Quantum theory of spontaneous emission in a one-dimensional optical cavity with two-side output coupling

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A quantum theory of spontaneous emission from an initially excited two-level atom in a one-dimensional optical cavity with output coupling from both sides is developed. Orthonormal mode functions with a continuous spectrum are employed, which are derived by imposing a periodic boundary condition on the whole space with a period much larger than the cavity length. The delay differential equation of the atomic state of Cook and Milonni [Phys. Rev. A **35**, 5081 (1987)] is re-derived in a strict manner, where the reflectivity of the cavity mirrors is included naturally in the mode functions. An approximate solution at a single-resonant-mode limit shows the results of "vacuum" Rabi oscillation in an underdamped cavity and enhanced spontaneous emission rate in an overdamped cavity. For the latter case, it is found that in the optical range the spontaneous emission rate is enhanced by a factor F (finesse of the cavity).

I. INTRODUCTION

It has been shown both theoretically and experimentally that spontaneous atomic emission in a cavity differs from the process in free space, due to the difference in the distribution of the modes. Enhanced spontaneous emission in a resonant cavity was first pointed out by Purcell,[1] while inhibited spontaneous emission in a small cavity was shown by Kleppner.[2] Experimentally, such effects have already been observed in microwave cavities[3,4] and also in optical cavities.[5-7]

It is of significant interest to develop a fully quantum-mechanical theory to account for these effects. Since the difference between the spontaneous emission in a cavity and that in free space arises from the change of the distribution of the field modes and depends on the Q (quality of the cavity) value[1] or the loss of the cavity, strict treatments of the field modes and the cavity loss are indispensable for such a theory.

So far, quantum theories of spontaneous emission in the cavity have been developed by a number of authors.[8-10] Sanchez-Mondragon, Narozhny, and Eberly studied this problem by considering the interaction of an atom with a lossless single-mode cavity.[8] They found, in an initially vacuum field, spectral line narrowing at large detuning and "vacuum Rabi splitting" at small detuning. Sachdev further developed the theory by including cavity damping with the reservoir method also in the single-mode context.[9] At the low-temperature limit, he obtained an enhanced spontaneous emission rate in an overdamped cavity in agreement with the prediction of Purcell,[1] while in an underdamped cavity he obtained damped Rabi oscillation. Cook and Milonni considered the interaction of an atom with a multimode Fabry-Pérot cavity in zero temperature and derived a delay-differential equation of the atomic state, where the damping of the cavity was introduced by two imperfect mirrors.[10] Their delay-differential equation was the first to treat a multimode cavity. In the single-mode limit they obtained the same results as those of Ref. 9. However, in their analysis, longitudinal normal-mode functions for a cavity with perfect mirrors were used, and the mirror reflectivity was introduced phenomenologically.

In this paper, we further develop the theory by using mode functions with a continuous spectrum[11-15] instead of the longitudinal normal-mode functions with a discrete spectrum. A one-dimensional cavity with dielectric medium inside and dielectric-vacuum coupling surfaces at the two ends is considered, and the continuous modes are derived by imposing a periodic boundary condition on the whole space with a period much larger than the cavity length. We analyze the interaction of an initially excited two-level atom with this cavity in the absence of any photons initially, and re-derive the Cook-Milonni delay-differential equation in a strict way. Here, the reflectivity at the coupling surfaces is naturally included in the mode functions, and the cavity modes are derived as resonant modes which give peaks in the continuous spectrum.

In Sec. II we derive the orthonormal mode functions of the cavity with the continuous spectrum. We then in Sec. III apply these mode functions to the analysis of the interaction of the atom with the cavity. The delay-differential equation, in agreement with that of Cook and Milonni,[10] will be obtained. It will also be shown that, in the optical range, the spontaneous emission rate in an overdamped single-mode cavity is F (finesse of the cavity) times, rather than Q times, faster than in a one-dimensional "free" space. A brief discussion about the approximations used in the present analysis is given in Sec. IV. Conclusions appear in Sec. V.

II. MODE FUNCTIONS AND FIELD QUANTIZATION

The cavity model used in this paper is illustrated in Fig. 1, where the z axis is taken along the longitudinal direction of the cavity. The cavity is filled with non-dispersive dielectric medium with the dielectric constant

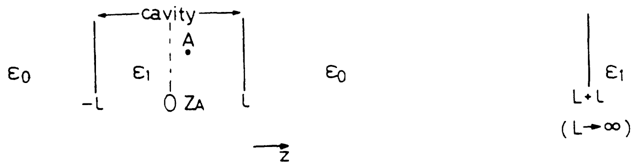

FIG. 1. The cavity model. The dielectric constants inside and outside the cavity are ϵ_1 and ϵ_0, respectively. The dielectric-vacuum coupling surfaces are at $z=-l$ and l. The emitting atom A is at $z=Z_A$.

ϵ_1, and outside the cavity is a vacuum with the dielectric constant ϵ_0. The dielectric-vacuum surfaces are at $z=-l$ and l. For mathematical convenience, we consider a periodic structure and a periodic boundary condition with one period from $z=-l$ to $z=L+l$, where the boundary $z=L+l$ will be set at an infinite distance later. The polarizations of the electric field and the magnetic field are assumed to be in the directions of the x axis and y axis, respectively. For such a one-dimensional cavity, we derive in this section normalized orthogonal (orthonormal) mode functions from the Maxwell equation with the boundary conditions at $z=-l$, l, and $L+l$, and then quantize the field with them.

A. Derivation of the mode function and the mode density

In our one-dimensional cavity, the wave equation for the vector potential $A(z,t)$ of the field obeys

$$\left[\frac{\partial}{\partial z}\right]^2 A(z,t)=\frac{1}{c^2}\left[\frac{\partial}{\partial t}\right]^2 A(z,t) , \qquad (2.1)$$

where c is the light velocity with the values of $c_1=(\mu\epsilon_1)^{1/2}$ inside the cavity and $c_0=(\mu\epsilon_0)^{1/2}$ outside the cavity. The above equation can be separated into two eigenequations:

$$A(z,t)=\sum_j U_j(z)Q_j(t) , \qquad (2.2)$$

$$\left[\frac{d}{dt}\right]^2 Q_j(t)=-\omega_j^2 Q_j(t) , \qquad (2.3)$$

$$\left[\frac{d}{dz}\right]^2 U_j(z)=-k_j^2 U_j(z) , \qquad (2.4)$$

where ω_j is a constant independent of the variables z and t, and $k_j=\omega_j/c$. The solution of (2.3) gives an oscillation with frequency ω_j, and (2.4) will give the mode function of the jth mode. With (2.2), the electric field $E(z,t)$ and the magnetic field $H(z,t)$ are expressed as

$$E(z,t)=-\frac{\partial}{\partial t}A(z,t)=-\sum_j U_j(z)P_j(t) , \qquad (2.5)$$

$$H(z,t)=\frac{1}{\mu}\frac{\partial}{\partial z}A(z,t)=\frac{1}{\mu}\sum_j U_j'(z)Q_j(t) , \qquad (2.6)$$

where the prime indicates differentiation with respect to z and

$$P_j(t)=\frac{d}{dt}Q_j(t) . \qquad (2.7)$$

The general solution of (2.4) for one period $(-l<z<L+l)$ can be written as

$$U_{j0}(z)=A_j e^{ik_{j0}z}+B_j e^{-ik_{j0}z} \quad (l<z<L+l) , \qquad (2.8a)$$

$$U_{j1}(z)=C_j e^{ik_{j1}z}+D_j e^{-ik_{j1}z} \quad (-l<z<l) , \qquad (2.8b)$$

where the subscripts 0 and 1 refer to the spatial regions $l<z<L+l$ and $-l<z<l$, respectively, and

$$k_{j0}=\omega_j/c_0, \quad k_{j1}=\omega_j/c_1 . \qquad (2.9)$$

Applying the continuous boundary conditions at two ends of the cavity $z=\pm l$ and the periodic boundary condition from $z=-l$ to $z=L+l$ to the electric and magnetic fields (2.5) and (2.6), we have

$$\begin{aligned} &U_{j1}(l)=U_{j0}(l) ,\\ &U_{j1}'(l)=U_{j0}'(l) ,\\ &U_{j1}(-l)=U_{j0}(L+l) ,\\ &U_{j1}'(-l)=U_{j0}'(L+l) , \end{aligned} \qquad (2.10)$$

The last two equations are obtained by combining the continuous boundary condition at $z=-l$ and the periodic boundary condition. With (2.8), these equations are

$$\begin{aligned} &C_j e^{ik_{j1}l}+D_j e^{-ik_{j1}l}=A_j e^{ik_{j0}l}+B_j e^{-ik_{j0}l} ,\\ &C_j k_{j1} e^{ik_{j1}l}-D_j k_{j1} e^{-ik_{j1}l}=A_j k_{j0} e^{ik_{j0}l}-B_j k_{j0} e^{-ik_{j0}l} ,\\ &C_j e^{-ik_{j1}l}+D_j e^{ik_{j1}l}=A_j e^{ik_{j0}(L+l)}+B_j e^{-ik_{j0}(L+l)} ,\\ &C_j k_{j1} e^{-ik_{j1}l}-D_j k_{j1} e^{ik_{j1}l}\\ &\qquad =A_j k_{j0} e^{ik_{j0}(L+l)}-B_j k_{j0} e^{-ik_{j0}(L+l)} \end{aligned} \qquad (2.11)$$

Here if the field of the jth mode exists, the constants A_j, B_j, C_j, and D_j will not all be zero, and the following equation must be satisfied:

$$\left[1-\frac{k_{j1}}{k_{j0}}\right]^2 \sin^2\left[k_{j1}l-\frac{k_{j0}L}{2}\right] =\left[1+\frac{k_{j1}}{k_{j0}}\right]^2 \sin^2\left[k_{j1}l+\frac{k_{j0}L}{2}\right] , \qquad (2.12)$$

or in the form of frequency ω_j, with (2.9)

$$\left[1-\frac{c_0}{c_1}\right]^2 \sin^2\left[\omega_j\left[\frac{l}{c_1}-\frac{L}{2c_0}\right]\right] =\left[1+\frac{c_0}{c_1}\right]^2 \sin^2\left[\omega_j\left[\frac{l}{c_1}+\frac{L}{2c_0}\right]\right] . \qquad (2.12')$$

Now, we consider the density of the modes determined by (2.12′). Sketching the curves of the two sides of Eq. (2.12′) as the functions of ω_j, one can find the solutions of (2.12′) at the intersections of the two curves. With this method, we find that there are two modes in each angular frequency interval $\Delta\omega$,

$$\Delta\omega=\frac{\pi}{l/c_1+L/(2c_0)} ,$$

so that the number of the modes in a unit angular frequency is

$$\rho(\omega_j)=\frac{2}{\Delta\omega}=\frac{2l}{\pi c_1}+\frac{L}{\pi c_0} \quad (L \text{ finite}) .$$

When $L\rightarrow\infty$, we can write the above mode density as

$$\rho(\omega_j)=\frac{L}{\pi c_0} \quad (L \text{ infinite}) . \tag{2.13}$$

This is equal to the mode density in a one-dimensional vacuum free space, and the modes given by such a mode density constitute a continuous spectrum.

Equation (2.12) can be further separated into the following two equations:

$$\tan(k_{j1}l)=-\frac{c_0}{c_1}\tan\left[\frac{k_{j0}L}{2}\right] \quad (a \text{ mode}) , \tag{2.14a}$$

or

$$\tan(k_{j1}l)=-\frac{c_1}{c_0}\tan\left[\frac{k_{j0}L}{2}\right] \quad (b \text{ mode}) . \tag{2.14b}$$

To facilitate the calculation, we use the form of k_j. We refer to the mode determined by (2.14a) as an a mode and by (2.14b) as a b mode. Solving (2.14) with a sketch, as that for (2.12′), we find that the a mode and the b mode always appear in pairs along the frequency axis. Therefore, the mode densities $\rho^a(\omega_j)$ of a modes and $\rho^b(\omega_j)$ of b modes are

$$\rho^a(\omega_j)=\rho^b(\omega_j)=\tfrac{1}{2}\rho(\omega_j)=\frac{L}{2\pi c_0} . \tag{2.15}$$

We also find that as $L\rightarrow\infty$ the a mode and b mode in one pair approach infinitely close to each other, so we can regard them as being degenerate.

Then we derive the mode functions of (2.8) from (2.11) and (2.14). The mode functions $U_j^a(z)$ of the a mode and $U_j^b(z)$ of the b mode are obtained as follows:

$$U_j^a(z)=\alpha_j\times\begin{cases}\sin(k_{j1}z) & (-l<z<l)\\ \sin(k_{j1}l)\cos[k_{j0}(z-l)]+\dfrac{c_0}{c_1}\cos(k_{j1}l)\sin[k_{j0}(z-l)] & (l<z<L+l)\end{cases} \tag{2.16a}$$

$$U_j^b(z)=\beta_j\times\begin{cases}\cos(k_{j1}z) & (-l<z<l)\\ \cos(k_{j1}l)\cos[k_{j0}(z-l)]-\dfrac{c_0}{c_1}\sin(k_{j1}l)\sin[k_{j0}(z-l)] & (l<z<L+l) .\end{cases} \tag{2.16b}$$

The undetermined parameters α_j and β_j will be determined by the normalization of $U_j(z)$.

B. Quantization and the orthonormal mode functions

With the expressions of electric and magnetic fields (2.5) and (2.6), the field energy H_f stored in one period $(-l<z<L+l)$ is obtained as

$$\begin{aligned}H_f&=\frac{1}{2}\int_{-l}^{L+l}[\epsilon E^2(z,t)+\mu H^2(z,t)]dz\\ &=\frac{1}{2}\int_{-l}^{L+l}\epsilon\left[\sum_j U_j(z)P_j(t)\right]^2dz\\ &\quad+\frac{1}{2}\int_{-l}^{L+l}\frac{1}{\mu}\left[\sum_j U_j'(z)Q_j(t)\right]^2dz .\end{aligned} \tag{2.17}$$

The integrals included in (2.17) can be calculated from the following orthonormal relations which can be readily proved with the wave equations of $U_j(z)$ and $U_j'(z)$ and the periodic boundary conditions (see Appendix A)

$$\int_{-l}^{L+l}\epsilon(z)U_i(z)U_j(z)dz=\delta_{ij} , \tag{2.18}$$

$$\int_{-l}^{L+l}\frac{1}{\mu}U_i'(z)U_j'(z)dz=\omega_j^2\delta_{ij} . \tag{2.19}$$

Here the normality in (2.18) for $i=j$ will be satisfied by determining the undetermined constants α_j and β_j of the mode functions (2.16). The field energy H_f of (2.17) becomes

$$H_f=\tfrac{1}{2}\sum_j[P_j^2(t)+\omega_j^2Q_j^2(t)] . \tag{2.20}$$

Thus the preparation for the field quantization has been completed. The quantization is accomplished by replacing P_j, Q_j, and H_f with the operators $\hat{P}_j$, $\hat{Q}_j$, and $\hat{H}_f$, and imposing the commutation relations as

$$[\hat{Q}_i,\hat{P}_j]=i\hbar\delta_{ij},\quad [\hat{Q}_i,\hat{Q}_j]=[\hat{P}_i,\hat{P}_j]=0 . \tag{2.21}$$

Introducing further the annihilation operator $\hat{a}_j$ and the creation operator $\hat{a}_j^\dagger$ as

$$\begin{aligned}\hat{a}_j&=(2\hbar\omega_j)^{-1/2}(\omega_j\hat{Q}_j+i\hat{P}_j) ,\\ \hat{a}_j^\dagger&=(2\hbar\omega_j)^{-1/2}(\omega_j\hat{Q}_j-i\hat{P}_j) ,\end{aligned} \tag{2.22}$$

which have the commutation relations derived from (2.21),

$$[\hat{a}_i,\hat{a}_j^\dagger]=\delta_{ij},\quad [\hat{a}_i,\hat{a}_j]=[\hat{a}_i^\dagger,\hat{a}_j^\dagger]=0 , \tag{2.23}$$

we obtained the Hamiltonian of the field as

$$\hat{H}_f = \sum_j \hbar\omega_j(\hat{a}_j^\dagger \hat{a}_j + \tfrac{1}{2}) \tag{2.24}$$

and the field operator $\hat{E}(z,t)$ from (2.5) as

$$\hat{E}(z,t) = i\sum_j (\hbar\omega_j/2)^{1/2} U_j(z)[\hat{a}_j(t) - \hat{a}_j^\dagger(t)] \ . \tag{2.25}$$

Here, all the a modes and the b modes derived in Sec. II A must be included in the summation $\sum_j$, and their mode functions are given in (2.16). Determining the factors α_j and β_j in (2.16) by the normality defined in (2.18), we have the following orthonormal mode functions in the whole space:

$$U_j^a(z) = \begin{cases} \left[\dfrac{2}{\epsilon_0 L}\right]^{1/2} \sin[k_{j0}(z+l) - \phi_a] & (z < -l) \\ \left[\dfrac{2}{\epsilon_1 L}\dfrac{1}{1-K\sin^2(k_{j1}l)}\right]^{1/2} \sin(k_{j1}z) & (-l < z < l) \\ \left[\dfrac{2}{\epsilon_0 L}\right]^{1/2} \sin[k_{j0}(z-l) + \phi_a], & (z > l) \end{cases} \tag{2.26a}$$

$$U_j^b(z) = \begin{cases} \left[\dfrac{2}{\epsilon_0 L}\right]^{1/2} \cos[-k_{j0}(z+l) + \phi_b] & (z < -l) \\ \left[\dfrac{2}{\epsilon_1 L}\dfrac{1}{1-K\cos^2(k_{j1}l)}\right]^{1/2} \cos^2(k_{j1}z) & (-l < z < l) \\ \left[\dfrac{2}{\epsilon_0 L}\right]^{1/2} \cos[k_{j0}(z-l) + \phi_b], & (z > l) \end{cases} \tag{2.26b}$$

where

$$\begin{aligned} &K = 1 - \epsilon_0/\epsilon_1, \quad 0 < K < 1 \ , \\ &\phi_a = \tan^{-1}\left[\frac{c_1}{c_0}\tan(k_{j1}l)\right] , \\ &\phi_b = \tan^{-1}\left[\frac{c_0}{c_1}\tan(k_{j1}l)\right] , \\ &0 \le \phi_a, \phi_b < \pi \end{aligned} \tag{2.27}$$

and the expressions for $z < -l$ are derived by the periodic boundary condition $U_j(z) = U_j(z+L+2l)$ and Eq. (2.14). The mode functions outside the cavity have been reformed in a simple form compared to (2.16). It shows in (2.26) that the amplitudes of the mode functions outside the cavity are a constant $(\epsilon_0 L)^{1/2}$ independent of the mode number j. This comes from the normalization of the mode functions where the value of the integral over the range $(-l < z < l)$ inside the cavity has been omitted compared to the integral value outside the cavity $(-L-l < z < l$ and $-l < z < L+l)$ by the assumption $L \gg l$. Under this condition, the amplitudes inside the cavity can be obtained simply by the continuous boundary conditions at $z = \pm l$. We can also see that the mode function of the a mode is odd symmetrical and that of the b mode is even symmetrical about the center $(z=0)$ of the cavity. It can be proved, by such parities, that the orthogonal relation still holds between the degenerate a mode and b mode (Appendix B).

The mode functions (2.26) combined with (2.24) and (2.25), are the main results of this section. They possess a number of important characteristics. First, the field amplitude reflectivity

$$r = \frac{c_0 - c_1}{c_0 + c_1} \tag{2.28}$$

at two ends of the cavity is naturally included in the mode functions, which can be seen clearly in the expansions of the normalization factors in the Fourier series as

$$\frac{1}{1-K\cos^2(k_{j1}l)} = \frac{2c_0}{c_1}\sum_{n=0}^{\infty}\frac{(-r)^n}{1+\delta_{0,n}}\cos(2nk_{j1}l) \ , \tag{2.29a}$$

$$\frac{1}{1-K\cos^2(k_{j1}l)} = \frac{2c_0}{c_1}\sum_{n=0}^{\infty}\frac{r^n}{1+\delta_{0,n}}\cos(2nk_{j1}l) \ , \tag{2.29b}$$

where $\delta_{0,n}$ is the Kronecker delta. On the other hand, according to the Mittag-Leffler theorem, the same factors can be expanded in another form as

$$\frac{1}{1-K\sin^2(k_{j1}l)} = \frac{2}{t_r}\frac{c_0}{c_1}\sum_{m=-\infty}^{\infty}\left[\frac{i}{\omega_j - (\omega_m^a - i\gamma_c)} + \text{c.c.}\right] , \tag{2.30a}$$

$$\frac{1}{1-K\cos^2(k_{j1}l)} = \frac{2}{t_r}\frac{c_0}{c_1}\sum_{m=-\infty}^{\infty}\left[\frac{i}{\omega_j - (\omega_m^b - i\gamma_c)} + \text{c.c.}\right] . \tag{2.30b}$$

It shows that ω_m^a and ω_m^b are the resonant modes giving peaks in the spectrum, and $\gamma_c\gamma_c$ is the halfwidth of these resonant modes. They are given as

$$\gamma_c = \frac{2}{t_r}\ln\left[\frac{1}{r}\right] = \frac{\omega_c}{\pi}\ln\left[\frac{1}{r}\right] , \tag{2.31}$$

$$\omega_m^a = (2m+1)\omega_c \ , \tag{2.32a}$$

$$\omega_m^b = 2m\omega_c \ , \tag{2.32b}$$

where

$$t_r = 4l/c_1 \ , \tag{2.33}$$

$$\omega_c = 2\pi/t_r \ , \tag{2.34}$$

t_r is the round trip time of the light propagating in the cavity, and ω_c is the separating interval of the resonant modes. It should be noted that although the a mode and the b mode are degenerate at any ω_j, as stated in Sec.

II A, their resonant modes ω_m^a and ω_m^b are separated with the interval ω_c. Considering the parities of the a mode and the b mode, it is clear that all these resonant modes give antinodes at both ends of the cavity. The well-used longitudinal normal-mode functions always give nodes at the ends of the cavity which cannot explain the output of the light from the cavity.

III. SPONTANEOUS EMISSION FROM AN ATOM IN THE CAVITY

We consider the spontaneous emission from an initially excited two-level atom with transition frequency ω_A which is located at $z=Z_A$ inside the cavity (Fig. 1). The cavity is assumed to be at zero temperature and no thermal photons will be excited. Using the results of the field quantization (2.24) and (2.25), we can write the Hamiltonian describing the system containing the atom and the field as

$$\hat{H}=\hat{H}_f+\hat{H}_A-\hat{\mu}\hat{E}(Z_A,t)$$
$$=\sum_j \hbar\omega_j\hat{a}_j^\dagger\hat{a}_j+\hat{H}_A-i\sum_j(\hbar\omega_j/2)^{1/2}U_j(Z_A)\hat{\mu}(\hat{a}_j-\hat{a}_j^\dagger)\ , \tag{3.1}$$

where $\hat{H}_A$ is the atom's Hamiltonian in the unbounded dielectric "free" space and $\hat{\mu}$ is the component of the atom's electric dipole operator in the polarization direction of the radiation field. The sum $\sum_j \hbar\omega_j/2$ contained in $\hat{H}_f$ of (2.24) is omitted here, since it will exert no influence on the motion of the atom.

In the Schrödinger picture, the total wave function can be written as

$$|\varphi(t)\rangle=C_u(t)|u\rangle|0\rangle e^{-i\omega_A t}+\sum_j C_{lj}(t)|l\rangle|1_j\rangle e^{-i\omega_j t}, \tag{3.2}$$

where $|u\rangle$ and $|l\rangle$ denote the upper and the lower atomic states, respectively, $|0\rangle$ denotes the state in which no photon exists in any modes, $|1_j\rangle$ denotes the state in which one photon exists in the jth mode while no photon exists in any of the other modes, and $C_u(t)$ and $C_{lj}(t)$ are the probability amplitudes of the states $|u\rangle|0\rangle$ and $|l\rangle|1_j\rangle$, respectively.

Substituting the Hamiltonian (3.1) and the wave function (3.2) into the Schrödinger equation, we have

$$\dot{C}_u(t)=-\sum_j\left[\frac{\omega_j}{2\hbar}\right]^{1/2}U_j(Z_A)\mu_A e^{-i(\omega_j-\omega_A)t}C_{lj}(t)\ , \tag{3.3a}$$

$$\dot{C}_{lj}(t)=\left[\frac{\omega_j}{2\hbar}\right]^{1/2}U_j(Z_A)\mu_A^* e^{i(\omega_j-\omega_A)t}C_u(t)\ , \tag{3.3b}$$

where $\mu_A=\langle u|\hat{\mu}|l\rangle$. Using the initial condition $C_{lj}(0)=0$, we get from the above equations

$$\dot{C}_u(t)=-\frac{|\mu_A|^2}{2\hbar}\int_0^t\left[\sum_j\omega_j U_j^2(Z_A)e^{i(\omega_j-\omega_A)(t'-t)}\right]\times C_u(t')dt'\ . \tag{3.4}$$

Since the summation $\sum_j$ here includes the degenerate, continuous a modes and b modes, it can be rewritten by an integral over the continuous modes with the mode density (2.15). Thus we have

$$\dot{C}_u(t)=\frac{\omega_A|\mu_A|^2}{4\hbar}\int_0^t dt'\int_{-\infty}^{\infty}d\omega_j\rho(\omega_j)\{[U_j^a(Z_A)]^2+[U_j^b(Z_A)]^2\}e^{i(\omega_j-\omega_A)(t'-t)}C_u(t')\ . \tag{3.5}$$

Here, two approximations have been used. One is that the factor ω_j has been taken out of the integral sign as a constant ω_A, by assuming that it varies much more slowly than other factors. The other is to allow the frequency ω_j to be negative for the convenience of calculation.

The reason for using these approximations is as follows: As for the integrand, $\rho(\omega_j)$ is merely a constant given by (2.15), and $[U_j^a(Z_A)]^2$ and $[U_j^b(Z_A)]^2$ are periodic functions of ω_j with the period ω_c. The remaining factor is the Fourier transform of $C_u(t)$ which gives a spectrum with a peak at ω_A or several peaks near ω_A. If the width $\Delta\omega_j$ of this spectrum is much smaller than ω_A, the integral can be performed approximately only over a small range $\Delta\omega_j \ll \omega_A$, and both of the above approximations will be allowed.

The width $\Delta\omega_j$ can be approximately given by $|\dot{C}_u/C_u|$. Therefore, $|\dot{C}_u/C_u| \ll \omega_A$ is required; i.e., the variation of the atomic state must be much slower than the periodic motion of the light. We solve (3.5) with two different methods to get a result in a general multimode cavity and a result in a single-mode cavity.

A. Expansion in the Fourier series and the delay differential equation of the atomic state

Substituting the expansion of the Fourier series (2.29) into the mode functions (2.26), we have for the range $-l<z(=Z_A)<l$,

$$[U_j^a(Z_A)]^2+[U_j^b(Z_A)]^2=\frac{4c_0}{\epsilon_1 c_1 L}\sum_{n=0}^{\infty}\frac{r^n}{1+\delta_{0,n}}\cos(2nk_{j1}l)\times\begin{cases}1 & \text{for } n \text{ even}\\ \cos(2k_{j1}Z_A) & \text{for } n \text{ odd}\ .\end{cases} \tag{3.6}$$

Then defining

$$t_1=\frac{2l-2Z_A}{c_1},\quad t_2=\frac{2l+2Z_A}{c_1}=t_r-t_1 \tag{3.7}$$

and reforming the summation of (3.6), we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} d\omega_j \rho(\omega_j)\{[U_j^a(Z_A)]^2+[U_j^b(Z_A)]^2\}e^{i(\omega_j-\omega_A)(t'-t)}\\
&=\frac{2}{\epsilon_1 c_1 \pi}\sum_{n=0}^{\infty} r^{2n}\int_{-\infty}^{\infty} d(\omega_j-\omega_A)\left[\frac{2}{1+\delta_{0,n}}\cos(\omega_j n t_r)+r\cos[\omega_j(nt_r+t_1)]+r\cos[\omega_j(nt_r+t_2)]\right]e^{i(\omega_j-\omega_A)(t'-t)}\\
&=\frac{2}{\epsilon_1 c_1}\sum_{n=0}^{\infty} r^{2n}\left[\frac{2}{1+\delta_{0,n}}e^{i\omega_A n t_r}\delta(t'-t+nt_r)+\frac{2}{1+\delta_{0,n}}e^{-i\omega_A n t_r}\delta(t'-t-nt_r)\right.\\
&\qquad +re^{i\omega_A(nt_r+t_1)}\delta(t'-t+nt_r+t_1)+re^{-i\omega_A(nt_r+t_1)}\delta(t'-t-nt_r-t_1)\\
&\qquad \left.+re^{i\omega_A(nt_r+t_2)}\delta(t'-t+nt_r+t_2)+re^{-i\omega_A(nt_r+t_2)}\delta(t'-t-nt_r-t_2)\right],
\end{aligned} \tag{3.8}$$

where $\delta(t)$ is the delta function, t_r is given by (2.33), and t_1 (t_2) is the round-trip time for the photon propagating between the atom and the right (left)-hand mirror. Substituting (3.8) into (3.5), we obtain a delay-differential equation for the probability amplitude of the upper atomic state $C_u(t)$ as follows:

$$\begin{aligned}
\dot{C}_u(t)=-\frac{A_0}{2}\Bigg[&C_u(t)H(t)+2\sum_{n=1}^{\infty} r^{2n}e^{i\omega_A n t_r}C_u(t-nt_r)H(t-nt_r)+\sum_{n=0}^{\infty} r^{2n+1}e^{i\omega_A(nt_r+t_1)}C_u(t-nt_r-t_1)H(t-nt_r-t_1)\\
&+\sum_{n=0}^{\infty} r^{2n+1}e^{i\omega_A(nt_r+t_2)}C_u(t-nt_r-t_2)H(t-nt_r-t_2)\Bigg],
\end{aligned} \tag{3.9}$$

where $H(t)$ is the unit step function and A_0 is the spontaneous emission rate in the one-dimensional "free" space of the unbounded dielectric medium, which is given by

$$A_0=\frac{\omega_A|\mu_A|^2}{\hbar\epsilon_1 c_1}. \tag{3.10}$$

Under the initial condition $C_u(0)=1$, the solution of (3.9) describes the transient process of the spontaneous emission from an initially excited atom in the cavity. Before the time $t=t_1$ ($<t_2$), (3.9) reads simply as $\dot{C}_u(t)=(A_0/2)C_u(t)$, which means an exponential decay such as that in the free space. After $t=t_1$, more and more terms appear at the right-hand side of (3.9), which can be regarded as the result of interactions of the atom with the radiations emitted by itself and multiply reflected from the cavity mirrors. From the viewpoint of the theory associated with image atoms generated by mirrors,[16] these terms can also be considered as the effect of cooperative decay of a chain of image atoms, which are here located outside the cavity at the distances $c_1(nt_r\pm t_1/2)$ from the right-hand mirror and $c_1(nt_r\pm t_2/2)$ from the left-hand mirror.

Equation (3.9) is the same as that derived by Cook and Milonni,[10] where the mode functions of only the resonant modes were used and a mirror reflectivity was introduced phenomenologically. We solved the same problem, but for a cavity with one perfect conducting wall instead of one of the mirrors in the present cavity.[15] There, a similar delay-differential equation was derived, in which the factor r^{2n} of (3.9) is replaced by $(-r)^n$; i.e., the multireflectivity r^n of one mirror is replaced by $(-1)^n$ of the conducting wall.

B. Expansion in terms of resonant modes and solution at a single-resonant-mode limit

Using the expansion (2.30) instead of (2.29) to express the mode functions in the right-hand side of (3.5), we have

$$\begin{aligned}
\dot{C}_u(t)=-\frac{A_0}{\pi t_r}\sum_{m=-\infty}^{\infty}\int_0^t dt'\int_{-\infty}^{\infty} d\omega_j\Bigg[&\left[\frac{i}{\omega_j-\Omega_m^a}+\text{c.c.}\right]\sin^2(\omega_j t_A)\\
&+\left[\frac{i}{\omega_j-\Omega_m^b}+\text{c.c.}\right]\cos^2(\omega_j t_A)\Bigg]e^{i(\omega_j-\omega_A)(t'-t)}C_u(t'),
\end{aligned} \tag{3.11}$$

where c.c. means the complex conjugation of the first term within the same parentheses, A_0 is given by (3.10), and $\Omega^a{}_m$ and $\Omega^b{}_m$ are the complex frequencies of the resonant modes given by

$$\Omega_m^a=\omega_m^a-i\gamma_c,\quad \Omega_m^b=\omega_m^b-i\gamma_c\ , \tag{3.12}$$

and

$$t_A=\frac{Z_A}{c_1}\ . \tag{3.13}$$

The right-hand side of (3.11) is given by the contributions from all the resonant modes, and can be used to get a single-resonant-mode limit by considering only one resonant mode and neglecting other resonant modes. When $t\gg t_r$, such a case for a resonant mode, for example, ω_m^a, can be obtained under the following single-resonant-mode limit conditions:

$$A_0\ll 1/t_r,\quad \gamma_c\ll\omega_c,\quad |\Delta|\ll\omega_c\quad (\Delta=\omega_m^a-\omega_A)\ , \tag{3.14}$$

i.e., the atom must decay sufficiently slowly during the round-trip time t_r to ensure the possibility of the single-resonant-mode interaction, and in order to get the single-resonant-mode limit both the halfwidth γ_c of the resonant modes and the detuning Δ should be much smaller than the resonant-mode interval ω_c. In such a limit, only the resonant mode ω_m^a contributes appreciably to the interaction with the atom, and the terms of other resonant modes can be omitted in (3.11). Then performing the integration $\int_{-\infty}^{\infty}d\omega_j$ by the residue theorem, we have

$$\begin{aligned}\dot C_u(t)&\simeq-\frac{A_0}{\pi t_r}\int_0^t C_u(t')e^{i\omega_A(t-t')}dt'\int_{-\infty}^{\infty}\left[\frac{i}{\omega_j-\Omega_m^a}+\text{c.c.}\right]\sin^2(\omega_j t_A)e^{i\omega_j(t'-t)}d\omega_j\\&=-\frac{A_0}{2t_r}\left[(2-e^{i2\Omega a_m t_A}-e^{-i2\Omega a_m t_A})\int_0^t C_u(t')e^{i(\Omega_m^a-\omega_A)(t'-t)}dt'+e^{i2\Omega_m^a t_A}\int_{t-2t_A}^{t}C_u(t')e^{i(\Omega_m^a-\omega_A)(t-t')}dt'\right.\\&\qquad\left.-e^{i2(\Omega_m^a)^*t_A}\int_{t-2t_A}^{t}C_u(t')e^{i[(\Omega_m^a)^*-\omega_A](t-t')}dt'\right]\ .\end{aligned}\tag{3.15}$$

Since $t\gg t_r>2t_A$, terms of the integral over the time $2t_A$ are negligible compared to the integral over much longer time t, and thus we have

$$\ddot C_u(t)+(\gamma_c+i\Delta)\dot C_u(t)+\left[\frac{\Omega}{2}\right]^2C_u(t)=0\ , \tag{3.16}$$

where Δ is the detuning defined in (3.14) and Ω is the "vacuum" Rabi frequency given by

$$\left[\frac{\Omega}{2}\right]^2=\left[\frac{2A_0}{t_r}\right]\sin^2(\Omega_m^a t_A)\simeq\frac{\omega_c A_0}{\pi}\sin^2(\omega_m^a t_A)\ . \tag{3.17}$$

where the approximation $\Omega_m^a=\omega_m^a-i\gamma_c\simeq\omega_m^a$ has been used, since $\gamma_c\ll\omega_c\le\omega_m^a$. The same equation as (3.16) was derived by Cook and Milonni[10] directly from the delay-differential equation. For the single-resonant-mode limit at ω_m^b, the same equation as (3.16) can be obtained with $\Delta=\omega_m^b-\omega_A$ and $(\Omega/2)^{1/2}=(2A_0/t_r)\cos^2(\omega_m^b t_A)$.

From (3.17) and the condition $A_0\ll 1/t_r$, we have $\Omega^2\le(4/\pi)\omega_c A_0\ll(4/\pi)\omega_c/t_r=(2/\pi^2)\omega_c^2<\omega_c^2$, i.e., in the single-resonant-mode limit, the Rabi frequency Ω is much smaller than the resonant-mode interval ω_c, as expected.

With the initial condition $C_u(0)=1$ and $\dot C_u(0)=0$ [i.e., $C_{lj}(0)=0$], (3.16) can be solved easily. In the limit $\Delta=0$ we have for an underdamped cavity $\gamma_c\ll\Omega$

$$|C_u(t)|^2=e^{-\gamma_c t}\cos^2\left[\frac{\Omega}{2}t\right]=\tfrac{1}{2}e^{-\gamma_c t}[1+\cos(\Omega t)]\ . \tag{3.18}$$

This shows a damped Rabi oscillation of the atomic state with frequency Ω. In the same limit $\Delta=0$, we have for an overdamped cavity $\gamma_c\gg\Omega$

$$\begin{aligned}|C_u(t)|^2&=\exp\left[-\frac{\Omega^2}{2\gamma_c}t\right]\\&=\exp\left[-\frac{2\omega_c A_0\sin^2(\omega_m^a t_A)}{\pi\gamma_c}t\right]\\&=\exp\left[-\frac{4}{\pi}FA_0 t\sin^2(\omega_m^a t_A)\right]\ ,\end{aligned}\tag{3.19}$$

where (3.17) has been used and $F=\omega_c/2\gamma_c$ is the finesse of the cavity. When $\sin^2(\omega_m^a t_A)\simeq 1$, i.e., the atom is near to the antinode, (3.19) gives

$$|C_u(t)|^2\approx e^{-FA_0 t}\ . \tag{3.20}$$

This means that the spontaneous emission rate is F times faster than in the one-dimensional free space of unbounded dielectric medium. If the atomic radiation frequency ω_A resonates with the lowest mode of the cavity, i.e., $\omega_A=\omega_c$, the finesse F just equals the quality factor Q of the cavity. For the usual optical cavity with cavity length much larger than the wavelength of the radiation, we have $\omega_A \gg \omega_c$ and the factor of the enhancement of the spontaneous emission is, correctly, F rather than Q.

IV. DISCUSSION

The quantum theory of spontaneous emission presented in this paper is based on the dipole approximation and the rotating-wave approximation, which are implicitly contained in the expressions of the Hamiltonian and the wave function. It has been assumed that there is no interaction between the atom and the cavity medium by considering that the cavity medium has much larger transition frequency than that of the radiating atom.

As stated in Sec. III, the approximations used in (3.5) require that the variation of the atomic state is much slower than the periodic motion of the radiation. Two main factors affect the motion of the atomic state: the spontaneous decay rate in the free space A_0 and the photon round-trip time in the cavity t_r. First, in the optical range, A_0 is of the order of 10^9 s^{-1}, which is much slower than 10^{14} s^{-1} of the radiation frequency. Second, if the cavity length is much longer than the radiation wavelength, i.e., the inverse $1/t_r$ is much smaller than the radiation frequency, the condition for the approximations will be satisfied.[15] When the cavity length is the order of the radiation wavelength, it is appropriate to consider only the case of the single-resonant-mode limit. In such a case, from the results of Sec. III B we have $|\dot{C}_u/C_u| \le (\omega_c A_0)^{1/2}/2 < 10^{-2}\omega_A$ (here $\omega_c = \omega_A$ and $A_0 \approx 10^{-5}\omega_A$) and the approximations used in (3.5) are therefore still valid.

With the help of the mode functions incorporating output coupling, the intensities of the radiated fields both inside and outside the cavity can be obtained. The calculation processes are similar to those given in a recent paper,[15] and have not been repeated here.

It should be noted that the results of the spontaneous emission obtained in this paper are all limited to one dimension, and of course they differ from those in three dimensions. Here, we compare our results in the one-dimensional cavity with that in the one-dimensional free space to get an image of the spontaneous emission in the one-dimensional world, which can help us to get an insight into those in the realistic three-dimensional world.

V. CONCLUSIONS

In conclusion, for the one-dimensional optical cavity with two-side output coupling, we have derived the orthonormal mode functions with a continuous spectrum, and quantized the field of the cavity with them. Two kinds of degenerate continuous modes have been derived, which are odd symmetrical and even symmetrical about the center of the cavity, respectively. The resonant modes are also derived associated with the poles of the mode functions, and all of them have antinodes at the ends of the cavity.

With the above mode functions, we have developed a quantum theory of the spontaneous emission from a two-level atom in the cavity, and rederived Cook-Milonni's delay-differential equation of the atomic state[10] in a rigorous method. In the single-resonant-mode limit, we have obtained, in an underdamped cavity, damped "vacuum" Rabi oscillation, and in an overdamped cavity, a spontaneous emission rate F times faster than in the free space.

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APPENDIX A

In this appendix, we show the derivation of the orthogonal relations included in (2.18) and (2.19). The wave equation of the mode function (2.4) gives

$$\frac{d^2U_j(z)}{dz^2} = -\omega_j^2\mu\epsilon(z)U_j(z) . \tag{A1}$$

The periodic boundary condition with a period from $z=-l$ to $z=L+l$ gives

$$U_j(-l)=U_j(L+l) , \tag{A2}$$

$$U_j'(-l)=U_j'(L+l) , \tag{A3}$$

for the electric and magnetic fields (2.6) and (2.7), respectively. Using (A1)–(A3), we have

$$\begin{aligned}\int_{-l}^{l+L}\epsilon(z)U_i(z)U_j(z)\,dz &= -\frac{1}{\omega_i^2\mu}\int_{-l}^{l+L}\frac{d^2U_i(z)}{dz^2}U_j(z)dz \\ &= -\frac{1}{\omega_i^2\mu}\left[\int_{-l}^{l+L}\frac{d}{dz}[U_i'(z)U_j(z)]dz - \int_{-l}^{l+L}U_i'(z)U_j'(z)\,dz\right] \\ &= -\frac{1}{\omega_i^2\mu}\left[[U_i'(z)U_j(z)]_{z=-l}^{z=l+L} - \int_{-l}^{l+L}U_i'(z)U_j'(z)\,dz\right] \\ &= \frac{1}{\omega_i^2\mu}\int_{-l}^{l+L}U_i'(z)U_j'(z)\,dz = \frac{1}{\omega_j^2\mu}\int_{-l}^{l+L}U_i'(z)U_j'(z)\,dz = 0 \quad (\text{if } \omega_i \ne \omega_j) .\end{aligned} \tag{A4}$$

Therefore, with the addition of the normal relations, we have

$$\int_{-l}^{l+L} \epsilon(z) U_i(z) U_j(z) dz = \delta_{ij} , \quad \text{(A5)}$$

and

$$\frac{1}{\mu} \int_{-l}^{l+L} U_i'(z) U_j'(z)\, dz = \omega_j^2 \int_{-l}^{l+L} \epsilon(z) U_i'(z) U_j'(z)\, dz = \omega_j^2 \delta_{ij} . \quad \text{(A6)}$$

APPENDIX B

In this appendix, we show the derivation of the orthogonal relation between the degenerate a mode and b mode. The result of the integration of a periodic function over one period is independent of the integration position. Since the integrand $\epsilon(z) U_j^a(z) U_j^b(z)$ is a periodic function with period $L+2l$, we can change the range of the integration from $(-l, l+L)$ to be $(-l-L/2, l+L/2)$. In the range $(-l-L/2, l+L/2)$, $\epsilon(z)$ and $U_j^b(z)$ are even functions, and $U_j^a(z)$ is an odd function, hence their product $\epsilon(z) U_j^a(z) U_j^b(z)$ is an odd function. Thus we have

$$\int_{-l}^{l+L} \epsilon(z) U_j^a(z) U_j^b(z)\, dz = \int_{-l-L/2}^{l+L/2} \epsilon(z) U_j^a(z) U_j^b(z)\, dz = 0 . \quad \text{(B1)}$$

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