\ , \tag{2.15}$$

or in terms of q and p,

$$\mathcal{H}(q,p)=2\omega k(p+1)+\lambda k^2(p^2+2p)\cos(2q)\ . \tag{2.16}$$

Finally, from Eqs. (2.10) we obtain the classical equations of motion

$$\dot q=2\omega+2\lambda k(p+1)\cos(2q)\ , \tag{2.17a}$$

$$\dot p=2\lambda k(p^2+2p)\sin(2q)\ . \tag{2.17b}$$

The solutions are described in the next section.

III. ENERGY SURFACES AND GROUND-STATE PHASE TRANSITIONS

A. Energy surfaces

We begin by locating the stationary or fixed points for the system of Eqs. (2.17). Only the case when $\lambda>0$ is to be considered.

The stationary points of the system are determined as solutions of Eq. (2.17) when $\dot q=0$ and $\dot p=0$, $p=0$,[26] i.e., for

$$2\omega+2\lambda k(p+1)\cos(2q)=0\ , \tag{3.1a}$$

$$2\lambda k(p^2+2p)\sin(2q)=0\ . \tag{3.1b}$$

From Eq. (3.1b) we obtain $p=0$ or $2q_s=n\pi$, $n=0,1,2,\ldots$. Consider the latter case first. When $q_s=n\pi/2$, from Eq. (3.1a) we obtain

$$\omega-\lambda k(p+1)\cos(n\pi)=0 \ . \tag{3.2}$$

For n even, there is no solution since we must have $p\geq 0$. For n odd we obtain

$$p_s=\frac{\omega}{\lambda k}-1 \quad \text{or} \ \cosh\theta_s=\frac{\omega}{\lambda k} \tag{3.3}$$

which has solutions if $\lambda<\lambda_c$, where $\lambda_c=\omega/k$, the critical value of the coupling constant. Thus, if $0<\lambda<\lambda_c$, we obtain a stationary point at $(n\pi/2,\lambda_c/\lambda-1)$ for n odd. We have performed an analysis of these fixed points and have found them to be hyperbolic, or saddle, points.

In Fig. 1 we display the energy surface for a case when $\lambda<\lambda_c$. Specifically, we set $\omega=1$, $k=\frac{1}{4}$ (the representation for the even-parity oscillator states) such that $\lambda_c=4$. We then set $\lambda=1$. The contours are spaced in energy by the amount 0.1. The saddle points are marked with plus signs and the phase space is shown only out to $q=2\pi$. Two types of orbits are apparent, namely periodic orbits of the rotational type and orbits for which the motion is unbounded. No librational orbits are found.

In a previous study,[13] a phase-integral quantization rule for SU(1,1) Hamiltonian systems was given in the form

$$\oint p\,dq=\frac{2\pi}{k}n, \quad n=0,1,2,\ldots \ . \tag{3.4}$$

Such a rule obviously can be applied only to the rotational orbits of Fig. 1. Apparently for $0<\lambda<\lambda_c$, this Hamiltonian has both bound and unbound states.

On the other hand, for $\lambda>\lambda_c$, the only possible solutions are, for q in the range $[0,2\pi]$, $p_s=0$ and

$$\begin{aligned} q_{s1}&=\tfrac{1}{2}\cos^{-1}\left[-\frac{\lambda_c}{\lambda}\right], \\ q_{s2}&=\pi-\tfrac{1}{2}\cos^{-1}\left[-\frac{\lambda_c}{\lambda}\right], \\ q_{s3}&=\tfrac{3}{2}\cos^{-1}\left[-\frac{\lambda_c}{\lambda}\right], \\ q_{s4}&=\frac{3\pi}{2}-\tfrac{1}{2}\cos^{-1}\left[-\frac{\lambda_c}{\lambda}\right]. \end{aligned} \tag{3.5}$$

These points are found along the line $p=0$ and are also saddle points. The energy surface in this situation for $\lambda=7$ is shown in Fig. 2. It appears that no periodic orbits are to be found indicating that the energy surface undergoes a qualitative change as λ is raised above the critical value λ_c. In fact it is easy to see that as λ approaches λ_c from below, the saddle points of Fig. 1 migrate along the lines $q=n\pi/2$ toward the line $p=0$, striking that line at $\lambda=\lambda_c$. As λ is raised above λ_c, each of these saddle points bifurcates into two saddle points located symmetrically about $\pi/2$ and $3\pi/2$.

In another sense, however, there is actually only one critical point in phase space if the phase space is parametrized as the SU(1,1) hyperboloid analogous to the Bloch sphere of SU(2). This comes about by noting from Eqs. (2.14) that

$$\langle K_0\rangle^2-\langle K_+\rangle\langle K_-\rangle=k^2(\cosh^2\theta-\sinh^2\theta)=k^2 \ .$$

The point $\theta=0$ (or $p=0$) is actually the fixed point and the hyperboloid is divided into four sections along the lines given by Eqs. (3.5).

B. Ground-state phase transition

We show here that the change in the energy surface as one passes from $\lambda<\lambda_c$ to $\lambda>\lambda_c$ is associated with a first-order transition in the ground-state energy.[27] Consider

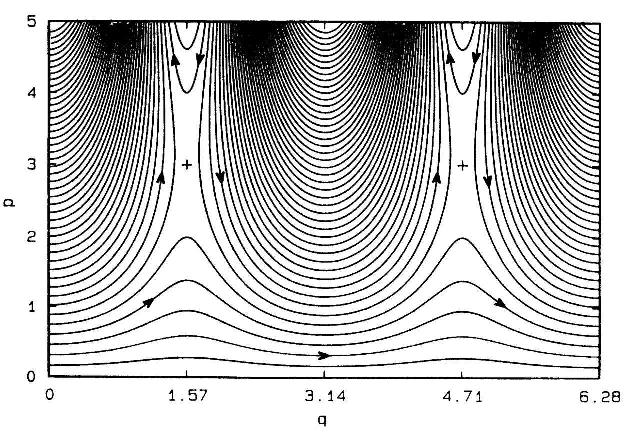

FIG. 1. Energy contours for $\lambda=1$. The saddle points are marked by + signs. The contour spacing is 0.1. Separatrices are not shown but are implied by the arrows.

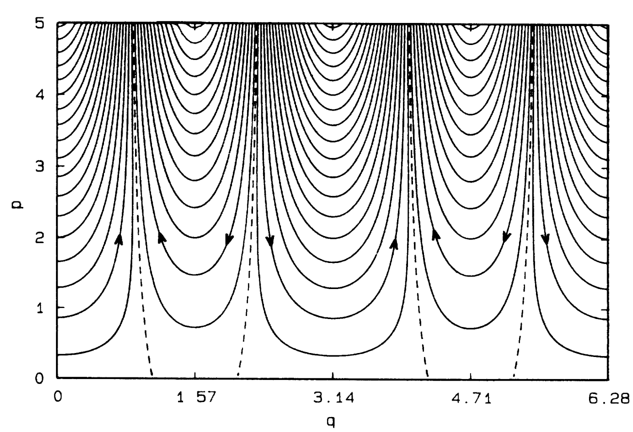

FIG. 2. Energy contours for $\lambda=7$. The contour spacing is 1.0. The dashed lines are the separatrices.

the Hamiltonian H from Eq. (2.15) as an energy functional

$$E(\theta,\phi)=2\omega k\cos\theta+\lambda k^2\sinh^2\theta\cos(2\phi)\ . \qquad (3.6)$$

We then set

$$\frac{\partial E}{\partial\phi}=-2\lambda k^2\sinh^2\theta\sin(2\phi)=0 \qquad (3.7)$$

to obtain $2\phi=n\pi$ as before. Then with n even or odd, we define

$$E_\pm(\theta)=2\omega k\cosh\theta\pm\lambda k^2\sinh^2\theta\ . \qquad (3.8)$$

Now we set

$$\frac{\partial E_\pm}{\partial\theta}=2\omega k\sinh\theta\pm2\lambda k^2\sinh\theta\cosh\theta=0\ . \qquad (3.9)$$

For the E_+ case, the only solution is $\theta=0$. For E_- we obtain the additional solution

$$\cosh\theta=\frac{\lambda_c}{\lambda}\ , \qquad (3.10)$$

provided $\lambda<\lambda_c$. For $\lambda>\lambda_c$ this solution disappears, and $\theta=0$ becomes a minimum at $2\phi=n\pi$, n even, and a maximum at $2\phi=n\pi$, n odd. However, for $\lambda<\lambda_c$, the points $\theta=\pm\cosh^{-1}(\lambda_c/\lambda)$ are relative maxima at $2\phi=n\pi$, n odd. Various E_- versus θ curves are illustrated in Fig. 3. As $\lambda\to\lambda_c$ from below the curve becomes very flat through $\theta=0$ such that at $\lambda=\lambda_c$, the second derivative

$$\frac{\partial^2E_-}{\partial\theta^2}=2\omega k\cosh\theta-2\lambda k^2(\sinh^2\theta+\cosh^2\theta) \qquad (3.11)$$

becomes zero at $\theta=0$. This indicates that a phase transition occurs at the critical value at $\lambda=\lambda_c$. The order of the transition is determined by the sign of the fourth-order derivative.[28] In fact the conditions under which the minimum at $\theta=0$ loses its stability as λ is increased may be discerned from the Taylor expansion

$$E_-(\theta)=E_-(0)+\frac{1}{2!}\theta^2\frac{\partial^2E_-}{\partial\theta^2}+\frac{1}{4!}\theta^4\frac{\partial^4E_-}{\partial\theta^4}\ , \qquad (3.12)$$

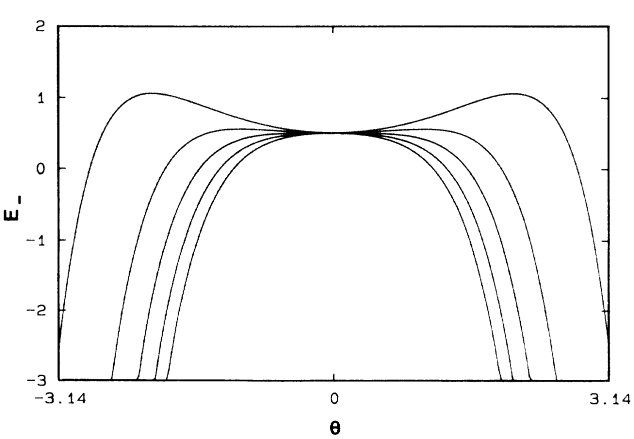

FIG. 3. E_- versus θ for various λ around $\lambda_c=4$.

where the derivatives are evaluated at $\theta=0$. [No third-order term appears because of the symmetry $E(\theta)=E(-\theta)$.] The minimum is locally stable if $\partial^2E_-/\partial\theta^2>0$, but stability disappears when $\partial^2E_-/\partial\theta^2=0$, which holds at $\lambda=\lambda_c$, as we have already shown. If the fourth-order derivative is negative at $\theta=0$, then the equilibrium there becomes metastable, and we thus have a first-order phase transition at $\lambda=\lambda_c$. This is indeed the case here since

$$\frac{\partial^4E_-}{\partial\theta_4}=2\omega k\cosh\theta-8\lambda k^2(\sinh^2\theta+\cosh^2\theta) \qquad (3.13)$$

which at $\theta=0$, $\lambda=\lambda_c=4$, $\omega=1$, and $k=\frac{1}{4}$ is evaluated to be $-\frac{3}{2}$.

We wish to point out that in considering this phase transition we have actually used the mean-field Hamiltonian of Eq. (2.15). A more accurate determination of the ground-state energy requires the exact expectation values of K_+^2 and K_-^2 which are

$$\langle K_+^2\rangle=\langle K_-^2\rangle^*=(k^2+\tfrac{1}{2}k)\sinh^2\theta\, e^{2i\phi}\ . \qquad (3.14)$$

Thus we have

$$E=\langle\xi,k|H|\xi,k\rangle$$
$$=2\omega k\cosh\theta+\lambda(k^2+\tfrac{1}{2}k)\sinh^2\theta\cos(2\phi)\ . \qquad (3.15)$$

The ground-state critical properties are the same in this case, but the more accurate quantum approximation to the ground-state energy is determined by minimizing E of Eq. (3.15). This method of determining the ground-state energy is closely related to the scaling variational method[29] which has been given a group-theoretical formulation and applied to a number of systems.[30] Note that if we designate the effective mean-field coupling constant for Eq. (2.15) as $g_{MF}=\lambda k^2$ and for Eq. (3.15) as $g_e=\lambda(k^2+\frac{1}{2}k)$, then

$$\frac{g_e}{g_{MF}}=1+\frac{1}{2k} \qquad (3.16)$$

which becomes unity as $k\to\infty$.

Finally we note the effect of the additional terms of Eq. (2.11) when $\gamma\neq0$. The mean-field Hamiltonian is now

$$\mathcal{H}=2\omega k\cosh\theta+k^2\sinh^2\theta[\gamma+\lambda\cos(2\phi)]\ . \qquad (3.17)$$

Apparently the only effect this has is to shift the value of p for the fixed point.

IV. CONCLUSIONS

In this paper we have studied the classical dynamics and phase transition of a model SU(1,1) Hamiltonian which may have some relevance to many-boson systems such as a four-photon interaction of interest in quantum optics. The energy surface was shown to undergo a qualitative change at the phase transition of the ground state, where rotational motion disappears for $\lambda\geq\lambda_c$. It is worthwhile to contrast this behavior with the compact analog of our Hamiltonian, the SU(2) LMG Hamiltonian, as discussed by Kan *et al.* (Ref. 4). In their case both local minima and maxima as well as saddle points are found simultaneously on the energy surface when the coupling constant is less than the critical value. Both librational as well as rotational orbits are present. Above the critical value only rotational orbits are present. The reason for the difference in the energy surfaces of the two models is simply that in SU(2) the momentum variable is $\cos\theta$ rather than $\cosh\theta$, which introduces more intrinsic "periodicity" into the effective Hamiltonian.

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