Stability of temporally periodic states of classical many-body systems

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We study the stability of spatially coherent, time-periodic states in noisy, classical, discrete-time, many-body systems with short-range interactions. Generic stability of periodic k cycles with $k > 2$ can be achieved only by rules carefully constructed to exploit lattice anisotropy and so suppress droplet growth. For ordinary rules which do not utilize spatial anisotropy in this way, periodic k cycles with periods $k > 2$ are metastable rather than stable under generic conditions, losing spatial coherence through nucleation and growth of droplets. The unusual dynamical properties of the periodic states stabilized by anisotropy are described.

Physical systems not driven by time-dependent external forces are typically described by dynamical equations invariant under time translation. However, dissipative many-body systems often break this time-translation symmetry spontaneously, their macroscopic properties varying periodically or even chaotically in time. The temporal oscillations of certain spatially averaged quantities (e.g., temperature fluctuations) in small-aspect-ratio Bénard convection cells are a familiar example.[1] Theoretical treatment of such broken-symmetry, macroscopically time-dependent states has been limited largely to mean-field-like approximations. In these schemes, only a finite (typically small) number of Fourier modes of the many-body system are retained, rendering the problem tractable, either analytically or numerically. Periodic or chaotic time variation of the Fourier amplitudes retained is a common result of this truncation (the Lorenz equations,[2] a three-mode approximant to convection, being an example), and is typically taken to imply the existence of stable nonstationary states in the full untruncated model.[3]

In a previous paper,[4] several of us studied the stability, with respect to fluctuations, of the macroscopically chaotic states predicted by mean-field theories. We argued that these fluctuations can restrict the growth of spatial correlations and so prevent the amplitudes of extended (e.g., Fourier) modes from varying chaotically in the thermodynamic limit, i.e., that they can destabilize the macroscopically chaotic states found in mean-field approximations. In this paper, we investigate the stability of *temporally periodic* states of many-body systems with respect to the nucleation and growth of droplet fluctuations. We restrict ourselves to discrete-time, synchronously updated, noisy systems with short-range interactions (e.g., cellular automata[5(a)] (CA) and coupled maps[4,5(a)]). Such models are often used[5(b)] as prototypes for the analysis of spatially extended systems such as chemical reaction-diffusion experiments. Our main results follow. (i) Periodic k-cycle states with periods k greater than 2 are typically *metastable* rather than stable under generic conditions; they can be stable only in the presence of special symmetries which force droplets to shrink under surface tension. (ii) There do exist dynamical rules which exploit lattice anisotropy to suppress droplet fluctuations and so stabilize periodic states of arbitrarily high period under *generic* conditions. (iii) Periodic states stabilized by anisotropy have unusual dynamical properties (e.g., in two dimensions (2D), autocorrelations that decay with anomalous rapidity), which may serve to distinguish them from long-lived metastable states.

Let us first discuss the main impediment to producing generically stable, temporally periodic states which spontaneously break a discrete time-translation invariance. Consider for concreteness a simple three-state CA, i.e., a regular lattice in d-space dimensions, on each site i on which sits a discrete variable $\theta_i(n)$ that takes on one of three values, 0, 1, or 2, say, and evolves in discrete-time steps n, according to some local stochastic rule; $\theta_i(n+1)$ depends probabilistically on the variables $\theta_j(n)$ on sites j in some neighborhood of i. Suppose one has constructed a CA rule which exhibits a stable 3-cycle. We imagine for convenience that the rule favors spatially uniform states, so that the spatial average M of all the variables θ_i assumes, in the thermodynamic limit, three distinct values, M_0, M_1, and M_2, in regular periodic fashion at

consecutive time steps. Because the rule is noisy, most, rather than all, of the variables take on a common value at any given time. No single variable experiences regular periodic evolution. To simplify the terminology, we shall ignore this fact and speak of the three states as if they consisted of all 0's, all 1's, and all 2's, respectively, i.e., as if $M_j \sim j$ for $j=0$, 1, or 2; this is close to the truth if the noise is small.

Take an initial state consisting of all 1's. The system is periodic only if the subsequent time evolution, monitored after every three time steps, appears stationary: the 1's, e.g., should occur at every third step. Since there is noise, however, droplets of 0's and 2's of arbitrarily large size will nucleate in the infinite sea of 1's. We now argue that sufficiently big droplets of either 0's or 2's will grow with time under generic conditions, eventually supplanting the 1's and so destabilizing the assumed periodic structure.

To see this, we write the phenomenological equation[6(a),6(b)] for a large, roughly spherical droplet (radius R), of 0's immersed in the sea of 1's, say,

$$\frac{dR}{dt} \sim -\frac{\sigma}{R} + h \ . \tag{1}$$

Equation (1) is familiarly used to study nucleation and growth in Ising-like models of equilibrium statistical mechanics.[6(b)] In that context, it describes the evolution of a droplet, radius R, of up spins, immersed in a sea of down spins, in the Ising model in a magnetic field h, at low temperature T, with surface tension σ. The first term, proportional to the droplet's curvature, $1/R$, expresses the tendency of finite droplets to shrink under the action of surface tension. The second term represents the system's preference for the state favored by the field. As $R \to \infty$ the domain wall becomes flat; the resulting equation, $dR/dt \sim h$, implies the uniform translation with velocity h of a flat wall between the two states. A similar interpretation holds in the locally periodic nonequilibrium situation considered here: σ/R represents the system's preference for spatially uniform states, while h represents the net translation velocity, averaged over the 3-cycle, of a flat domain wall between any two of the states. Note that Eq. (1) assumes that the sign of h is independent of domain wall orientation, as is the case, for example, in isotropic systems. We shall return to this point later. Recall that in equilibrium systems, even anisotropic ones where a flat wall's translational speed may depend on orientation, the sign of the velocity h is always independent of orientation, since it must produce growth of the phase of lower thermodynamic potential.

Equation (1) with $h>0$ describes[6] the growth (shrinking) of droplets of radius R greater (less) than a critical size $R_c \equiv \sigma/h$. If our CA rule favors 0's over 1's ($h>0$), the noise will eventually nucleate a finite density of droplets of 0 with $R>R_c$; these will expand linearly with t, supplanting the sea of 1's and destroying the system's assumed temporal periodicity. A similar breakdown of periodicity occurs if $h<0$, so that 1's are favored over 0's.

Only if h is strictly zero in (1) do droplets of arbitrary size and type shrink and so fail to undermine the system's periodicity. One can construct rules for which $h=0$, and the 3-cycle therefore stable, either by symmetry or by clever choice of parameters. Under generic conditions, however, h is *always* nonzero: The set of parameters for which the periodic state can be stable has zero measure in the total parameter space of the rules [just as the Ising model's coexistence curve has zero measure in the (h-T) phase diagram]. Thus the overwhelmingly probable situation is the one without special symmetries to stabilize cycles.

Under generic conditions, then, the 3-cycle we have tried to create is at best *metastable*, rather than stable. The *stable* state of the system is actually stationary. To see this, suppose that our initial state of 1's is destabilized, after many 3-step cycles, by the growth of droplets of 0's. Since $0 \to 1 \to 2 \to 0$ under the rule, the instability of 1's with respect to 0's implies the instability of 0's with respect to 2's, and of 2's with respect to 1's. Thus even as the initial sea of 1's is being undermined by large, growing droplets of 0's, the 0 droplets themselves are being supplanted by droplets of 2's, which in turn are supplanted by 1's, etc.[7] As $t \to \infty$, therefore, one expects the system to consist, at any instant, of an incoherent mixture of states 0, 1, and 2, i.e., to be stationary in time. Numerical simulations in two dimensions indicate that this steady state contains a stationary density of phase dislocations (places where all three phases meet) each surrounded by a Zhabotinsky-type spiral wave containing all three phases. We cannot rule out the possibility of the system setting into a stable periodic state with an *irrational* period, though we have seen no numerical evidence for such an occurrence.

Similar arguments can clearly be applied to rule out generically stable k cycles for any k greater than 2 in any spatial dimension d. Period-2 systems are an exception because in these systems the domains separated by a flat wall exchange identities with each time step. Any translation of the wall in the first time step is therefore balanced by an equal and opposite translation in the second. Thus there is no net translation in a full cycle, so droplets in period-2 systems are generically described by Eq. (1) with $h=0$. They therefore shrink in time regardless of their initial size, and so do not destabilize the periodic state.

The foregoing arguments implicitly assume that the sign of h is independent of domain-wall orientation. This need not be the case: Generic conditions guarantee only that a flat, infinite wall separating two states translates with nonzero velocity; under certain generic rules[6(a),8] the direction of the translation can vary with the wall's orientation with respect to special (e.g., lattice axis) directions. Droplets evolve rather remarkably under such rules, shrinking in certain directions while growing in others. In the sample 2D rule given below, e.g., walls oriented along (at 45° to) the lattice axes move to favor 0's over 1's (1's over 0's). In consequence, a large droplet of 0's in a sea of 1's, viewed at three-step intervals, expands (contracts) along those portions of its boundary oriented parallel (at 45°) to the lattice axes. Thus it distorts into a diamond shape and shrinks away [Fig. 1(a)]. Similarly, a droplet of 1's in a sea of 0's distorts into a square oriented

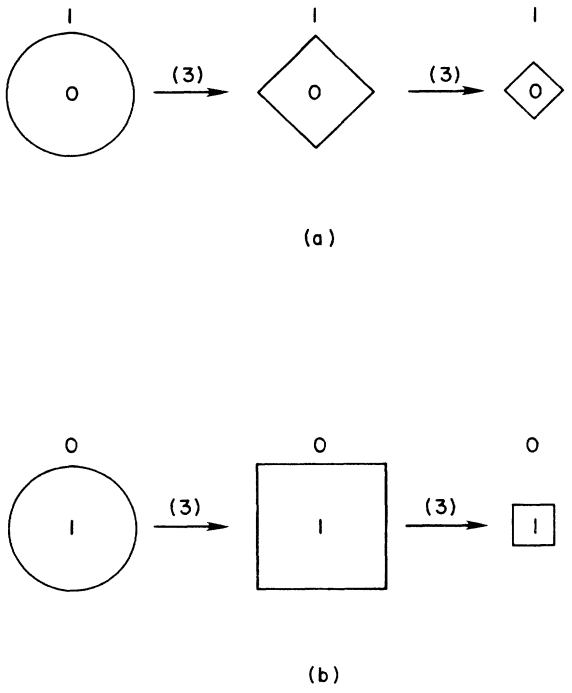

FIG. 1. Droplet of (a) 0's in a sea of 1's, (b) 1's in a sea of 0's, seen at three-step intervals. Droplet distorts into a square oriented (a) at 45° to the lattice axes and (b) with the lattice axes, and then shrinks.

parallel to the axes, thereupon likewise shrinking away [Fig. 1(b)]. The system thus eliminates droplets of *any* type and size; i.e., $R_c = \infty$. This anisotropy-driven elimination is described by $dR/dt \sim -h$, where $h(>0)$ represents the velocity of flat domain walls. It therefore proceeds very swiftly, the lifetime of a large droplet being linear in its original radius rather than quadratic, as it is in the more familiar case where $h=0$ and elimination is driven by surface tension. Note that this rapid annealing mechanism requires not only anisotropy, but also irreversibility.[6(a)] For reversible dynamics (i.e., governed by an underlying Hamiltonian), the sign of domain-wall motion is determined by the thermodynamic potentials of the two phases, and so is independent of orientation. Of course the speed of the motion can still be orientation dependent.

One concludes that anisotropy-driven annealing can stabilize periodic states at generic points in the parameter space of the transition probabilities. Though we have argued this only for 3-cycles in 2D, the extension to 3D and to cycles of arbitrary period is straightforward.[9] While isotropic systems such as fluids[10] cannot directly make use of this stabilization mechanism, they can undergo phase transitions (e.g., solidification) which spontaneously break the rotational invariance, and then exploit the resulting anisotropy to stabilize periodic states.

The time $[\tau_c \sim \exp(R_c^{d-1})]$ required for the nucleation of a droplet of critical size and the consequent disruption of periodic states which are metastable rather than stable can be astronomical if R_c is large. In such cases, metastable states can readily masquerade as stable ones.[11] At least in 2D, however, long-lived metastable periodic states can be distinguished from periodic states which achieve stability by using anisotropy, since the unusual efficiency with which droplets are eliminated in truly stable states gives rise to autocorrelations [i.e., the correlation function $C(t) \equiv \langle \theta_i(t)\theta_i(0)\rangle - \langle \theta_i(t)\rangle\langle \theta_i(0)\rangle$, in our CA terminology], which decay with anomalous rapidity. To see this, note that in order to contribute to $C(t)$ at large t, a given variable must be part of a droplet whose radius $R(t)$ is so large that the droplet persists[12] for time t. The probability of this occurring is roughly the probability of nucleating a droplet of radius $R(t)$, i.e., $\sim \exp[-R(t)^{d-1}]$. In metastable periodic states, droplets evolve "conventionally," i.e., according to Eq. (1). On time scales $t \ll \tau_c$, the nucleation time for a critical-size droplet, h can be neglected in (1), whereupon $dR/dt \sim \sigma/R$, or $R(t) \sim t^{1/2}$. Hence in 2D one recovers the stretched exponential decay, $C(t) \sim \exp[-(t/\tau)^{1/2}]$, characteristic of equilibrium Ising systems.[12] (Here τ is the correlation time.) By contrast, droplets in systems with rules which stabilize periodic states shrink according to $dR/dt \sim -h$, and so persist only for times t proportional to their size R. Thus $R(t) \sim t$, so rather than stretched exponentials, large droplets give rise to the ordinary exponential decay, $C(t) \sim \exp(-t/\tau)$, expected to arise from small droplets. For $d \geq 3$, $C(t) \sim \exp(-t/\tau)$ both for stable and metastable periodic states.

We close by describing a CA rule which illustrates the arguments in this paper. This prototype CA has three states per site, and is defined on a square lattice. The states represent three equally spaced points on the unit circle, labeled by the angle θ, which assumes the values 0, $2\pi/3$, or $-2\pi/3$, or equivalently, by the complex number $z \equiv \exp(i\theta)$. The value of the angle $\theta_i(n+1)$ on the ith site at time $n+1$ is determined by the argument $\bar{\theta}_i(n) \equiv \arg[\bar{z}_i(n)]$ of the sum, $\bar{z}_i(n) \equiv \sum_j z_j(n)$ (the sum on j running over a neighborhood consisting of i and its four nearest neighbors), according to the following algorithm: Let $\tilde{\theta}_i(n) \equiv \bar{\theta}_i(n) + \eta_i(n)$, where $\eta_i(n)$ is a random noise variable chosen from some appropriate (e.g., Gaussian) distribution $P(\eta)$. Divide the unit circle into three regions: (I) $\pi/3+\epsilon_1 \leq \tilde{\theta}_i(n) < \pi+\epsilon_3$; (II) $-\pi/3+\epsilon_2 \leq \tilde{\theta}_i(n) < \pi/3+\epsilon_1$; and (III) $-\pi+\epsilon_3 \leq \tilde{\theta}_i(n) < -\pi/3+\epsilon_2$, where $-\pi/3 < \epsilon_i < \pi/3$ for $i=1,2,3$, and the convention $-\pi+\epsilon_3 \leq \tilde{\theta} < \pi+\epsilon_3$ is adopted. If $\tilde{\theta}_i(n)$ falls in region I, II, or III, then $\theta_i(n+1)$ is 0, $-2\pi/3$, or $2\pi/3$, respectively.

Inspection of this rule yields the following conclusions. (i) For spatially uniform initial conditions in the deterministic limit, the system remains spatially uniform, executing the clockwise 3-cycle $0 \rightarrow -2\pi/3 \rightarrow 2\pi/3 \rightarrow 0$. (ii) In the symmetric, "nongeneric" case where all the ϵ_i's are zero and $P(\eta)$ is even, so that the noise tends to accelerate or retard the clockwise rotation with equal probability, flat domain walls between any two of the states are forbidden from translating. Stable 3-cycles can therefore occur if the noise is sufficiently small. (iii) In the more generic situation where the ϵ_i's are nonzero, this

symmetry is gone, even when $P(\eta)$ is even, since positive and negative ϵ's, respectively, tend to accelerate and retard the rotation. One can show by inspection, e.g., that for zero noise, $-\pi/3<\epsilon_1<\tan^{-1}(3^{1/2}/7)-\pi/3$, and $\epsilon_2=\epsilon_3=0$, flat walls of any orientation separating 0's from $2\pi/3$'s, translate by one lattice spacing to favor the $2\pi/3$'s with each 3-step cycle (i.e., retardation of the rotation is favored). Thus h in Eq. (1) is one lattice spacing per cycle in the deterministic limit. Computer simulations in this limit show that the critical radius R_c required for the $2\pi/3$'s to supplant the 0's is only about 2 for $\epsilon_1=-0.9$, $\epsilon_2=\epsilon_3=0$. Correspondingly, simulations with very small noise show the system setting into a stationary state, demonstrating the expected inability of the rule to stabilize a 3-cycle in this case. (iv) When, however, $\epsilon_1=-0.9$, $\epsilon_2=\epsilon_3=0.5$, and $\eta=0$, e.g., flat domain walls separating 0's and $2\pi/3$'s, and oriented with (at 45° to) the lattice axes move one lattice spacing to favor $2\pi/3$ (to favor 0) in each 3-step cycle. We therefore anticipate that $R_c=\infty$, i.e., that droplets of *any* size and type shrink (linearly) with time. This expectation has been verified in simulations of droplets in zero noise. Moreover, simulations of the rule at reasonable noise levels show clear 3-cycles consistent with the anisotropy-driven stabilization of periodic states.

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[7]Concentric growing droplets called "target patterns" are readily observed in thin-film experiments on the Belousov-Zhabotinsky chemical reaction; see, e.g., C. Vidal, A. Pagola, J. M. Bodet, P. Hanusse, and E. Bastardie, J. Phys. (Paris) **47**, 1999 (1986). It is interesting to speculate that such patterns, whose occurrence is not well understood, are related to the droplet growth mechanism discussed here.

[8]A. L. Toom, in *Multicomponent Random Systems*, Vol. 6 of *Advances in Probability*, edited by R. L. Dobrushin (Dekker, New York, 1980), pp. 549–575.

[9]It should be possible to stabilize periodic states in 1D, but rules which do this are probably extremely complicated. See P. Gacs, J. Comput. Syst. Sci. **32**, 15 (1986).

[10]We emphasize that our conclusions have been derived only for discrete-time systems. In continuous-time systems like real fluids, domain walls are described by a nonlinear diffusion equation [see Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, Berlin, 1985)], and tend to spread out, becoming ill defined. Thus the arguments given here do not apply directly. One expects stabilizing periodic states which break a continuous symmetry to be at least as difficult as in the discrete case, but this has not yet been established.

[11]For example, Bohr *et al.* (Ref. 4) reported the numerical observation of apparently stable cycles with long periods in coupled map lattices of up to 200×200 in size, with nonzero noise. We now believe that the couplings used in that reference were incapable of stabilizing anything beyond the 2-cycle; the higher cycles are presumably metastable. From numerical studies of droplet evolution in the 4-cycle of Ref. 4, we estimate $R_c\sim\sigma/h$ to be of $O(3000)$, a size too big for simulation. Numerical studies, to be reported elsewhere, indicate that this anomalously large R_c is due to a spontaneous reconstruction of domain walls which is probably peculiar to cycles that arise through period doubling.

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