Properties of displaced number states

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Recent developments in quantum optics have led to new proposals to generate number states of the electromagnetic field using conditioned measurement techniques or the properties of atom-field interactions in microwave cavities in the micromaser. The number-state field prepared in such a way may be transformed by the action of a displacement operator; for the microwave micromaser state this could be implemented by the action of a classical current that drives the cavity field. We evaluate some properties of such displaced number states, especially their description in phase space. The photon number distribution is shown to display unusual oscillations, which are interpreted as interference in phase space, analogous to Franck-Condon oscillations in molecular spectra. The possibility of detecting these oscillations is discussed, through the photodetection counting statistics of the displaced number states. We show that the displaced-number-state quantum features are relatively robust when dissipation of the field energy is included.

I. INTRODUCTION

Recent developments in quantum optics have led to suggestions of how nonclassical states of light, particularly number states of the electromagnetic field, may be prepared.[1] At microwave frequencies the Rydberg-atom micromaser[2] is highly sensitive to the quantized nature of the radiation field in a cavity and has "trap" states where the field approaches a number state with a large degree of sub-Poisson photon statistics.[3] These trap states are those in which the quantum field has the required photon number to generate multiples of full Rabi cycles in subsequent atoms entering the cavity, leaving the field, in consequence, unchanged. A distribution of interaction times, analogous to normal laser pump fluctuations, will wash out such effects.[4] Experimental observation of sub-Poissonian trap states, with the cavity field prepared to good approximation in a number state, has been reported when care was taken to ensure uniform atom-field interaction times in the micromaser.[5]

Localized one-photon states have been constructed by optical shutter techniques using photons generated in pairs in parametric down conversion,[6] where the signal photon opens a photoelectric detection gate to the idler photon. Related aspects of such conditioned measurements have been proposed for photon number-state preparation in nondegenerate parametric amplification.[7]

Given that photon number states (at least those with modest occupation numbers) can be generated, it is natural to ask whether they can be amplified. Rather than address this problem directly, we turn our attention to the simpler problem of *displacing* a number state by a translation operator $\hat{D}(\alpha)$,[8] the generator of coherent states from the vacuum. A displacement of a field state (usually the vacuum) may be implemented by driving the quantized field by a classical current. To this end, we have in mind a microwave cavity field relevant to the micromaser, which can indeed be driven by a current with essentially negligible quantum fluctuations. Had we prepared the microwave cavity initially in a vacuum state, such a classical current would displace the vacuum to create a coherent state. In this paper, we examine the consequences of driving a cavity field, initially prepared in a number state, by a classical current, or, in other words, of displacing the number state.

We will call the states derived by acting on the number state with a displacement operator the *displaced number state*. The idea of a displaced number state can be found in earlier work, for example, in the work of Cahill and Glauber,[9] where it was defined as the eigenstate of an operator used to expand functions of boson annihilation and creation operators in a given order. They did not associate these states with a physical state of intrinsic interest. Boiteux and Levelut[10] have studied what is equivalent to the displaced number state in their work on generalized coherent states. More recently, Roy and Singh[11] and Venkata Satyanarayana[12] have discussed the displaced number state extensively and have related it to the general form of the transition probability found independently by Feynman and Schwinger.[13] In one sense, the displaced number state is obtained from a number state by adding a nonzero value to the field amplitude. We will show in this paper that the displaced number states have interesting and unusual physical properties. The number state is determined by its photon number while the phase is completely random. The amplitude of the field has a zero expectation value. By displacing in phase space, a field amplitude is added to this state, and the photon number has now a contribution from the coherent component of the field. The state becomes phase dependent because of the phase of the displacement, is centered around a new origin located at the coherent amplitude position, and is invariant by a rotation around this point, with phase-independent fluctuations. Despite this apparent simplicity, several interesting consequences emerge from this study. First, although the field is nonclassical and does not possess a well-behaved diagonal coherent-state P distribution, we will see that the fluctuations of the field quadratures and the photon number can be greater than the vacuum fluctuations. This is an important point, because we usually associate a nonclassical field with having either of these observables with *less* fluctuations than the vacuum. Second, this state shows very striking oscillations in the photon number distribution,[9,11,12] related to the Franck-Condon oscillations observed in molecular transitions between displaced oscillator states.[14] We interpret these photon distribution oscillations as the result of interference in phase space. Furthermore, using the arguments of Ref. 15, because the regions interfering in phase space can be very close to each other (as we will show), the probability of measuring such oscillations increases. One final importance of the displaced number state is that it provides the generalization of the coherent state (which is a displaced vacuum state) to states obtained by displacing photon number states originally with at least one photon present.[10–12] If only one photon were initially present, this displaced one-photon state and the coherent states would be two states which would be microscopically close; through the measurement of their photon number distributions, one could have a macroscopic distinction between these states. We demonstrate that such a measurement is possible, by investigating the influence of dissipation, or equivalently of nonunit quantum efficiency in these oscillations, showing that the displaced number states are much less sensitive to dissipation and imperfect detection than the squeezed states.

The plan of this paper is as follows. First, we define the displaced number state and describe some of its properties. We calculate the quasiprobability functions and the deviation from a Poissonian field. The relationship between squeezing and displacement of any state is discussed, and as illustrations we compare the squeezed number states with the displaced number states. The photon number distribution is discussed, and we find that it displays some unusual oscillations, which are explained in terms of phase-space interference.[14]

II. DISPLACED NUMBER STATE

In this section we summarize various properties of the displaced number state necessary in later sections. The displaced number state is defined by

$$|\alpha,n\rangle=\hat{D}(\alpha)|n\rangle\ , \tag{2.1}$$

where $\hat{D}(\alpha)$ is the displacement operator,[8] given by

$$\hat{D}(\alpha)=\exp(\alpha\hat{a}^{\dagger}-\alpha^*\hat{a})\ . \tag{2.2}$$

For $n=0$, the displaced number state, Eq. (2.1), reduces to the well-known coherent state, introduced by Schrödinger.[16] Some properties of the displaced number state can be derived using the transformation of the annihilation and creation operators under a displacement

$$\begin{aligned}\hat{D}(\alpha)\hat{a}\hat{D}^{\dagger}(\alpha)&=\hat{a}-\alpha\ ,\\ \hat{D}(\alpha)\hat{a}^{\dagger}\hat{D}^{\dagger}(\alpha)&=\hat{a}^{\dagger}-\alpha^*\ .\end{aligned} \tag{2.3}$$

We define the quadrature operators by

$$\hat{X}_1=\hat{a}+\hat{a}^{\dagger}\ , \tag{2.4}$$

$$\hat{X}_2=-i(\hat{a}-\hat{a}^{\dagger})\ . \tag{2.5}$$

With the use of Eqs. (2.1)–(2.5), the variances $\langle(\Delta\hat{X}_i)^2\rangle=\langle\hat{X}_i^2\rangle-\langle\hat{X}_i\rangle^2$ of the quadrature operators in the state $|\alpha,n\rangle$ are

$$\langle(\Delta\hat{X}_i)^2\rangle=\langle\alpha,n|(\Delta\hat{X}_i)^2|\alpha,n\rangle=(2n+1)\quad(i=1,2)\ . \tag{2.6}$$

These variances are independent of the coherent amplitude of the state, but linearly dependent of the initial photon number n. The photon number variance $\langle(\Delta\hat{n})^2\rangle$ is

$$\langle(\Delta\hat{n})^2\rangle=(2n+1)|\alpha|^2\ , \tag{2.7}$$

and is always greater than that for a number state and a coherent state. With the use of Eqs. (2.7) and (2.8), we find the Mandel Q parameter,[17] measuring the deviation from a Poisson statistics for the state $|\alpha,n\rangle$

$$Q=\frac{\langle(\Delta\hat{n})^2\rangle-\langle\hat{n}\rangle}{\langle\hat{n}\rangle}=n\left[\frac{2|\alpha|^2-1}{n+|\alpha|^2}\right]\ , \tag{2.8}$$

where the average photon number

$$\langle\hat{n}\rangle=n+|\alpha|^2\ , \tag{2.9}$$

in which the contributions from the numberlike and coherentlike characters of the field are explicitly displayed. For Poissonian statistics, $Q=0$. If $Q<0$, the light is said to be sub-Poissonian, otherwise, it is super-Poissonian. From Eq. (2.8), we find that the state $|\alpha,n\rangle$ has sub-Poissonian photon statistics if $|\alpha|^2<\frac{1}{2}$, in other words, if the coherent contribution adds more than half a photon to the average photon number, the state is super-Poissonian independent of the initial photon number n.

III. QUASIPROBABILITY FUNCTIONS

A quantum quasiprobability is defined in terms of the displacement operator $\hat{D}(\lambda)$ [Eq. (2.2)] as[9]

$$W(\beta,p)=\pi^{-2}\int\mathrm{Tr}[\hat{\rho}\hat{D}(\lambda)]e^{p|\lambda|^2}\times\exp(\beta\lambda^*-\beta^*\lambda)d^2\lambda\ . \tag{3.1}$$

For $p=1$, we obtain the P distribution; $p=0$ gives the Wigner distribution and $p=-1$ gives the Q function. Based on this definition, very general conclusions can be reached about a given new state of light, if such state can be demonstrated to be obtained by a transformation from a previous one. We denote the original functions and operators with a subscript 0 and the displaced ones with a subscript d. Let us consider the effect of the displacement $\hat{D}(\alpha)$ on the original state described by the density matrix $\hat{\rho}_0$. The displaced quasiprobabilities are given by

$$W_d(\beta,p)=\pi^{-1}\int \mathrm{Tr}[\hat{\rho}_0\hat{D}(\lambda)]e^{p|\lambda|^2}\times\exp[(\beta-\alpha)\lambda^*-(\beta^*-\alpha^*)\lambda]d^2\lambda$$

or

$$W_d(\beta,p)=W_0(\beta-\alpha,p)\ , \tag{3.2}$$

and we see that for any ordering the new quasiprobability is obtained by a simple displacement of the quasiprobability of the original state. This is a very important result because we can immediately say that if a state does not have a well-behaved given quasiprobability, then a displacement of that state will produce a new state which does not have that quasiprobability as well. From this, we conclude that the displaced number state does not have a well-behaved P function. One could argue that this should be expected, but curiously, we know from the results of Sec. II that this state can simultaneously be super-Poissonian and have more quadrature fluctuations than the vacuum.

We will derive the effect of squeezing for comparison purposes. In this case, the new state density matrix $\hat{\rho}_s$ is obtained from the original density matrix $\hat{\rho}_0$ by the transformation

$$\hat{\rho}_s=\hat{S}(r)\hat{\rho}_0\hat{S}^{\dagger}(r)\ , \tag{3.3}$$

where the squeezing operator is given by[18]

$$\hat{S}(r)=\exp(\tfrac{1}{2}r\hat{a}^{2}-\tfrac{1}{2}r\hat{a}^{\dagger 2})\ . \tag{3.4}$$

We have assumed a real squeezing parameter r for simplicity. As a result of the application of $\hat{S}(r)$, a Bogoliubov transformation of the annihilation and creation operators is obtained[18]

$$\begin{aligned}\hat{S}(r)\hat{a}\hat{S}^{\dagger}(r)&=\hat{a}\cosh r+\hat{a}^{\dagger}\sinh r\ ,\\ \hat{S}(r)\hat{a}^{\dagger}\hat{S}^{\dagger}(r)&=\hat{a}^{\dagger}\cosh r+\hat{a}\sinh r\ .\end{aligned} \tag{3.5}$$

We denote the initial functions and operators with a subscript 0 and the squeezed ones with a subscript s. Using Eq. (3.5), it can be shown[20] that $\hat{S}\hat{D}(\lambda)\hat{S}^{\dagger}=\hat{D}(\zeta)$, where $\zeta=\lambda\cosh r+\lambda^*\sinh r$. Using Eq. (3.2) in Eq. (3.1), we obtain

$$\begin{aligned}W_s(\beta,p)&=\pi^{-2}\int \mathrm{Tr}[\hat{\rho}_0\hat{D}(\zeta)]e^{p|\lambda|^2}\times\exp(\beta\lambda^*-\beta^*\lambda)d^2\lambda\\ &=\pi^{-2}\int \mathrm{Tr}[\hat{\rho}_0\hat{D}(\zeta)]e^{p|\lambda|^2}\times\exp(\bar{\beta}\zeta^*-\bar{\beta}^*\zeta)d^2\zeta\ ,\end{aligned} \tag{3.6}$$

where $\bar{\beta}=\beta\cosh r+\beta^*\sinh r$. Comparison of Eq. (3.6) with Eq. (3.1) shows no simple connection between them because of the presence of the term $e^{p|\lambda|^2}$, which is not transformed like the other terms. But for $p=0$, this factor disappears, and we can see that $W_s(\beta,0)=W_0(\bar{\beta},0)$, i.e., the result is a scaling transformation, with lengths in one direction being contracted and the orthogonal direction being expanded.[21] The Wigner function is the only quasiprobability allowing a simple description of such transformation, probably due to the fact that it is associated with Hermitian symmetrical and antisymmetrical combinations of the annihilation and creation operators. It might be thought that we could obtain all the quasiprobabilities from the simpler ones by a simple unitary transformation. What we have shown is that this is valid only for the Wigner function. We will relate this result to previous ones in what follows. In the following, we will be concerned only with the Wigner and the Q functions, associated, respectively, with the symmetrical and the antinormal orderings.

A. Q function

The Q function for a pure state $|\psi\rangle$ can be defined by[9]

$$\pi Q(\beta)=|\langle\psi|\beta\rangle|^2\ , \tag{3.7}$$

where $|\beta\rangle$ is a coherent state. For the displaced number state,[11]

$$|\psi\rangle=|\alpha,n\rangle=\hat{D}(\alpha)|n\rangle=\frac{(a^{\dagger}-\alpha^*)^n}{\sqrt{n}}|\alpha\rangle\ ,$$

so that we obtain the Q function $Q_{dn}(\beta)$ for the displaced number state in terms of $Q_n(\beta)$, the Q function for the number state

$$Q_{dn}(\beta)=Q_n(\beta-\alpha)=\pi^{-1}e^{-|\beta-\alpha|^2}\frac{|\alpha-\beta|^{2n}}{n!}\ . \tag{3.8}$$

This function is zero for $\beta=\alpha$ and is dominated by the Gaussian factor for points distant enough from the center of symmetry (Fig. 1). The maximum value is obtained for values of β located at the circle of radius $|\beta-\alpha|=n^{1/2}$ with center at $\beta=\alpha$. It is seen that the average value of the amplitude $\langle\beta\rangle=\alpha$ and the fluctuations in β are phase independent.

B. Wigner function

The Wigner function $W_{dn}(\beta)$ for the displaced number state can be calculated by using the displaced version of

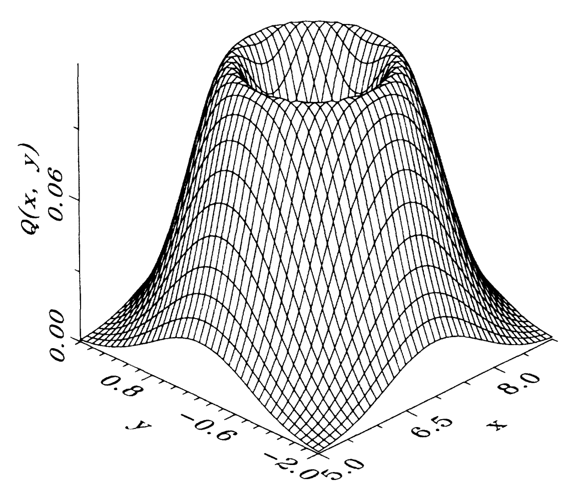

FIG. 1. Q function for the displaced number state $|7,1\rangle$.

the well-known Wigner function for the number state $W_n(\beta)$ (see, e.g., Ref. 22 and references therein),

$$W_{dn}(\beta)=W_n(\beta-\alpha)=\frac{2}{\pi}\exp(-2|\beta-\alpha|^2)(-1)^n \times\mathcal{L}_n(4|\beta-\alpha|^2) , \quad (3.9)$$

where $\mathcal{L}_n(x)$ is the Laguerre polynomial of order n.[23] We see that there will be oscillations in this function, and negative values occur in some regions. In the following discussion, we will use $\mathcal{L}_1(x)=1-x$. Displacing a single-photon state to obtain $|\alpha,1\rangle$, the resulting Wigner function have negative values inside the circle $|\beta-\alpha|<\frac{1}{2}$, centered at α, but is positive outside this circle. The maximum value is reached at the circle centered at α of radius $|\alpha-\beta|=1.5$. This function is displayed in Fig. 2 for the state $|\alpha=7,n=1\rangle$.

C. Comparison with the squeezed number state

As expected from the discussion in the beginning of this section, we see that Figs. 1 and 2 have the same form as for the number state. Comparing Fig. 1 with Fig. 2(b) of Ref. 20, representing the Q function for the squeezed number state, we see that a simple contraction cannot transform one Q function into the other. Nevertheless, it is easy to see that Fig. 2(b) of Ref. 20, for the Wigner function of the squeezed number state, is exactly the expected contraction of our Fig. 2. The Wigner function for the squeezed number state has the same maximum value, in any direction of the phase space, but for points located in an ellipse rather than a circle.

IV. PHOTON NUMBER STATISTICS

A. Wave functions and number distributions

The coordinate space wave functions of the number state is that of the harmonic oscillator

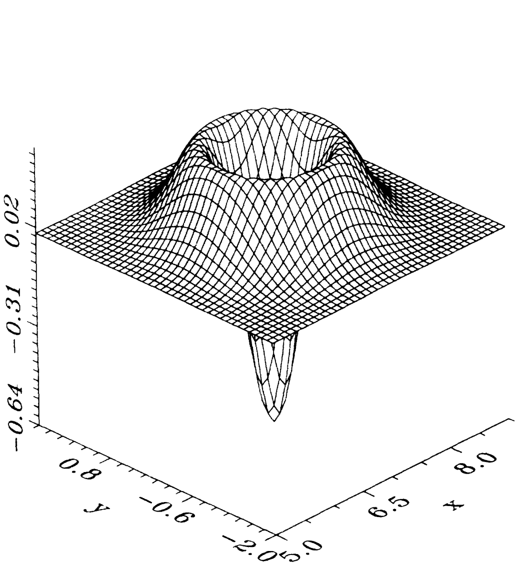

FIG. 2. Wigner function for the displaced number state $|7,1\rangle$.

$$\Psi_n(x)=\frac{1}{\pi^{1/4}}\frac{1}{(2^n n!)^{1/2}}H_n(x)\exp(-\tfrac{1}{2}x^2) , \quad (4.1)$$

where $H_n(x)$ is a Hermite polynomial. The displacement operator for a *real* displacement translates $x\rightarrow x-\sqrt{2}\alpha$,[24] thus the wave function $\Psi_{dn}(x)$ for the displaced number state is

$$\Psi_{dn}(x)=\frac{1}{\pi^{1/4}}\frac{1}{(2^n n!)^{1/2}}H_n(x-\sqrt{2}\alpha) \times\exp[-\tfrac{1}{2}(x-\sqrt{2}\alpha)^2] . \quad (4.2)$$

The number distribution $P_{dn}(m)$ of the displaced number state is one of the quantities associated with this simple state with the most interesting properties. For the displaced number state, the number distribution is defined by

$$P_{dn}(l)=\left|\int_{-\infty}^{\infty}dx\,\Psi_{dn}(x)\Psi_n(x)\right|^2 , \quad (4.3)$$

which is directly analogous to the Franck-Condon overlap in molecular transitions responsible in that context for the oscillations which are such a noticeable feature of vibrational substructure in molecular electronic transitions. Assuming $l\geq n$, it is found that[9,11-13]

$$P_{dn}(l)=\frac{n!}{l!}\alpha^{2(l-n)}e^{-\alpha^2}[\mathcal{L}_n^{(l-n)}(\alpha^2)]^2 , \quad (4.4)$$

where $\mathcal{L}_n^{(l-n)}(x)$ is an associated Laguerre polynomial.[23] From Eq. (4.1), it is readily seen that this number distribution is symmetrical in n and l, i.e., the probability of l photons in state $|\alpha,n\rangle$ is equal to the probability of n photons in state $|\alpha,l\rangle$.

There is a simple relationship between the displaced number state $|\alpha,n\rangle$ and the Hamiltonian of the harmonic oscillator. By applying the displacement operator to the eigenvalue equation

$$\hat{H}_0|n\rangle=(n+\tfrac{1}{2})\hbar\omega|n\rangle ,$$

we obtain

$$\hat{H}_d|\alpha,n\rangle=(n+\tfrac{1}{2})\hbar\omega|\alpha,n\rangle , \quad (4.5)$$

where $\hat{H}_d=\hat{D}(\alpha)\hat{H}_0\hat{D}^\dagger(\alpha)=h\omega(\hat{a}^\dagger-\alpha^*)(\hat{a}-\alpha)+\frac{1}{2}$. The displaced number state is seen to be an eigenstate of the displaced Hamiltonian for the harmonic oscillator.

More generally, for a complex displacement we can use the number states directly rather than the position space wave functions. The photon number distribution of the displaced number state $P_{dn}(l)$ can be obtained from

$$P_{dn}(l)=|\langle l|\alpha,n\rangle|^2=|\langle l|\hat{D}(\alpha)|n\rangle|^2 , \quad (4.6)$$

where now α is a complex number. From this equation, it is seen that this photon distribution is symmetrical in n and l, as in the previous case. We find for the matrix elements of $\hat{D}(\alpha)$, when $l\geq n$,

$$\langle l|\hat{D}(\alpha)|n\rangle=(n!/l!)^{1/2}\alpha^{l-n}e^{-1/2|\alpha|^2} \times\mathcal{L}_n^{(l-n)}(|\alpha|^2) , \quad (4.7)$$

in agreement with earlier results,[9,11,12] so that we obtain explicitly

$$P_{dn}(l)=\frac{e^{-|\alpha|^2}|\alpha|^{2(l-n)}}{n!l!}\times\left|\sum_{k=0}^{n}\frac{n!l!(-1)^k|\alpha|^{2(n-k)}}{k!(n-k)!(l-k)!}\right|^2 . \tag{4.8}$$

For $\alpha=0$, $P_{dn}(l)=\delta_{ln}$, as it expected. In the squared modulus, we have a polynomial in l of degree n, so extending this function for real values of l, one would have eventually n zeros. Therefore it can be expected that $P_{dn}(l)$ have up to n minima between $(n+1)$ maxima. In Figs. 3(a)–3(d), we plot $P_{dn}(l)$ for $\alpha=7$, and $n=1,2,3,10$. We see that there is a strong similarity between these functions and the Hermite polynomials of order n. Let us discuss the case with $n=1$. In this case, it is readily seen that Eq. (4.8) can be rewritten as

$$P_{dn}(l)=(l!)^{-1}e^{-|\alpha|^2}|\alpha|^{2(l-1)}(l-|\alpha|^2)^2 . \tag{4.9}$$

This distribution have a zero at $l=|\alpha|^2$ if $|\alpha|^2$ is an integer. It is interesting to note that adding a coherent amplitude to a one-photon state, the probability of having zero photons becomes nonzero. We see in Fig. 3(a) that there are two peaks, around the minimum, as opposed to a coherent state, where there is only one peak value for the number of photons. For $n=2$, we have a second minimum appearing [Fig. 3(b)], and the number of maxima increases, being four for $n=3$, and eleven for $n=10$ [Figs. 3(c) and 3(d)].

B. Interpretation of the oscillations

We now interpret the oscillations in the photon number distributions. Figure 4 is a pictorial representation of the phase-space contour for number states and displaced number states. The small circle encloses the area representing the displaced vacuum state, i.e., the coherent state. Between this circle and the next one is a phase-space band corresponding to the displaced one-photon state, and so on. The other curves crossing the displaced number state circles are parts of the number-state contours. According to Eq. (4.3), the distribution $P_{dn}(l)$ is given by the overlap between the displaced number state $|\alpha,n\rangle$ and the number state $|0,l\rangle$.[14] As shown in Fig. 4, there is only one overlap between the coherent-state circle and the number-state bands. The central overlap is larger than the side ones. This explains the Poissonian peak of $P_{dn}(l)$ for the coherent state $|\alpha,0\rangle$.

Now, let us turn our attention to the state $|\alpha,1\rangle$, corresponding to the band adjacent to the central circle, in

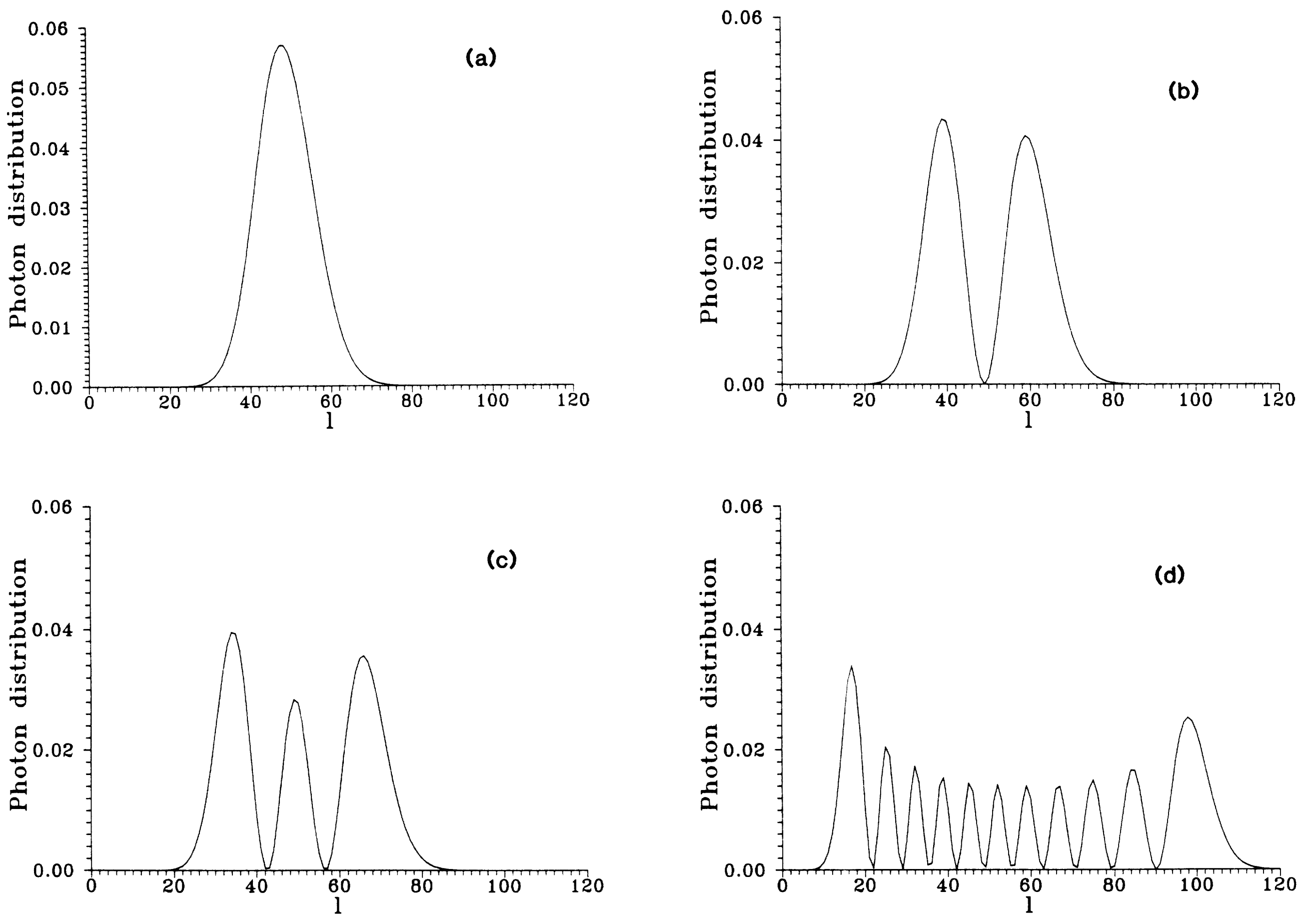

FIG. 3. Photon distribution for the displaced number state $|7,n\rangle$: (a) $n=1$; (b) $n=2$; (c) $n=3$; (d) $n=10$.

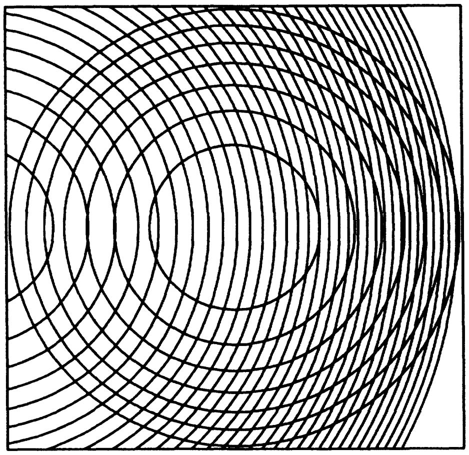

FIG. 4. Pictorial representation of number states and displaced number states in phase space.

Fig. 4. The number distribution is displayed in Fig. 3(b). Now, we have two kinds of overlaps with the number-state bands: single overlaps in the left and right hand sides of the $|\alpha,1\rangle$ band and two overlaps between the upper and the lower parts of the $|\alpha,1\rangle$ band and the number-state bands. For the single overlaps, the photon number distribution is simply proportional to the overlap areas. When there are two overlaps, the photon number distribution is the sum of the overlap areas with appropriate phases. This results in the interference between the contributions from the two overlaps. They constructively or destructively interfere and the photon number distribution gives rise to oscillations. Notice that the single overlaps between the number-state bands and the $|\alpha,1\rangle$ circles on the left-hand side are thicker than the ones on the right-hand side. Because of this, a higher value of $P_{dn}(l)$ is expected for the peak occurring at small values of l. Also, there is a larger number of $|0,l\rangle$ bands crossing $|\alpha,1\rangle$ on the right than on the left, so that the peak for larger l's must be broader than the first peak, occurring for small l's. The minimum between these peaks is explained by the destructive interference between the upper and lower overlaps.

For the other states $|\alpha,n\rangle$, with $n>1$, most of the preceding analysis can be applied with the same conclusions about the position and the widths of the external peaks [see Figs. 3(c) and 3(d)]. The last peak will be broader and lower than the first peak. But because there is a larger number of double overlaps than in the previous case of $|\alpha,1\rangle$, a possibility of constructive interference between the upper and lower overlaps is introduced in the middle part [between the two external peaks of Fig. 3(c), for example], and new peaks and minima are introduced as n increases. Indeed, the central peak of the distribution $P_{dn}(l)$ for $|\alpha,2\rangle$ occurs around the value of l, which is giving the minimum $P_{dn}(l)$ for $|\alpha,1\rangle$ [see Fig. 3(c)]. This alternation of minimum and maximum as n increases is analogous to what happens for the squeezed number states.[19]

The oscillations in $P_{dn}(l)$ can also be understood as the Franck-Condon oscillations occurring in molecular emission of light due to electronic transitions.[20,25] There, the electric dipole is proportional to the Franck-Condon factors, which are essentially the overlaps of the wave functions corresponding to the vibrational sublevels of the two electronic levels of the transition. In our case, we have the two electronic levels being replaced by the two parabolas, one for a harmonic oscillator, the other one for the displaced harmonic oscillator, and the Franck-Condon factors replaced by the overlaps between the number state and the displaced number states. The transitions are more probable at the turning points of the harmonic oscillator. This explains the external peaks of $P_{dn}(l)$.

V. EFFECT OF DISSIPATION ON PHOTON NUMBER DISTRIBUTIONS

Milburn and Walls[15] discussed recently the behavior of the number distribution, when a harmonic oscillator interacts with a vacuum heat bath. This is also a model for the effect of imperfect photodetection[26] discussed by Mollow, and more recently by Srinivas and Davies, and by Mandel. The results of Milburn and Walls describe the possibility of an experimental detection of oscillations in the photon number distribution for the squeezed states. Their argument is very general and uses the Q function to describe the oscillations and their detection. The condition for the existence of oscillations for a squeezed-state number distribution is that the system must be strongly squeezed; the oscillations come from the interference between two quite distant regions in phase space. Because the coherence between these two regions is strongly damped by imperfect detection, this implies that such a detection would be extremely improbable. But we know from the previous section that the oscillations in the photon distribution for the displaced number state come from the interference between two regions which may be very close indeed for some states, and the state $|\alpha,1\rangle$ is an obvious candidate to study such a question.

A. General expression

We can analyze the effect of dissipation on the oscillatory behavior of the photon number distribution,[15,26] using the model developed by Milburn and Walls. The cavity mode is assumed to be decaying to a heat bath at zero temperature, so that using standard techniques the field reduced density operator is shown to have the evolution described by the following equation:

$$\frac{d\hat{\rho}}{dt}=-\frac{i}{\hbar}[\hat{H}_0,\hat{\rho}]+\frac{\gamma}{2}(2\hat{a}\,\hat{\rho}\,\hat{a}^{\dagger}-\hat{a}^{\dagger}\hat{a}\,\hat{\rho}-\hat{\rho}\,\hat{a}^{\dagger}\hat{a}) , \quad (5.1)$$

where $\hat{H}_0$ is the free Hamiltonian and γ is the damping constant. The solution of Eq. (3.1) is given by[15,26]

$$P(l;\mu)=\sum_{k=l}^{\infty}P(k)\begin{bmatrix}k\\l\end{bmatrix}\mu^l(1-\mu)^{k-l} , \quad (5.2)$$

where $\mu=e^{-\gamma t}$. Equation (5.2) describes how the photon distribution $P(l;\mu)$ of one cavity mode decays in time from an initial value $P(l;1)\equiv P(l)$, and is valid in general. As time goes to infinity, μ tends to zero, and we conclude that for any $l>0$, the probability vanishes, and the final state is the vacuum field, with $P(l,0)=\delta_{l0}$. Before analyzing the general expression for $P(l;\mu)$, it is instructive to consider the effect of dissipation on a number state $|n\rangle$ and a coherent state $|\alpha\rangle$.

B. Dissipation of a number state

For an initial number state $|n\rangle$, $P(l,1)=\delta_{ln}$. In this case, we obtain from Eq. (5.2) a result given explicitly by Srinivas and Davies,[26] generalizing the Bernoulli form of the counting distribution[27] to include decay of the field elements,

$$P(l,\mu)=\begin{cases}\begin{bmatrix} n \\ l \end{bmatrix}\mu^l(1-\mu)^{n-l} & (l\leq n)\\ 0 \quad (l>n)\ . \end{cases} \tag{5.3}$$

According to this equation, the probability of finding fewer photons than the initial number becomes nonzero, and simultaneously the probability of finding the initial photon number decreases, as expected, because the cavity is losing energy. For each particular $l<n$, there is a different value of $\mu=\mu_0$ (and consequently of time) in which the associated probability reaches the maximum and then decays to zero, where $\mu_0=l/n$. We see that this value depends also on the initial state $|n\rangle$. It takes more time for the probability of small l's to reach the maximum value than for the probability of large l's. Also, for the same l, the maximum is reached later for an initial state with a large n, than for small initial value of n.

C. Dissipation of a coherent state

For an initial coherent state $|\alpha\rangle$, we obtain

$$P(l,\mu)=e^{-\mu|\alpha|^2}\frac{(\mu|\alpha|^2)^l}{l!}\ . \tag{5.4}$$

The distribution remains Poissonian, but the position of the maximum shifts towards zero.[28]

D. Dissipation of the displaced number state

The initial photon number distribution is given by Eq. (4.8). Let us consider the case $n=1$. In this case, the time evolution of the photon number distribution is given by (after rearranging and redefining the index of the summation)

$$P(l;\mu)=\frac{\mu^l|\alpha|^{2(l-1)}}{l!}\exp(-|\alpha|^2)\times\sum_{k=0}^{\infty}[k^2+2k(l-|\alpha|^2)+(l-|\alpha|^2)^2]\frac{x^k}{k!}\ , \tag{5.5}$$

where $x=(1-\mu)|\alpha|^2$. Using the identity

$$\sum_{k=0}^{\infty}(k^2+2kb+b^2)\frac{x^k}{k!}=[x+(x+b)^2]e^x\ , \tag{5.6}$$

we obtain, finally,

$$P(l;\mu)=\frac{e^{-\mu|\alpha|^2}}{|\alpha|^2}\frac{(\mu|\alpha|^2)^l}{l!}\times[(1-\mu)|\alpha|^2+(l-\mu|\alpha|^2)^2]\ . \tag{5.7}$$

For $t=0$, we have $\mu=1$, and Eq. (5.7) reduces correctly to Eq. (4.9). For $t\to\infty$, we correctly obtain a unity probability of the vacuum state. In Fig. 5, we show the photon distribution $P(l;\mu)$ for the initial displaced number state $|7,1\rangle$ for $\mu=0.9,0.7,0.5,0.3$. We see that even for a dissipation, or a quantum efficiency, $\mu=0.5$, the two peaks are still resolved, as we anticipated above.

VI. CONCLUSIONS

We have studied several properties of the displaced number states of the harmonic oscillator. We obtained the Q and Wigner functions and introduced a general argument which allowed us to relate these functions with those distribution functions for the number and the squeezed number states. This argument shows that if two states are related by a squeezing (Bogoliubov) transformation, then only the Wigner functions of the two states are related by a contraction, this being not true for all the other possible quasiprobability functions, including the Q function and the P function. The analysis for the photon number distribution showed that this function exhibits oscillations in the photon number distribution, as expected from a general argument based on interference in phase space. This argument was further developed in a qualitative way, and several properties of the photon number distribution were explained by an inspection of phase-space contours. We investigated the possibility of detection of the oscillations of the photon distribution of the displaced number state, by using a model of detection as a damping mechanism, and our conclusion is that even

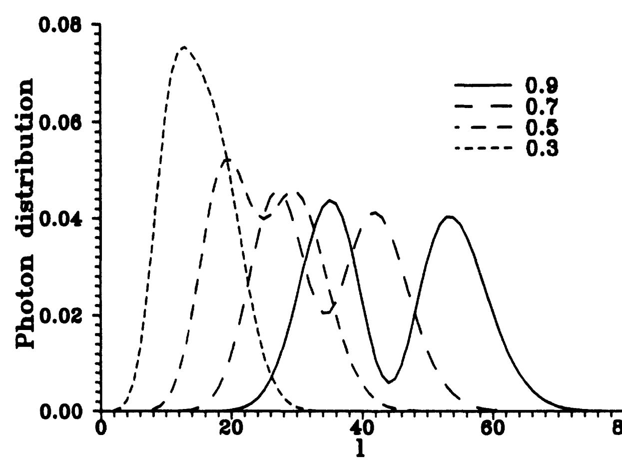

FIG. 5. Effect of dissipation, or, equivalently, of nonunity quantum efficiency μ for the detection of the oscillations of the photon distribution for the displaced number state $|7,1\rangle$. We plot the photon distribution corresponding to $\mu=0.9,0.7,0.5,0.3$. The oscillations are still resolved for $\mu=0.5$.

with a strongly imperfect detection system the oscillation in the photon number distribution of the displaced number state could be experimentally demonstrated. This is rather encouraging, recalling that recently a one-photon state has been produced in the laboratory. Adding a coherent contribution to the amplitude of this state should, in principle, pose no insurmountable practical barrier. This experiment would also be of general importance in demonstrating that the coherent component added to the amplitude of a one-photon state is capable of making this state macroscopically different from its neighbors, which differ from it only microscopically, such as the displaced vacuum (coherent state, without oscillations) and the three-peaked two-photon displaced state (which should not be confused with the two-photon coherent state, better known as the squeezed coherent state).

We have recently become aware of the work of Král,[29] who, using a different approach to problems related to those discussed here, has also found interesting nonclassical properties of the displaced number state.

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