Modulation instability of obliquely modulated ion-acoustic waves in an unmagnetized plasma

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The stability of oblique modulation of ion-acoustic waves in an unmagnetized plasma due to nonlinear interaction with a slow-response quasistatic plasma is studied. A nonlinear Schrödinger equation is derived for the system. It is found that there exists a wide domain in the k-θ plane in which the ion-acoustic waves would be modulationally unstable. It is also found that when $\theta > 60°$, the waves are unstable for all values of k. However, for $\theta < 60°$, the waves are unstable only for $0 < k < k_{\max}$, where $k_{\max}$ is given by $\tan\theta/(3-\tan^2\theta)^{1/2}$.

The modulational instability of ion-acoustic waves in a dispersive and weakly nonlinear plasma has been a topic of significant interest.[1–11] Accounting for harmonic-generation nonlinearities, several authors[1–8] derived a nonlinear Schrödinger equation that governs the dynamics of nonlinear ion-acoustic wave packets for different types of plasmas with parallel and oblique modulation.

Recently, nonlinear parallel modulation of ion-acoustic waves in an unmagnetized plasma due to nonlinear interaction with a slow-response quasistatic plasma was studied by Shukla.[9] Then the case of a magnetized plasma was discussed by Bharuthram and Shukla[10] and Bharuthram.[11] In the present paper we studied the modulational instability of obliquely modulated ion-acoustic waves in an unmagnetized plasma due to the nonlinear interaction with a slow-response quasistatic plasma. We derived the nonlinear Schrödinger equation for the system. It is found that there exists a wide domain in the k-θ plane in which the ion-acoustic waves would be modulationally unstable in contrast to the case of parallel modulation where the waves remain modulationally stable.[9] The results of our investigation reduce to those obtained by Shukla[9] in the limit of parallel modulation.

We consider an obliquely modulated ion-acoustic wave traveling in the x-y plane in a collisionless unmagnetized plasma. We further assume that the modulated amplitude of the ion-acoustic wave varies in the x direction. The nonlinear interaction of finite-amplitude ion-acoustic waves with an unmagnetized background plasma is governed by the following set of normalized equations:

$$\frac{\partial n_i}{\partial t}+\nabla\cdot(n_i\mathbf{V}_i)=0 , \tag{1}$$

$$\frac{\partial \mathbf{V}_i}{\partial t}+(\mathbf{V}_i\cdot\nabla)\mathbf{V}_i=-\nabla\phi-\left[\frac{T_i}{T_e}\right]\frac{1}{n_i}\nabla n_i^{\gamma} , \tag{2}$$

$$\nabla\phi=\frac{1}{n_e}\nabla n_e , \tag{3}$$

$$\nabla^2\phi=n_e-n_i , \tag{4}$$

where $\mathbf{V}_i=(V_{ix},V_{iy},0)$, $\nabla=(\partial/\partial x,\partial/\partial y,0)$. n_i, n_e, $\mathbf{V}_i$, and ϕ are, respectively, the normalized ion density, electron density, ion-fluid velocity, and electrostatic potential. T_i and T_e are, respectively, the ion temperature and electron temperature. Finally $\gamma=2$ for the adiabatic ions and $\gamma=1$ for the isothermal ions. In Eq. (3) we have neglected the electron inertia and assumed that the electrons are isothermal. In the above equations the densities are normalized with respect to the unperturbed plasma density n_0, the velocity with respect to ion-acoustic velocity, $C_s=(T_e/m_i)^{1/2}$, the electrostatic potential with respect to the electron thermal potential, (T_e/e), the length with respect to Debye length λ_D, and time with respect to the inverse of the ion plasma frequency, ω_{pi}^{-1}.

As we are interested in investigating the slow response of the quasistatic plasma to the ion-acoustic waves, we write the field quantities in normalized form as follows:

$$n_j=1+n_j^h+n_j^l , \tag{5}$$

$$\mathbf{V}_j=\mathbf{V}_j^h+\mathbf{V}_j^l , \tag{6}$$

$$\phi=\phi^h+\phi^l , \tag{7}$$

where $n_j^{h(l)} \ll 1$. The superscripts h and l represent the corresponding quantities associated with the ion wave and with the quasistatic plasma slow motion, respectively.

Now taking Eq. (3) and using Eqs. (5)–(7) the electron density perturbation associated with the ion-acoustic waves in the presence of the plasma slow motion is given by

$$n_e^h=(1+n_e^l)\phi^h . \tag{8}$$

Combining Eqs. (1) and (2), introducing Eqs. (4)–(7), we obtain a nonlinear equation for the ion-acoustic wave in the presence of the plasma slow response

$$\left[\left[1-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right]\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right]\phi^h - \left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right]n_e^l\phi^h=0 . \tag{9}$$

In deriving Eq. (9), the ions are assumed to be much colder than the electrons, i.e., $T_i/T_e \ll 1$, the plasma slow response is assumed to be quasineutral and quasistatic, i.e., $n_i^l = n_e^l$, and $\mathbf{V}_i^l = \mathbf{V}_e^l \approx 0$. In the absence of nonlinear interaction, linearization of (9) yields the following dispersion relation:

$$\omega^2 = \frac{k^2}{1+k^2} , \tag{10}$$

where $k^2 = k_x^2 + k_y^2$, k_x and k_y being the x and y components of the wave vector $\mathbf{k}$ of the ion-acoustic wave. The modulation group velocity (i.e., the velocity with which the modulation propagates) of the wave is given by

$$V_g = \frac{\partial \omega}{\partial k_x} = \frac{\omega^3}{k^3}\cos\theta = \frac{\partial \omega}{\partial k}\cos\theta , \tag{11}$$

which is the component of the group velocity $(\partial\omega/\partial k)$ along the direction of modulation, where θ is the angle between the wave vector of the ion-acoustic wave and the x axis, the direction in which the modulation of the wave amplitude propagates.

Now we calculate the electron density perturbation n_e^l associated with the quasistatic plasma slow motion. When the phase velocity of the modulation is much smaller than the electron and ion thermal velocities, then taking the x component of the momentum balance equations, using Eqs. (5)–(7) and averaging over the ion-acoustic wave periods, we get

$$\frac{1}{2}\frac{\partial}{\partial x}\langle V_{ix}^{h\,2} \rangle = -\frac{\partial \phi^1}{\partial x} - \gamma \left[\frac{T_i}{T_e} \right] \frac{\partial n_i^l}{\partial x} \tag{12}$$

and

$$\frac{\partial \phi^1}{\partial x} = \frac{\partial n_e^l}{\partial x} , \tag{13}$$

where the angular brackets denote averaging over the ion-acoustic wave period. The left-hand side of Eq. (12) represents the ion ponderomotive force. Now from the x component of the ion momentum equation, we get

$$V_{ix}^h = \frac{k_x}{k}(1+k^2)^{1/2}\phi^h . \tag{14}$$

Substituting the value of V_{ix}^h from Eq. (14) in Eq. (12), and eliminating ϕ^l assuming $n_i^l = n_e^l$, we get

$$n_e^l = -\frac{1+k^2}{2(1+\gamma\sigma)}(\cos^2\theta)|\phi^h|^2 , \tag{15}$$

where $\sigma = T_i/T_e$ is the ratio of the ion to electron temperatures.

Substituting Eq. (15) in Eq. (9), we get

$$\left[\left[1 - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right] \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right] \phi^h - \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right] \frac{(1+k^2)}{2(1+\gamma\sigma)}(\cos^2\theta)|\phi^h|^2\phi^h = 0 . \tag{16}$$

Assuming that the nonlinear interaction of slow-response quasistatic plasma with ion-acoustic waves gives rise to an envelope of a wave whose amplitude varies on time and space scales much more slowly than those of ion-acoustic oscillations, we let

$$\phi^h = \epsilon^{1/2}\phi^h(\xi,\tau)\exp(-i\omega t + ik_x x + ik_y y) + \text{c.c.} , \tag{17}$$

where ϵ indicates the magnitude of small but finite amplitude ϕ^h and ξ and τ are defined such that

$$\xi = \epsilon^{1/2}(x - V_g t) \tag{18a}$$

and

$$\tau = \epsilon t . \tag{18b}$$

Substituting Eqs. (17) and (18) in Eq. (16), using Eqs. (10) and (11), we get to $O(\epsilon^{3/2})$

$$i\frac{\partial \phi^h}{\partial \tau} + \frac{\omega^3}{2k^4}[1-(1+3\omega^2)\cos^2\theta]\frac{\partial^2\phi^h}{\partial \xi^2} + \frac{\omega k^2}{4(1+\gamma\sigma)}(\cos^2\theta|\phi^h|^2\phi^h = 0$$

or

$$i\frac{\partial \phi^h}{\partial \tau} + P\frac{\partial^2\phi^h}{\partial \xi^2} + Q|\phi^h|^2\phi^h = 0 , \tag{19}$$

which is the nonlinear Schrödinger equation. In Eq. (19), the P and Q are the dispersive and nonlinear terms, respectively. The dispersion term $2P$ is the component of the modulation group velocity dispersion $(\partial\omega/\partial k)$ along the direction of modulation

$$P = \frac{1}{2}\frac{\partial V_g}{\partial k}\cos\theta$$
$$= \frac{1}{2}\frac{\partial V_g}{\partial k_x} = \frac{1}{2}\frac{\partial^2\omega}{\partial k_x^2} = \frac{1}{2}\left[(\cos^2\theta)\frac{\partial^2\omega}{\partial k^2} + (\sin^2\theta)\frac{1}{k}\frac{\partial\omega}{\partial k} \right] \tag{20a}$$

or

$$P = \frac{1}{2}\frac{\omega^3}{k^4}[1-(1+3\omega^2)\cos^2\theta] \tag{20b}$$

and

$$Q = \frac{\omega k^2}{4(1+\gamma\sigma)}\cos^2\theta . \tag{21}$$

From the above expressions we note that the coefficient P of the dispersive term is the same as that of Kako and Hasegawa,[3] while the coefficient Q of the nonlinear term differs from that of Kako and Hasegawa, where the authors included only the second-harmonic nonlinearities in the study of ion-acoustic wave modulation. Our expressions for P and Q (for $\gamma = 1$) reduce to those obtained by Shukla[9] in the limit of parallel modulation, i.e., $\theta = 0$.

The amplitude of an obliquely modulated ion-acoustic wave, defined by the nonlinear Schrödinger equation (19), will be modulationally unstable or stable according as $PQ > 0$ or $PQ < 0$.[12]

For parallel modulation the coefficient P is always negative irrespective of the wave number. Similarly the coefficient Q is always positive irrespective of the wave number, which implies that PQ is always less than zero, i.e., $PQ < 0$. Therefore the ion-acoustic waves remain modulationally stable in the case of parallel modulation.

In the case of oblique modulation, Eq. (20a) shows that P consists of two parts; one of them is proportional to $\cos^2\theta$ and depends upon the group velocity dispersion $(\partial^2\omega/\partial k^2)$. The other one is proportional to $\sin^2\theta$ and depends upon the group velocity [i.e., on $(1/k)(\partial\omega/\partial k)$]. Now

$$\frac{\partial^2\omega}{\partial k^2} = -\frac{3k}{(1+k^2)^{5/2}} \tag{22}$$

and

$$\frac{1}{k}\frac{\partial\omega}{\partial k} = \frac{1}{k}\frac{1}{(1+k^2)^{3/2}} \;. \tag{23}$$

Thus the term proportional to $\cos^2\theta$ is negative for all values of k, whereas the term proportional to $\sin^2\theta$ is always positive. The ratio of these two terms is $3k^2/(1+k^2)$, which increases monotonically from 0 to 3 as k increases from 0 to ∞. Hence the value of P will be positive for lower values of k and negative for higher values of k when $\tan^2\theta < 3$, i.e., $\theta < 60°$. However, when θ is greater than 60°, i.e., $\tan^2\theta > 3$, the value of P becomes always positive for all values of k. Physically one can say that the contribution to P for $\theta > 60°$, refer to Eq. (20a), from the group velocity dispersion is smaller than the contribution from the group velocity term for all values of k. While for $\theta < 60°$, the contribution to P from the group velocity dispersion and the group velocity term are such that for small values of k, the first contribution is smaller than the second but for the larger values of k, the reverse is true. For a given value of k, P changes its sign from negative to positive when θ passes through a value

$$\cos^{-1}\left[\frac{1+4k^2}{1+k^2}\right]^{-1/2} .$$

We have plotted the $P=0$ curve on a polar graph, in Fig. 1, which separates the regions of positive and negative values of P. However, in the oblique case also, the value of Q remains positive throughout the whole region. Therefore for values of k-θ in the domain lying below the curve $P=0$, the wave will be stable. And for values of k-θ in the domain lying above the curve $P=0$, the wave will be unstable. This is shown in Fig. 1. It is obvious from Fig. 1 that for $\theta > 60°$, the wave is unstable for all values of k. However, for $\theta < 60°$, the wave in unstable only for $0 < k < k_{\max}$, where $k_{\max}$ is given by $\tan\theta/(3-\tan^2\theta)^{1/2}$.

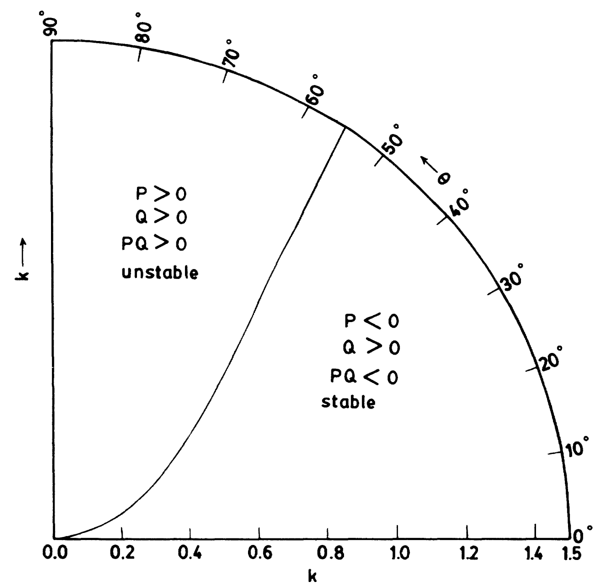

FIG. 1. The plot of $P=0$ in the k-θ plane. Q is always positive. The upper domain represents the modulationally unstable domain for the ion-acoustic wave.

A comparison with the Kako and Hasegawa[3] work in which they have considered the second-harmonic generation nonlinearities, shows that an instability exists in both cases of oblique modulation. But the instability region comes out to be different in the two cases. The change of the instability region is due to the consideration of different types of nonlinearities.

To summarize, we have investigated the modulational instability of obliquely modulated ion-acoustic waves in an unmagnetized plasma due to nonlinear interaction with a slow-response quasistatic plasma. It is found that there exits a wide domain in the k-θ plane in which ion-acoustic waves would be modulationally unstable. It is also found that when $\theta > 60°$, the waves are unstable for all values of k, whereas for $\theta < 60°$, the waves are unstable only for $0 < k < k_{\max}$, where $k_{\max}$ is given by $\tan\theta/(3-\tan^2\theta)^{1/2}$.

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