Phase transitions in a description of multifractals generated by a logistic map

Piotr Cieśliński and Marek Napiórkowski
Institute of Theoretical Physics, Warsaw University, Hoża 69, 00-681 Warszawa, Poland
(Received 29 August 1989)

A logistic map generates multifractal sets in its periodic windows. Using the thermodynamic formalism of Halsey *et al.* [Phys. Rev. A **33**, 1141 (1986)], we present two methods of calculating their generalized dimensions D_q, and we show that there is a phase transition in a period-6 window, i.e., D_q is a nonanalytic function of q.

I. INTRODUCTION

Inhomogeneous fractals appear in a variety of physical problems, and standard examples consist of fractal growth processes[1] and turbulence;[2] chaotic dynamic systems also generate multifractal measures.[3–10] The fractal measures can be characterized by the spectrum of generalized dimensions D_q (Ref. 11) and singularity dimensions $f(\alpha)$.[3] Halsey *et al.*[3] introduced a convenient method of calculating functions D_q and $f(\alpha)$ [$f(\alpha)$ is the Legendre transform of D_q]. This approach, which generalizes Hausdorff construction, is called thermodynamic formalism because it is based on calculating a certain "partition function." D_q and $f(\alpha)$ are interpreted as thermodynamic functions[4–9] whereas nonanlyticities of D_q are called phase transitions.[7–10]

In this note we analyze multifractals generated by the logistic map

$$f_\mu(x)=1-\mu x^2,\quad x\in I,\quad \mu\in[0,2] \tag{1}$$

in its periodic windows.[12] We introduce a convenient method of calculating the spectrum of generalized dimensions D_q characterizing these multifractals. This method makes use of the thermodynamic formalism and, in particular, we find the existence of a phase transition in one of the period-6 windows. We also indicate the mathematical mechanism of this phase transition.

II. THE MULTIFRACTAL SETS AND THEIR PARTITION FUNCTIONS

Assume that the nonlinearity parameter μ belongs to a certain period-p window. Then the p-fold iterate of the logistic map

$$f_\mu^{[p]}(x)=f_\mu\cdot f_\mu^{[p-1]}(x),\quad f_\mu^{[0]}(x)\equiv x,\quad p\in\mathbb{N} \tag{2}$$

has p distinct attractors.[13,14] Depending on the μ value they can be fixed points, cycles, Feigenbaum attractors, or a set of intervals. Our multifractal set consists of the boundary between the basins of attraction of these attractors. It can be constructed by a reccurrence procedure in the following way.

To each of the p attractors there corresponds an interval Δ_j, $j=1,\ldots,p$, called the primary basin of attraction.[14] Δ_j is the largest open interval containing only one local extremum of $f_\mu^{[p]}(x)$ and is mapped to itself under the p-fold iterate (Fig. 1). The interval I in (1) (see also Fig. 1) is the smallest interval containing all primary basins of attraction for given value of μ. Between the intervals Δ_j there are $p-1$ closed intervals I_k, $k=1,\ldots,p-1$. The points belonging to the given interval Δ_j as well as to all its preimages under $f_\mu^{[p]}(x)$ form the basin of attraction of the jth attractor. The analyzed boundary, which is denoted by B, is contained in the intervals I_k, $k=1,\ldots,p-1$.

Let us now consider the hierarchy of sets of intervals,

$$\begin{aligned} B_0&=\{I_k,k=1,\ldots,p-1\}\ ,\\ B_n&=\{\text{closed intervals } l_i^{(n)}\subset I;\ f_\mu^{[np]}(l_i^{(n)})=I_k,\ k=1,\ldots,p-1\} \end{aligned} \tag{3}$$

or shortly $B_n=f_\mu^{[-p]}(B_{n-1})$. Then $B=\lim_{n\to\infty}B_n$.

Now we apply the thermodynamic formalism to calculate the generalized dimensions of set B. In this formalism, at the nth step of construction, we cover the analyzed multifractal by N_n open balls of diameter l_i and calculate the partition function[3,4]

$$\Gamma_n(q,\tau)=\sum_{i=1}^{N_n}\frac{p_i^q}{l_i^\tau}\ , \tag{4}$$

where p_i is the measure of the ith ball. Next, to calculate the generalized dimensions D_q we use the condition[3]

$$\lim_{n\to\infty}\Gamma_n(q,\tau)=\Gamma(q,\tau)=1\ , \tag{5}$$

which is fulfilled at a certain value $\tau(q)$ and then we obtain $D_q=\tau(q)/(q-1)$. The Legendre transform of $\tau(q)$ gives the singularity dimensions $f(\alpha)$. In order to calculate the partition function $\Gamma_n(q,\tau)$ (4) we use the intervals

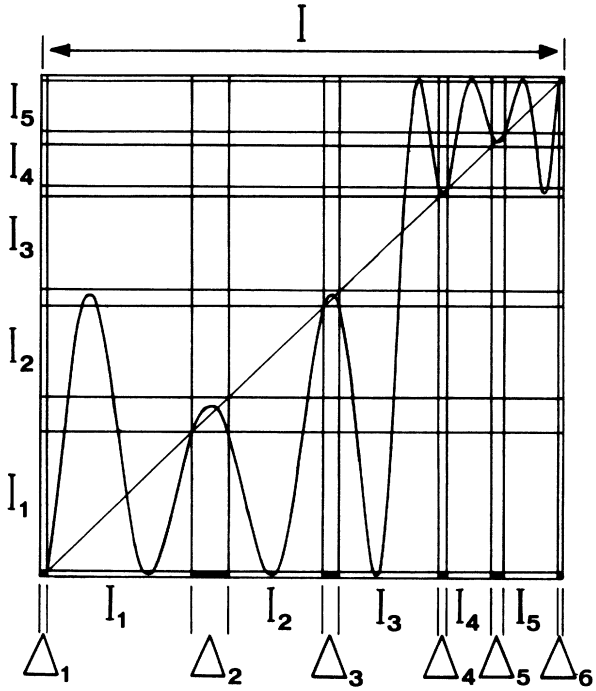

FIG. 1. The sixth iterate $f_\mu^{[6]}(x)$ for $\mu=1.480$. Within the interval I there are six primary basins of attraction Δ_j, $j=1,\ldots,6$ and five intervals I_k, $k=1,\ldots,5$.

$l_i^{(n)}\in B_n$ (at the nth step of construction) and to each of them we ascribe the same measure $p_i=1/N_n$ as none of the intervals $l_i^{(n)}$ is distinquished by the construction.[15] After inserting these data into condition (5) we obtain in the limit of large n the following implicit equation for $\tau(q)$:

$$N_n^q=\sum_{i=1}^{N_n}|l_i^{(n)}|^{-\tau}\,. \tag{6}$$

III. APPROXIMATE METHODS AND RESULTS

To present the main steps of our analysis we choose the period-6 window which appears out of chaos at $\mu=1.474\,695\,378$. We shall use two different methods to calculate the function $\tau(q)$ from Eq. (6). In the first method—called the mean-slope approximation—to each monotonic branch of $f_\mu^{[6]}(x)$ in a given box $I_k\times I_{k'}$ (Fig. 2) we attribute its mean slope denoted by s_i, $i=1,\ldots,25$. Now let the intervals $l_i^{(n)}$ cover the boundary B at the nth step of its construction. Using the fact that the analyzed set is contained in the separate intervals I_k, $k=1,\ldots,5$, we can write the right-hand side of Eq. (6) in the following form:

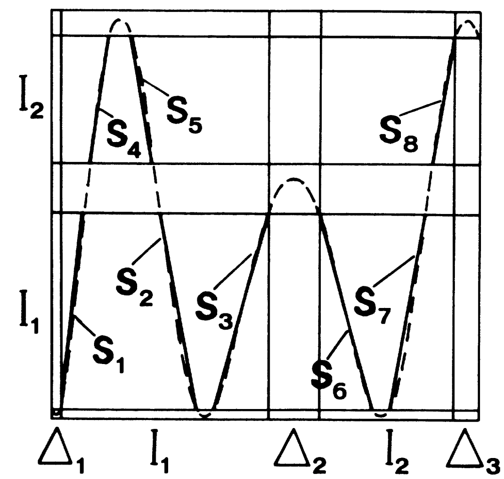

FIG. 2. Fragment of Fig. 1 in which monotonic branches of $f_\mu^{[6]}(x)$ (dashed line) are replaced by straight lines (solid line) according to the mean-slope approximation.

$$S=\sum_{i=1}^{N_n}|l_i^{(n)}|^{-\tau}=\sum_{i=1}^{5}W_{(n)}^k\,, \tag{7}$$

where

$$W_{(n)}^k=\sum_{i=1}^{N_{(n)}^k}|l_i^{(n)}|^{-\tau}$$

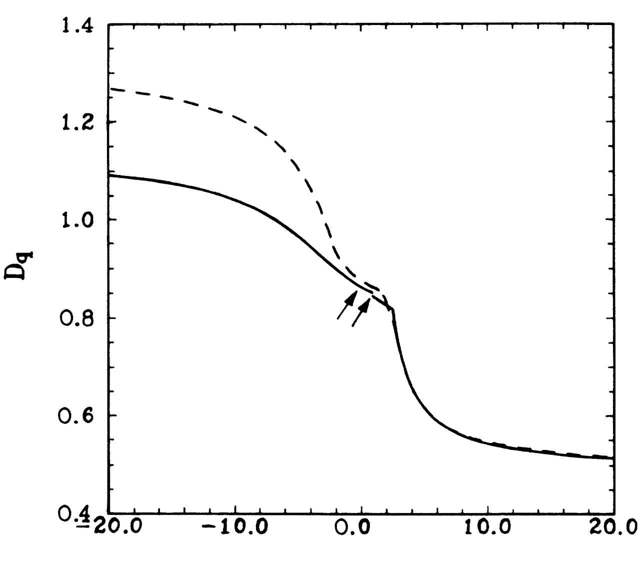

FIG. 3. Generalized dimensions D_q obtained from the mean-slope approximation (solid line) and direct evaluation (dashed line). The direct evaluation method indicates the existence of one point of nonanalyticity at $q\approx2.3$. The mean-slope approximation gives three points of nonanalyticity at $q=0.187$, 1.0, and 2.475; two of them, which are not clearly seen at this scale, are indicated by arrows.

represents the sum over $N^k_{(n)}$ intervals $l_i^{(n)} \in B_n$ located in I_k. At the $(n+1)$ step of construction the covering of the set B consists of preimages of the intervals $l_i^{(n)}$ under the map $f_\mu^{[6]}(x)$. Approximating each branch of the function $f_\mu^{[6]}(x)$ in boxes $I_k \times I_{k'}$ by straight lines with slopes s_i (Fig. 2) we are able to represent each sum $W^k_{(n+1)}$ as a linear combination of $W^k_{(n)}$.

For example,

$$\begin{aligned} W^1_{(n+1)} &= \sum_{i=1}^{N^1_{(n+1)}} |l_i^{(n+1)}|^{-\tau} \\ &= \sum_{i=1}^{N^1_{(n)}} \left[\left| \frac{l_i^{(n)}}{s_1} \right|^{-\tau} + \left| \frac{l_i^{(n)}}{s_2} \right|^{-\tau} + \left| \frac{l_i^{(n)}}{s_3} \right|^{-\tau} \right] + \sum_{i=1}^{N^2_{(n)}} \left[\left| \frac{l_i^{(n)}}{s_4} \right|^{-\tau} + \left| \frac{l_i^{(n)}}{s_5} \right|^{-\tau} \right] \\ &= (s_1^\tau + s_2^\tau + s_3^\tau) W^1_{(n)} + (s_4^\tau + s_5^\tau) W^2_{(n)} \ . \end{aligned} \tag{8}$$

After introducing the auxiliary vector $\mathbf{W}_n = (W^k_{(n)})$, $k=1,\ldots,5$, we obtain the recurrence formula

$$\mathbf{W}_{n+1} = T \cdot \mathbf{W}_n \ , \tag{9}$$

where $T=(t_{kk'})$, $k,k'=1,\ldots,5$ is the transfer matrix, e.g., $t_{11}=s_1^\tau+s_2^\tau+s_3^\tau$, $t_{12}=s_4^\tau+s_5^\tau$, etc. From Eq. (9) one gets $\mathbf{W}_n = T^n \cdot \mathbf{W}_0$. Therefore the right-hand side of Eq. (7) will asymptotically grow proportionally to the nth power of the largest eigenvalue $\delta(\tau)$ of the transfer matrix T,

$$S \propto |\delta(\tau)|^n \ . \tag{10}$$

The number of intervals N_n covering the boundary B at the nth setp of construction grows with n according to the asymptotic law[14]

$$N_n \propto |\lambda|^n \ , \tag{11}$$

where λ is the largest eigenvalue of the kneading matrix.[14] Thus asymptotically Eq. (6) reduces to the following implicit expression for the function $\tau(q)$:

$$|\lambda|^q = |\delta(\tau)| \ . \tag{12}$$

The eigenvalues of the transfer matrix T are calculated from the equation

$$(t_{33}-\delta)[(t_{11}-\delta)(t_{22}-\delta)-t_{12}t_{21}] \times [(t_{44}-\delta)(t_{55}-\delta)-t_{45}t_{54}] = 0 \ .$$

Hence $\delta(\tau)=\max\{\delta_1(\tau),\delta_2(\tau),\delta_3(\tau)\}$ where

$$\begin{aligned} \delta_1(\tau) &= \tfrac{1}{2}[t_{11}+t_{22}+(t_{11}^2+t_{22}^2+4t_{21}t_{12}-2t_{11}t_{22})^{1/2}] \ , \\ \delta_2(\tau) &= \tfrac{1}{2}[t_{44}+t_{55}+(t_{44}^2+t_{55}^2+4t_{54}t_{45}-2t_{44}t_{55})^{1/2}] \ , \\ \delta_3(\tau) &= t_{33} \ . \end{aligned} \tag{13}$$

It turns out that a phase transition (in the above described sense) takes place when the largest eigenvalue $\delta(\tau)$ of T becomes degenerate. This happens at certain values of parameter τ. Numerical analysis of Eq. (13) showed that

$$\delta(\tau) = \begin{cases} \delta_2, & -\infty < \tau \le -0.700 \\ \delta_1, & -0.700 < \tau \le 0.000 \\ \delta_2, & 0.000 < \tau \le 1.210 \\ \delta_3, & 1.210 < \tau \le \infty \ . \end{cases} \tag{14}$$

$\delta(\tau)$ is not differentiable at these three points. Hence $q(\tau)=\ln|\delta(\tau)|/\ln|\lambda|$ and $D_q=\tau(q)/(q-1)$ are nonanalytic at these points (Fig. 3).

In the second method we do not use the mean-slope approximation and we make the direct evaluation of the sum s in Eq. (7). For large values of n one has

$$\frac{\Gamma_{n+1}(q,\tau)}{\Gamma_n(q,\tau)} = 1 \tag{15}$$

which can be rewritten as

$$|\lambda|^q = \frac{\sum_{i=1}^{N_{n+1}} |l_i^{(n+1)}|^{-\tau}}{\sum_{i=1}^{N_n} |l_i^{(n)}|^{-\tau}} \ . \tag{16}$$

Equation (16) was analyzed numerically for increasing values of n (up to $n=6$) and the input parameters $l_i^{(n)}$ were calculated directly, without using the mean-slope approximation. The function D_q obtained in this way is presented in Fig. 3.

IV. DISCUSSION

The generalized dimensions D_q calculated by both methods, i.e., in the mean-slope approximation and via direct evaluation, show nonanalytic behavior. In the mean-slope approximation we obtain three points of nonanalyticity while the direct evaluation indicates only one point of nonanalyticity. In order to determine whether the existence of the remaining two points is confirmed or not in the direct evaluation one should carry on the calculations for much higher values of n which unfortunately is beyond our numerical possibilities.

The mathematical mechanism responsible for this nonanalytic behavior is, at least in the mean-slope approximation, quite transparent: similarly to phase transitions in physical model systems[16] it corresponds to degeneracy of the largest eigenvalue of a certain transfer matrix. This observation tells us that perhaps the presently analyzed nonanalytic behavior of D_q could be understood in terms of certain lattice spin models[5–8] into which our calculations could be translated.

Finally let us add that a similar analysis was also performed for period-3 and -4 windows and nonanalyticities (in any of the methods) were not observed.

ACKNOWLEDGMENTS

The work of one of us (M.N.) was supported by the Polish Ministry of National Education.

[1]T. C. Halsey, P. Meakin, and I. Procaccia, Phys. Rev. Lett. **56**, 854 (1986); I. Procaccia and R. Zeitak, *ibid.* **60**, 2511 (1988); T. Bohr, P. Cvitanović, and M. H. Jensen, Europhys. Lett. **6**, 445 (1988).

[2]B. B. Mandelbrot, J. Fluid. Mech. **62**, 331 (1974); I. Procaccia, J. Stat. Phys. **36**, 649 (1984).

[3]T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986).

[4]M. Kohmoto, Phys. Rev. A **37**, 1345 (1988).

[5]M. J. Feigenbaum, M. H. Jensen, and I. Procaccia, Phys. Rev. Lett. **57**, 1503 (1986).

[6]M. H. Jensen, L. P. Kadanoff, and I. Procaccia, Phys. Rev. A **36**, 1409 (1987).

[7]D.Katzen and I. Procaccia, Phys. Rev. Lett. **58**, 1169 (1987).

[8]T. Bohr and M. H. Jensen, Phys. Rev. A **36**, 4904 (1987).

[9]E. Ott, C. Grebogi, and J. A. Yorke, Phys. Lett. A **135**, 343 (1989).

[10]G. H. Gunaratne and I. Procaccia, Phys. Rev. Lett. **59**, 1377 (1987).

[11]H. G. E. Hentschel and I. Procaccia, Physica D **8**, 435 (1983).

[12]P. Collet and J.-P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems* (Birkhäuser, Boston, 1980).

[13]S. Thomae and S. Grossmann, J. Stat. Phys. **26**, 485 (1981).

[14]M. Napiórkowski and S. Thomae, Europhys. Lett. **4**, 1247 (1986).

[15]T. Tel, Institute for Theoretical Physics, Budapest, Report No. 462, 1987 (unpublished).

[16]M. Kac, *Mathematical Mechanisms of Phase Transitions,* (Gordon & Breach, New York, 1966); C. J. Thompson, J. Math. Phys. **9**, 1059 (1968).