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Generalized ϵ expansion for self-avoiding tethered manifolds

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We present a scheme of interpreting the generalized ϵ expansion encountered in the renormalization-group analysis of D-dimensional self-avoiding tethered manifolds embedded in d-dimensional external space. Owing to the nonlinear nature of the parameter ϵ, $O(\epsilon)$ results of previous studies of the ϵ expansion are somewhat ambiguous. Our scheme resolves ambiguities and gives optimal numerical results to every order in ϵ. The $O(\epsilon)$ optimal values of the radius-of-gyration exponent ν are calculated for polymers in two and three dimensions, and for membranes in various dimensions. The optimized polymer exponents are better than the $O(\epsilon)$ results obtained from the traditional ϵ expansion. The exponent for two-dimensional membranes embedded in high-dimensional space, i.e., for $D=2$ and $d\to\infty$, is $4/d$ in agreement with the Flory exponent. We further explore the ϵ-expansion properties in the limit $d,D\to 0$ and find multiple-body excluded-volume interactions to be relevant. We show that these interactions cannot be ignored in the physically relevant case of two-dimensional membranes in three-dimensions.

One of the major challenges in theoretical physics today is to understand the properties and behaviors of surfaces and membranes.[1] There are appreciable interest and applications of this new branch of physics in fields ranging from cell membrane interactions in biology[2] to world-sheet dynamics in string theory.[3] Progress in studies of these surfaces has been limited, however, due to enormous mathematical complexities. To gain more insight and knowledge, we investigate the macroscopic properties of membranes in equilibrium. A simple model for interacting membranes is the self-avoiding tethered manifold (SATM);[4] this model describes systems such as polymerized membranes in solution. A formalism based on renormalization-group analysis has been developed,[5-7] and some universal properties of the membrane, e.g., the radius-of-gyration exponent ν and the dimensionless second-virial coefficient g are calculated in a generalized ϵ expansion to $O(\epsilon)$. One-loop renormalizability of the SATM theory has been recently proved,[8] and the $O(\epsilon)$ results are verified via direct calculations.[9] In this paper, after providing a brief account of the motivations for the SATM model, I will discuss in detail some aspects of the generalized ϵ expansion which have not been fully appreciated in previous studies. Some new numerical estimates for the exponent ν are obtained as a consequence. As will be seen, these estimates are good for membranes embedded in high-dimensional space, but they fail when the dimension of the embedding space is reduced because multiple-body excluded volume interactions become relevant.

A good starting point to study self-avoiding tethered membranes is the Edwards model,[10] which has been very successful in describing self-avoiding linear polymers.[11,12] By extending the one-dimensional *internal* connectivity of polymers to two dimensions, we have the following partition function

$$Z=\int \mathcal{D}^d\mathbf{r}(\mathbf{x})\exp\left[-\frac{1}{2}\int d^2\mathbf{x}[\nabla\mathbf{r}(\mathbf{x})]^2 - \frac{v}{2}\int d^2\mathbf{x}_1 d^2\mathbf{x}_2\delta^d[\mathbf{r}(\mathbf{x}_1)-\mathbf{r}(\mathbf{x}_2)]\right], \tag{1}$$

describing a self-avoiding tethered membrane embedded in d-dimensional space. Here the first term is the entropy-generated elasticity, and the second term is the two-body excluded-volume interaction whose strength is parametrized by v.

It is clear that the interaction term in Z is badly divergent as $\mathbf{x}_1\to\mathbf{x}_2$, so that a microscopic cutoff is required. Renormalizations must then be used to extract macroscopic properties that are independent of microscopic cutoffs. For polymers, it was shown[12] that an ϵ expansion exists below a critical dimension of 4. However, such an expansion fails to exist for membranes described by (1) because the corresponding critical dimension is shown[5] to

be at $d=\infty$. This problem is circumvented by generalizing the internal connectivity of the network from 2 to D, i.e.,

$$Z=\int \mathcal{D}^d\mathbf{r}(\mathbf{x})\exp\left[-\frac{1}{2}\int d^D\mathbf{x}[\nabla\mathbf{r}(\mathbf{x})]^2 - \frac{v}{2}\int d^D\mathbf{x}_1 d^D\mathbf{x}_2\delta^d[\mathbf{r}(\mathbf{x}_1)-\mathbf{r}(\mathbf{x}_2)]\right] . \tag{2}$$

Such a partition function describes a D-dimensional self-avoiding tethered *manifold*; $D=1$ corresponds to a linear polymer, $D=2$ to a polymerized membrane, and $D=3$ to a gel.

Naive dimension counting of (2) has

$$v\sim|\mathbf{x}|^{-2D+d(2-D)/2} ,$$

suggesting that self-avoidance begins to be relevant at the line of critical dimensions $4D=d(2-D)$. Indeed, a direct calculation of the two-point correlation function from perturbation series[8] gives to one-loop order

$$\langle[\mathbf{r}(\mathbf{x})-\mathbf{r}(\mathbf{x}')]^2\rangle = 2d\frac{|\mathbf{x}-\mathbf{x}'|^{2-D}}{S_D(2-D)} \times\left[1+\left[I_1+\frac{d}{2}I_2\right]\frac{v}{\epsilon}K|\mathbf{x}-\mathbf{x}'|^{\epsilon/2}\right]^{I_2/[I_1+(d/2)I_2]} , \tag{3}$$

with

$$\epsilon(d,D)=4D-d(2-D) , \tag{4}$$

and the constants $S_D=2\pi^{D/2}/\Gamma(D/2)$ and $K=S_D^2[S_D(2-D)/4\pi]^{d/2}$. The integrals are

$$I_2=\frac{1}{2}\frac{2-D}{2D}\int_0^1 d\xi\,\xi^{-1+\epsilon/2}=\frac{1}{\epsilon}\frac{2-D}{2D}=\frac{1}{\epsilon}\frac{2}{d}+O(\epsilon^0) , \tag{5a}$$

$$I_1=\frac{1}{2}\int_0^1 d\xi\,\xi^{D-1}\int_0^1 d\eta\,\eta^{D-1}(\xi^{2-D}+\eta^{2-D})^{-d/2} . \tag{5b}$$

For $d,D\neq 0$, the leading ϵ divergence of I_1 can be easily extracted

$$I_1=\frac{1}{\epsilon}J(d)+O(\epsilon^0) ,$$

where

$$J(d)=\frac{4+d}{8}\Gamma^2\left[\frac{d}{4}\right]\Big/\Gamma\left[\frac{d}{2}\right] .$$

In the limit of large $|\mathbf{x}-\mathbf{x}'|$, the correlation function (3) becomes $\langle[\mathbf{r}(\mathbf{x})-\mathbf{r}(\mathbf{x}')]^2\rangle\sim|\mathbf{x}-\mathbf{x}'|^{2\nu}$. The universal exponent ν is $(2-D)/2$ in the ideal region $\epsilon<0$. But in physically relevant cases where $\epsilon>0$, we have

$$\nu(d,D)=\frac{2-D}{2}+\frac{1}{2d}\frac{\epsilon(d,D)}{1+J(d)}+O(\epsilon^2) . \tag{6}$$

The $O(\epsilon^2)$ terms are contributions to ν from subleading terms of one-loop interactions, as well as from leading terms of two-loop interactions which have not yet been calculated. In this study, we will concentrate only on $O(\epsilon)$ results.

If our interest lies only in the numerical values of the exponent ν, then we can simply use the Flory expression[13] for the exponent as a guide,

$$\nu_F=\frac{D+2}{d+2} . \tag{7}$$

(This has been shown[5] to agree very well with all available best numerical estimates and exact results for polymers; the Flory exponent for tethered membranes is also close to some numerical estimates.) However, the Flory expression cannot be extended to study other universal quantities; nor can it be used to calculate scaling functions, determine relevances of other interactions, etc., which can be studied systematically using the ϵ expansion. In this paper, we use the exponent ν as a case study to explore the ϵ expansion properties of all universal quantities. We then use the Flory exponent as a guide against which we check results of the ϵ expansion.

It is important to recognize that, as in all renormalizable theories, the expression for the correlation function in (3) is merely a reorganization of the perturbation series. And the small parameter ϵ, which emerged naturally from perturbative calculations, is nothing more than a mathematical convenience which can be exploited to organize the expansion of exponents. Normally ϵ is a linear function of the spatial dimension d only, for instance, $\epsilon=4-d$ in the theory of polymers. Expansion in powers of ϵ is therefore conceptually very simple. In our case, however, we find that by generalizing the manifold dimension to D, we are left with an ϵ which is a *nonlinear* function of d and D. Since a universal quantity such as the exponent ν is some arbitrary function of d and D, it usually cannot be written in terms of $\epsilon(d,D)$ alone. If we still want to take advantage of the expansion parameter ϵ we need to make a transformation of variables from d,D to $\epsilon=\epsilon(d,D)$, $\delta=\delta(d,D)$, such that $\nu(d,D)=\tilde{\nu}_\delta(\epsilon(d,D),\delta(d,D))$. The subscript δ is a reminder that the function $\tilde{\nu}$ will depend on the form of $\delta(d,D)$ used. In this way, ν can be written as a double expansion in ϵ and δ.

While the form of $\epsilon(d,D)$ is given by (4), we have at our disposal an infinite number of invertible transformations $\delta(d,D)$, all of which will lead to the same values of exponent $\nu(d,D)$. This freedom in choosing $\delta(d,D)$ was a cause of concern in previous studies,[5] and the ϵ expansion was thought to be ambiguous at least to first order since different forms of $\delta(d,D)$ should lead to different values of ν at $O(\epsilon)$. I will show in the following that it is exactly this freedom that provides us with a guide to resolve the apparent ambiguities.

We would like to choose a transformation $\delta(d,D)$ that is either physically meaningful or mathematically convenient. One very simple choice is to have the point of interest, say $(\hat{d},\hat{D})$, be on the line $\delta(d,D)=0$. In this case $\tilde{\nu}_\delta(\epsilon,\delta)$ becomes a single power series in ϵ. We limit our discussion to $\delta(d,D)$'s such that $\delta=0$ are straight lines in

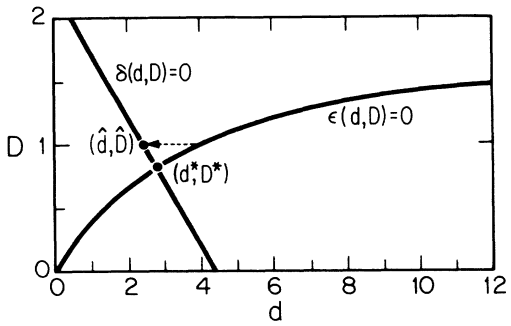

FIG. 1. Lines $\epsilon(d,D)=0$ and $\delta(d,D)=0$ in the (d,D) plane. The path of the traditional ϵ expansion is marked by the arrow.

the (d,D) plane. [We will soon see that a straight line is all we need for $O(\epsilon)$ calculations.] Suppose the straight line $\delta(d,D)=0$ intersects the curve $\epsilon(d,D)=0$ at (d^*,D^*) as shown in Fig. 1, then the transformation $\delta(d,D)$ is completely specified by the point (d^*,D^*) for a given $(\hat{d},\hat{D})$, i.e.,

$$\delta(d,D)=D(d^*-\hat{d})-d(D^*-\hat{D})+D^*\hat{d}-\hat{D}d^* , \quad (8)$$

with $4D^*=d^*(2-D^*)$. The point (d^*,D^*) is called the expansion point. Since d^* (or D^*) serves as a parameter in $\delta(d,D)$, we write $\nu(\hat{d},\hat{D})=\tilde{\nu}_{d^*}(\epsilon(\hat{d},\hat{D}),\delta(\hat{d},\hat{D})=0)$. Note that the case $d^*=4\hat{D}/(2-\hat{D})$ (or $D^*=\hat{D}$, see dashed line in Fig. 1) corresponds to the traditional ϵ expansion.

Equations (4) and (8) can then be inverted, with

$$D(\epsilon,\delta=0)=D^*+\epsilon\frac{D^*(D^*-\hat{D})}{(D^*)^2(d^*-\hat{d})+2(D^*\hat{d}-d^*\hat{D})}+O(\epsilon^2) , \quad (9)$$

and a similar expression for d: $d(\epsilon,\delta=0)=d^*[1+O(\epsilon)]$. Substituting these expressions for d and D into (6) and keeping everything to $O(\epsilon)$, we can compute the value of exponent $\nu(\hat{d},\hat{D})=\tilde{\nu}_{d^*}(\epsilon,\delta=0)$ as a function of the parameter d^*. The results for $(\hat{d},\hat{D})=(3,2)$, (3,1), and (2,1) are plotted in Figs. 2(a)–2(c). As is clear from these figures, ν is not a monotonically decreasing function of d^*, in contradiction to a finding in Ref. 5. There, $D(\epsilon,\delta)=\hat{D}$ was inadvertently used [instead of Eq. (9)] in evaluating ν. Due to the nonlinear form of $\epsilon(d,D)$, it is easy to see that in general $D(\epsilon,\delta)\neq\hat{D}$ to $O(\epsilon)$, except in the special cases when $d^*=\hat{d}$ or $d^*=4\hat{D}/(2-\hat{D})$.

There is now an apparent ambiguity in values of ν due to its dependency on d^*. To find the "optimal" ν to $O(\epsilon)$, we recall that $\nu(\hat{d},\hat{D})$ is independent of the choice of $\delta(d,D)$. Therefore $\partial\nu(\hat{d},\hat{D})/\partial d^*=0$. It follows that

$$\nu_{\rm opt}=\tilde{\nu}_{\bar{d}}, \quad \text{where} \quad \left.\frac{\partial\tilde{\nu}_{d^*}}{\partial d^*}\right|_{d^*=\bar{d}}=0 .$$

We see that ambiguity in ν may be removed to a large degree by choosing the extremum value of $\tilde{\nu}_{d^*}$. This is readily applied to higher orders in ϵ and more complicated curves $\delta(d,D)=0$. In the latter case, if the curve is

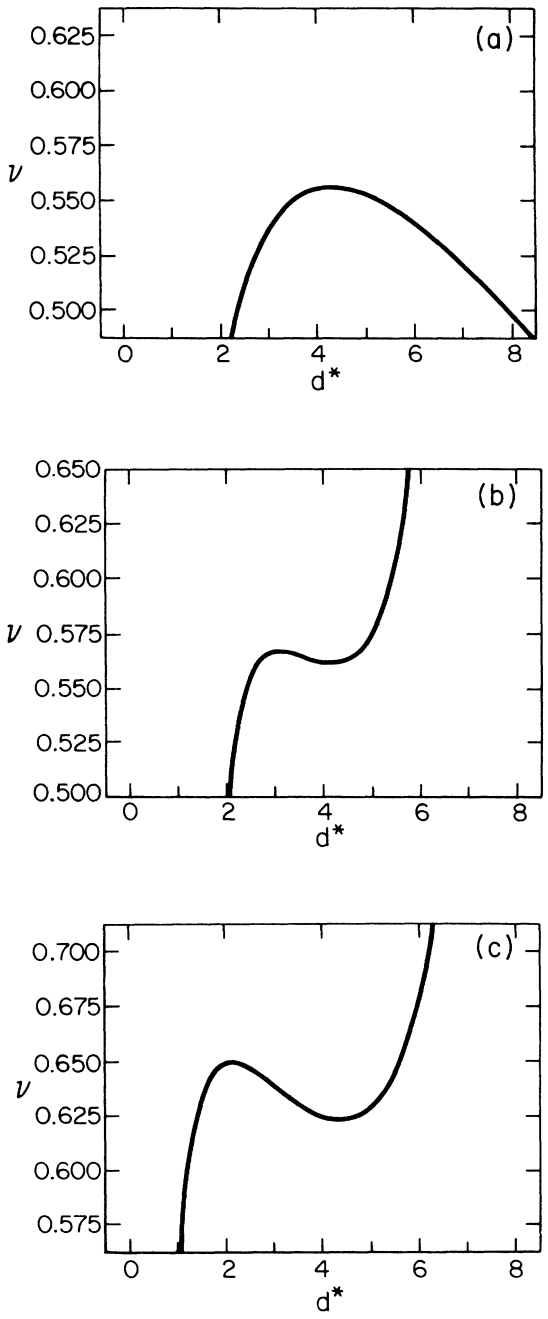

FIG. 2. Numerical results of $\tilde{\nu}_{d^*}$ for (a) membranes in three dimensions, (b) polymers in three dimensions, and (c) polymers in two dimensions.

specified by n parameters $p_1,\ldots,p_n$, then there will be n conditions $\partial\tilde{\nu}/\partial p_i=0$ to fix every parameter. However, to $O(\epsilon)$, the dependence of $\tilde{\nu}$ on p_i's comes only from its dependence on $D(\epsilon,\delta)$ in (6). Suppose $D=D^*+f_1(d^*,p_i)\epsilon+O(\epsilon^2)$, then $\partial\tilde{\nu}/\partial p_i=0$ implies that $\partial D/\partial p_i=0$, or $f_1(d^*,p_i)=f(d^*)$. Hence straight line (8) is the only family of curves needed to be searched for the optimized value of ν to $O(\epsilon)$.

Applying our optimization scheme to the exponent values plotted in Fig. 2, we find the $O(\epsilon)$ estimate for

TABLE I. Values of the exponent ν calculated by (a) the optimization method described in text to $O(\epsilon)$ ($\bar{d}$ is the optimal expansion point), (b) the traditional ϵ expansion to $O(\epsilon)$, (c) exact calculation or numerical estimates, and (d) the Flory expression.

$(\hat{d},\hat{D})$	$\bar{d}$	(a)	(b)	(c)	(d)
(3,2)	4.3	0.556		0.80	0.800
(4,2)	5.3	0.517			0.667
(5,2)	6.2	0.484			0.571
(6,2)	6.9	0.454			0.500
(7,2)	7.7	0.426			0.444
(8,2)	8.4	0.401			0.400
$(\hat{d}\to\infty,2)$	$\hat{d}$	$4/\hat{d}$			$4/\hat{d}$
(3,1)	3.2	0.567	0.562	0.591	0.600
(2,1)	2.2	0.650	0.625	0.750	0.750

tethered membranes in three dimensions to be $\nu(\hat{d}=3,\hat{D}=2)=0.556$; the optimal expansion point [the maximum in Fig. 2(a)] is at $\bar{d}\approx 4.3$. The exponents for membranes in higher embedding spatial dimensions are also obtained by this method. Their values are listed in Table I. The corresponding exponent values as calculated from the Flory expression (7) are also listed for ease of comparison. For polymers in three and two dimensions [Figs. 2(b) and 2(c)], two extrema are present. We choose the maxima values since the exponent values are underestimated in both cases;[14] these results are again listed in Table I. Since $\bar{d}\neq 4$, we discover that the traditional $\epsilon=4-d$ expansion for polymer is not the "optimal" expansion; our scheme is thus an improvement over the traditional method.

There is clearly a trend in Table I: Estimates of exponent values for manifolds in high-dimensional embedding space ($\hat{d}\gg\hat{D}$) are quite good; but it starts to deviate from best estimate, exact results as embedding spatial dimension $\hat{d}$ is reduced. Unfortunately, the physically relevant situation of membranes in three dimensions is the worst case. It should be noted that this trend is not related to the value of ϵ. For two-dimensional membranes, we have $\epsilon=8$ in any $\hat{d}$, yet the $O(\epsilon)$ estimate of ν is very sensitive to $\hat{d}$. In fact, it is easy to see (at $d^*=\hat{d}$) that ν of (6) and ν_F of (7) have the same limit $4/\hat{d}$ for two-dimensional membranes embedded in very high-dimensional space, i.e., for $\hat{d}\to\infty$. (With a little algebra, we can show that $\nu_{\rm opt}$ also goes to the limit $4/\hat{d}$.) This, however, is not true for manifolds with $D\neq 2$.

For manifolds embedded in low-dimensional space, can the situation be improved if we are able to compute and include higher-order terms in ϵ? To answer this question, we go to the extreme situation where $\hat{d}\approx\hat{D}$. It is known[5] on physical grounds that the manifold is "stretched," i.e., $\nu=1$ for $\hat{d}=\hat{D}$. Therefore, if we do a double expansion in $\epsilon(d,D)$ and $\delta(d,D)=d-D$, we expect to have $\nu(\epsilon,\delta)$ be independent of ϵ at small δ. In particular, we must have $\nu(\epsilon,\delta=0)=1$ for all ϵ.[15]

In the vicinity of $\epsilon\approx 0$ and $\delta\approx 0$, we have to the lowest order $D=\delta+\epsilon/2$ and $d=2\delta+\epsilon/2$. We cannot simply substitute these expressions for D and d into (6) to obtain ν because (6) is not valid in the limit $d,D\to 0$. To obtain ν in the region of interest, we need to go back to the full expression with I_1 and I_2 in (5), and extract the relevant divergences in them. This has been done in the Appendix, with the result

$$I_1=\frac{2}{\epsilon D}+O\left[\frac{d}{\epsilon}\right].$$

Combined with

$$I_2=\frac{1}{\epsilon D}\frac{2-D}{2},$$

we have

$$\nu=\frac{2-D}{2}+\frac{\epsilon}{4}\frac{I_2}{I_1+\frac{d}{2}I_2}$$

$$=1-\frac{\delta}{2}-\frac{\epsilon}{8}+O(\delta\epsilon,\epsilon^2,\delta^2).$$

The above expression clearly shows that $\nu\neq 1$ at $\delta=0$ for any nonzero ϵ, underestimating the exact result $\nu=1$. The dependence of $\nu(\epsilon,\delta=0)$ on ϵ will persist even if all higher-order ϵ terms are included. We thus have to conclude that the partition function (2), from which we obtained the correlation function (3) and the above expression for ν, does not fully describe the self-avoiding manifold in the region $\delta\approx 0$. A moment of reflection suggest that as the dimension of the embedding space approaches that of the manifold dimension, the manifold is "squeezed" and n-body excluded-volume interactions of the form

$$-\frac{v_n}{n!}\int d^D\mathbf{x}_1\cdots d^D\mathbf{x}_n$$

$$\times\delta^d[\mathbf{r}(\mathbf{x}_1)-\mathbf{r}(\mathbf{x}_2)]\cdots\delta^d[\mathbf{r}(\mathbf{x}_{n-1})-\mathbf{r}(\mathbf{x}_n)]$$

become increasingly important. These terms should be included in (2) for $\delta\to 0$; they will then tend to "stretch" the network and increase ν.

Dimension counting of the n-body interaction tells us that v_n is relevant if

$$nD-(n-1)\nu(d,D)d>0. \tag{10}$$

Since $\nu(d,D)\leq 1$ for any d and D, then at $d=D$ we always have $n-(n-1)\nu>0$. There, all v_n's are relevant, and it is not surprising that (2) underestimates ν.

To estimate the region of validity for (2), we use the Flory expression for $\nu(d,D)$ in (10). We find that the three-body interaction v_3 becomes relevant for $D>4d/(6+d)$. So the calculation for polymer exponents in two and three dimensions are valid, but for membranes below six dimensions, the three-body interaction must be taken into account. According to the Flory exponent (7),

$$D=\frac{2(n-1)d}{2n+d}$$

is the line when n-body interaction v_n becomes relevant (Fig. 3). For membranes in three dimensions, the Flory exponent $\nu_F=0.80$ is known to agree with numerical esti-

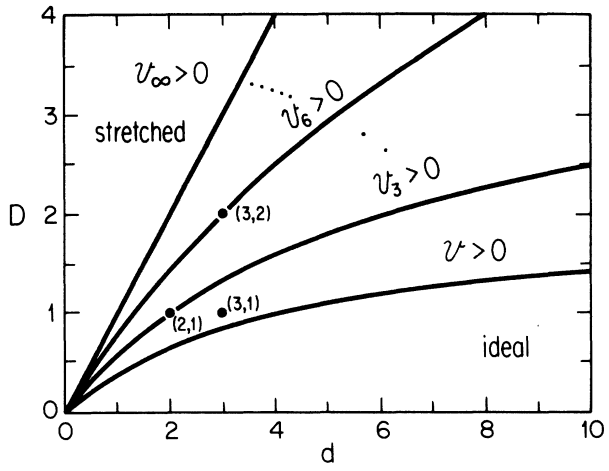

FIG. 3. Different regions in the (d,D) plane where various multiple-body excluded volume interactions are relevant (estimated using Flory exponents).

mate of Monte Carlo simulations[4] (see Table I); the above equation then tells us that we must consider up to five-body interactions. In this case, we cannot obtain the correct exponent values for membranes by working with the two-body excluded volume interaction alone, even if we are able to carry out the ϵ expansion to all orders. This leaves room for the possibility that membranes may be flat in three dimensions, as suggested by recent results of molecular-dynamics simulations.[16]

To summarize, we attempted to calculate the radius of gyration exponent ν for tethered membranes embedded in d-dimensional space by generalizing the Edwards model for polymers. We are forced to consider a generalized D-dimensional manifold, and associated with it, a generalized ϵ expansion. We have presented a scheme of interpreting this expansion. This scheme resolves ambiguities and gives optimal numerical results to a given order in ϵ. The $O(\epsilon)$ estimate of ν for polymers is an improvement over the traditional $\epsilon=4-d$ expansion. The estimates obtained for membranes are good in high-dimensional embedding space, but become not as good as the embedding spatial dimension is reduced. In the physically relevant case of membranes in three dimensions, multiple-body excluded-volume interactions may be important.

The next task is to develop a systematic way of incorporating these multiple-body interactions. Also, the structure of the optimized ϵ expansion awaits to be understood. For example, is it an asymptotic series as in the case of the traditional ϵ expansion,[17] or is it a more converging series? And finally, we would like to investigate the possibility of applying this optimization scheme to other problems that are presently treated by the traditional approach.

I would like to thank M. Kardar for initiating my interest in this problem and for many stimulating discussions and helpful suggestions. This research was supported by the National Science Foundation through Grant No. DMR-86-20386 and by IBM.

APPENDIX

We want to extract relevant poles in I_1 as $\epsilon\to 0$ and $\delta\to 0$. Let us first change integrations variables and split the integrals into two parts:

$$I_1=\tfrac{1}{2}\int_0^1 dx\, x^{-1+\epsilon/2}\int_0^1 dy\, y^{D-1}(1+y^{2-D})^{-d/2} + \tfrac{1}{2}\int_0^1 dx\, x^{-1+\epsilon/2}\int_1^{1/x} dy\, y^{D-1}(1+y^{2-D})^{-d/2} .$$

Let the first integral be I_a and the second be I_b. We compute I_a first

$$I_a=\frac{1}{\epsilon}\int_0^1 dy\, y^{D-1}+\frac{1}{\epsilon}\int_0^1 dy\, y^{D-1}[(1+y^{2-D})^{-d/2}-1] .$$

We recall that $D\approx\delta+\epsilon/2$ and $d\approx 2\delta+\epsilon/2$. Then in the limit $\epsilon\to 0$, the second integral above becomes

$$\int_0^1 dy\, y^{\delta-1}[(1+y^{2-\delta})^{-\delta}-1] \approx -\delta\int_0^1 dy\, y^{-1}\ln(1+y^2)=-\delta(\pi^2/24) ,$$

while in the limit $\delta\to 0$, it is

$$\int_0^1 dy\, y^{\epsilon/2-1}[(1+y^{2-\epsilon/2})^{-\epsilon/4}-1] \approx -\frac{\epsilon}{4}\int_0^1 dy\, y^{-1}\ln(1+y^2)=-\frac{\epsilon}{4}(\pi^2/24) .$$

Therefore $I_a=1/(\epsilon D)+O(d/\epsilon)$.

To extract leading divergences out of I_b, we rewrite it as follows:

$$I_b=\tfrac{1}{2}\int_0^1 dx\, x^{-1+\epsilon/2}\int_1^{1/x} dy\, y^{D-1}y^{-(2-D)d/2} + \tfrac{1}{2}\int_0^1 dx\, x^{-1+\epsilon/2} \times\int_1^{1/x} dy\, y^{D-1}[(1+y^{D-2})^{-d/2}-1] .$$

Using the definition of ϵ in (4), we easily find the first term above to be $1/(\epsilon D)$. The divergence in the remaining integral comes from the limit $x\to 0$. If we again split the integrals into two parts

$$\tfrac{1}{2}\int_0^1 dx\, x^{-1+\epsilon/2} \times\left[\int_1^\infty dy\, y^{D-1}[(1+y^{D-2})^{-d/2}-1] -\int_{1/x}^\infty dy\, y^{D-1}[(1+y^{D-2})^{-d/2}-1]\right] ,$$

then the second part is well behaved in the limit $x\to 0$, and the first part is easily worked out to be of order d/ϵ.

Combining all of the above results, we have

$$I_1=\frac{2}{\epsilon D}+O\left[\frac{D}{\epsilon}\right] .$$

[1]See, e.g., *Statistical Mechanics of Membranes and Surfaces,* Proceedings of the Fifth Jerusalem Winter School, edited by D. R. Nelson, T. Piran, and S. Weinberg (World Scientific, Singapore, in press).

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[14]This choice seems to be rather arbitrary; however, there is another justification: Of the two extrema, the maximum is close to $\hat{d}$ while the minimum is close to $4\hat{D}/(2-\hat{D})$ in both cases. Using (4) and (9), we see that $d(\epsilon,\delta=0)=\hat{d}$, and $D(\epsilon,\delta=0)=D^*+\epsilon D^*/2d^*=\hat{D}$ when $d^*=\hat{d}$. There is no term higher than $O(\epsilon)$ generated from expansion of $d(\epsilon,\delta)$ and $D(\epsilon,\delta)$ in (6). So we expect the extremum close to $\hat{d}$ (maximum in both cases) to be closer to the exact value of the exponent.

[15]The limit of small ϵ and small δ is similar to the limit $d,D\rightarrow 0$. The latter case has been considered by R. C. Ball (private communication).

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