PHYSICAL REVIEW A VOLUME 41, NUMBER 3 1 FEBRUARY 1990

Rapid Communications

The Rapid Communications section is intended for the accelerated publication of important new results. Since manuscripts submitted to this section are given priority treatment both in the editorial office and in production, authors should explain in their submittal letter why the work justifies this special handling. A Rapid Communication should be no longer than 3½ printed pages and must be accompanied by an abstract. Page proofs are sent to authors, but, because of the accelerated schedule, publication is not delayed for receipt of corrections unless requested by the author or noted by the editor.

Quantum-mechanical models of position measurements

Masanao Ozawa*

Department of Electrical Engineering and Computer Science, Northwestern University, Evanston, Illinois 60208

(Received 10 July 1989)

A systematic construction of exactly solvable models of interactions between the measuring apparatus and the particle whose position is to be measured is shown. The two types of root-mean-square errors are calculated for these models and many new models with better statistics than the standard von Neumann model are found. The best among them is the model which has been shown to circumvent the standard quantum limit for the position monitoring. A feature of the models similar to Bondurant's improvement of the interferometric position measurement is also pointed out.

Since von Neumann posed a quantum-mechanical model of an interaction for a position measurement,[1] that model has been the only available model for rigorous analysis until quite recently. The recent revival[2-4] of the study of the von Neumann model has emerged as a result of the interest in the gravitational-wave detection[5] and has revealed considerable features of the model. Contrary to the consensus among most researchers that those features of the model represent the most conceivable properties of position measurements, however, the possibility of a different kind of statistics of a position measurement was presented previously;[6] the von Neumann model satisfies the standard quantum limit (SQL) for monitoring the free-mass position[2] but a model which circumvents the SQL has been successfully constructed.[6] (See Ref. 7 for a complete and rigorous discussion about the SQL.) Although the new model realizes Yuen's idea[8] to break the SQL in rigorous calculations, we have known the details of the statistics of position measurements about only these two models so far.[9] From such a state of affairs we can hardly judge whether the new model is an exceptional one or whether these two are only two of many different possibilities. Thus, for the purpose of theoretical considerations on precision-measurement technology, it seems to be an important task to list all possible models of position measurements, because the less mathematical models give us the poorer physical intuition. The present Rapid Communication is a step towards this program. I will give a list of exactly solvable linear coupling models of position measurements which fulfill certain reasonable requirements for their statistics to be good position measurements. The list will give a full interpolation between the von Neumann model[1] and my previous model.[6] The calculations of the previously introduced types of the root-mean-square errors[6] of these models will demonstrate that there exist continuously many coupling models of position measurements which give still better statistics than the von Neumann model, and that the model in Ref. 6 is the best among them.

Consider a one-dimensional, nonrelativistic, quantum-mechanical system called an *object*, with position x, momentum p ($[\hat{x},\hat{p}]=i\hbar$), and Hamiltonian $\hat{H}_{\text{obj}}$; the caret stands for the operator on a Hilbert space corresponding to an observable. The object is coupled with a measuring apparatus designed to measure the object position x. The measuring apparatus is supposed to consist of two parts called the *probe* and the *detector*. The probe is also supposed to be a one-dimensional quantum system with canonical variables X and P. The object is directly coupled with the probe, which can be regarded as the first stage of the macroscopic measuring apparatus. The detector is a macroscopic system and, after the micro-micro coupling between the object and the probe, the probe is coupled with the detector as subsequent stages of the measuring apparatus. The micro-macro coupling between the probe and the detector makes an arbitrarily precise measurement of the coordinate X of the probe. The outcome of the measurement of x is recorded as the output $\bar{X}$ of the detector, which is the outcome of the measurement of X actually carried out by the detector.

The object-probe coupling is turned on from time $t=0$ to $t=\tau$. The total Hamiltonian for the object and the probe is taken to be

$$\hat{H}_{\text{tot}}=\hat{H}_{\text{obj}}+\hat{H}_{\text{pro}}+K\hat{H}\,, \tag{1}$$

where $\hat{H}_{\text{obj}}$ and $\hat{H}_{\text{pro}}$ are the free Hamiltonians of the object and the probe, respectively, $\hat{H}$ is the interaction, and K is the coupling constant. We assume for mathematical simplicity that the coupling is so strong ($K\gg 1$) that the free Hamiltonians $\hat{H}_{\rm obj}$ and $\hat{H}_{\rm pro}$ can be neglected and that the duration τ of the coupling is so small $(0<\tau\ll 1)$ that we can choose $K\tau\sim 1$.

We suppose that, possibly by the linear approximation, the interaction Hamiltonian is given by

$$\hat{H}_{\rm tot}=K[\alpha(\hat{x}\hat{p}-\hat{X}\hat{P})+\beta\hat{X}\hat{p}+\gamma\hat{x}\hat{P}] , \tag{2}$$

where α, β, $\gamma\in \underline{R}$. We shall write

$$\underline{Z}=\begin{pmatrix}\alpha & \beta\\ \gamma & -\alpha\end{pmatrix}. \tag{3}$$

For suitably chosen K and τ, the case where $\alpha=0$ and $\beta=0$ is reduced to the von Neumann model[1] and the case where $\alpha=1$, $\beta=-2$, and $\gamma=2$ is the model discussed in Ref. 6; in the sequel we shall refer to this model as the $(1,-2,2)$ model.

By the Heisenberg equations of motion, we have

$$d\hat{x}(t)/dt=K(\alpha\hat{x}(t)+\beta\hat{X}(t)) , \tag{4}$$

$$d\hat{X}(t)/dt=K(\gamma\hat{x}(t)-\alpha\hat{X}(t)) . \tag{5}$$

Thus the solution is given by

$$\hat{x}(t)=a_{11}(t)\hat{x}+a_{12}(t)\hat{X} , \tag{6}$$

$$\hat{X}(t)=a_{21}(t)\hat{x}+a_{22}(t)\hat{X} , \tag{7}$$

$$\begin{pmatrix}a_{11}(t) & a_{12}(t)\\ a_{21}(t) & a_{22}(t)\end{pmatrix}=\exp(Kt\underline{Z}) .$$

The determinant of $\underline{Z}$ is $|\underline{Z}|=-(\alpha^2+\beta\gamma)$. The explicit forms of the solutions are as follows.

(a) The case $|\underline{Z}|=0$. In this case $\underline{Z}$ is nilpotent and hence $\exp(Kt\underline{Z})=\underline{I}+Kt\underline{Z}$, where $\underline{I}$ is the unit matrix. Thus the solution is obtained by

$$\begin{aligned} a_{11}(t)&=\alpha Kt+1 ,\\ a_{12}(t)&=\beta Kt ,\\ a_{21}(t)&=\gamma Kt ,\\ a_{22}(t)&=-\alpha Kt+1 . \end{aligned} \tag{8}$$

(b) The case $|\underline{Z}|>0$. In this case $\underline{Z}$ has eigenvalues $\pm\sqrt{-1}D$, where $D=|\underline{Z}|^{1/2}$. The solution is periodic and given by

$$\begin{aligned} a_{11}(t)&=(\alpha/D)\sin DKt+\cos DKt ,\\ a_{12}(t)&=(\beta/D)\sin DKt ,\\ a_{21}(t)&=(\gamma/D)\sin DKt ,\\ a_{22}(t)&=(-\alpha/D)\sin DKt+\cos DKt . \end{aligned} \tag{9}$$

(c) The case $|\underline{Z}|<0$. In this case $\underline{Z}$ has eigenvalues $\pm E$, where $E=(-|\underline{Z}|)^{1/2}$. The solution is obtained by

$$\begin{aligned} a_{11}(t)&=(\alpha/E)\sinh EKt+\cosh EKt ,\\ a_{12}(t)&=(\beta/E)\sinh EKt ,\\ a_{21}(t)&=(\gamma/E)\sinh EKt ,\\ a_{22}(t)&=(-\alpha/E)\sinh EKt+\cosh EKt . \end{aligned} \tag{10}$$

Assume now that the probe is prepared in a fixed pure state with normalized wave function $\varphi(X)$ just prior to the measurement $(t=0)$. For simplicity, we assume

$$\langle\varphi|\hat{X}|\varphi\rangle=0 \text{ and } \varphi(X)=\varphi(-X) . \tag{11}$$

We shall write $\Delta X=\langle\varphi|\hat{X}^2|\varphi\rangle^{1/2}$ for the uncertainty of the prior probe coordinate. Let $\psi(x)$ be the normalized wave function of the object just prior to the measurement. Then, just after the object-probe interaction $(t=\tau)$, the joint wave function of the object and the probe is given by

$$\begin{aligned} &\Psi(x,X)=\psi(dx-bX)\varphi(-cx+aX) ,\\ &a=a_{11}(\tau) ,\ b=a_{12}(\tau) ,\\ &c=a_{21}(\tau) ,\ d=a_{22}(\tau) . \end{aligned} \tag{12}$$

The probability density to obtain the outcome $\bar{X}$ of this measurement for the object prior state ψ is given by

$$p(\bar{X}|\psi)=\int_{-\infty}^{\infty}|\Psi(x,\bar{X})|^2dx . \tag{13}$$

The conditional wave function $\psi_{\bar{X}}(x)=\psi(x|\bar{X})$ of the object just after the measurement, given the outcome $\bar{X}$, is obtained (up to normalization) by

$$\psi(x|\bar{X})=p(\bar{X}|\psi)^{-1/2}\Psi(x,\bar{X}) . \tag{14}$$

Let $G(\bar{X},x)$ be defined as

$$G(\bar{X},x)=\begin{cases}|d|^{-1}|\varphi(d^{-1}(cx-\bar{X}))|^2 \text{ if } d\neq 0 ,\\ \delta(cx-\bar{X}) \text{ if } d=0 .\end{cases} \tag{15}$$

Then from Eqs. (12)-(13) and assumption (11) we have

$$p(\bar{X}|\psi)=\int_{-\infty}^{\infty}G(\bar{X},x)|\psi(x)|^2dx . \tag{16}$$

This shows[10] that if the object position is $\bar{x}$ just prior to the measurement, then the conditional probability density of the outcome $\bar{X}$ of this measurement is given by $G(\bar{X},\bar{x})$.

In Ref. 6, the following two measures of the noise of the measurement are introduced. The *precision* $\varepsilon(\psi)$ of the measurement for the prior state ψ is defined by

$$\begin{aligned} \varepsilon(\psi)^2&=\int_{-\infty}^{\infty}\varepsilon(x)^2|\psi(x)|^2dx ,\\ \varepsilon(x)^2&=\int_{-\infty}^{\infty}(\bar{X}-x)^2G(\bar{X},x)d\bar{X} , \end{aligned} \tag{17}$$

and the *resolution* $\sigma(\psi)$ of the measurement for the prior state ψ is defined by

$$\begin{aligned} \sigma(\psi)^2&=\int_{-\infty}^{\infty}\sigma(\bar{X})^2p(\bar{X}|\psi)d\bar{X} ,\\ \sigma(\bar{X})^2&=\int_{-\infty}^{\infty}(x-\bar{X})^2|\psi(x|\bar{X})|^2dx . \end{aligned} \tag{18}$$

Then an ideal position measurement with $p(\bar{X}|\psi)=|\psi(\bar{X})|^2$ and $|\psi(x|\bar{X})|^2=\delta(x-\bar{X})$ is characterized by the quantitative condition $\varepsilon(\psi)=\sigma(\psi)=0$ for all ψ. In the case of the von Neumann model we shall see $\varepsilon(\psi)=\sigma(\psi)=\Delta X$ for all ψ and in the case of the $(1,-2,2)$ model we shall see $\varepsilon(\psi)=0$ and $\sigma(\psi)=\Delta X$ for all ψ. It is shown[11] that the SQL holds for measurements with $\sigma(\psi_1)\le\varepsilon(\psi_2)$ for all ψ_1, ψ_2.

Now we shall consider the criteria which should be satisfied by any plausible object-probe interaction for position measurements. Our first criterion is that the noise of the measurement should be unbiased in the sense that the mean value of the outcome should be identical to the mean position of the object just prior to the measurement; i.e., we should require

$$\int_{-\infty}^{\infty} \bar{X} p(\bar{X} \mid \psi) d\bar{X} = \langle \psi \mid \hat{x} \mid \psi \rangle . \tag{19}$$

From Eq. (12) [or Eq. (7)] and assumption (11), for the present model, we have

$$\int_{-\infty}^{\infty} \bar{X} p(\bar{X} \mid \psi) d\bar{X} = c \langle \psi \mid \hat{x} \mid \psi \rangle . \tag{20}$$

This requires $c=1$. Our next criterion is that if ΔX tends to 0 then $\varepsilon(\psi)$ should converge to 0 for all ψ. By Eqs. (15) and (17) and assumption (11), we have

$$\varepsilon(\psi)^2 = (1-c)^2 \langle \psi \mid \hat{x}^2 \mid \psi \rangle + d^2 (\Delta X)^2 . \tag{21}$$

This requires also $c=1$. Our last criterion is that if ΔX tends to 0 then $\sigma(\psi)$ should converge to 0 for all ψ. By Eqs. (14), (16), and (18) and assumption (11), we have

$$\sigma(\psi)^2 = (a-c)^2 \langle \psi \mid \hat{x}^2 \mid \psi \rangle + (b-d)^2 (\Delta X)^2 . \tag{22}$$

Thus the last criterion requires $a=c$. According to the fact that $|\exp(Kt\underline{Z})|=1$ for all t, we have $ad-bc=1$ and hence these criteria require

$$a=c=1 \text{ and } b=d-1 . \tag{23}$$

The obvious physical meaning of the second and third criteria is that if the probe coordinate just prior to the measurement could be precisely prepared at 0, then the measurement would be an ideal one.

Now we shall consider the case where these criteria are satisfied so that $a=c=1$ and $b=d-1$. In this case every model is characterized by a single parameter d which determines the errors of the measurement as $\varepsilon(\psi) = |d| \Delta X$ and $\sigma(\psi) = \Delta X$ for all ψ. This simple result has the following remarkable conclusions: (i) These errors do not depend on the prior object state ψ. (ii) The resolution is the same for all d for a fixed probe preparation. (iii) The $(1,-2,2)$ model corresponds to the case $d=0$ and has the best possible errors $[\varepsilon(\psi)=0,\ \sigma(\psi)=\Delta X]$ among all d. (iv) There are continuously many different possibilities ($|d|<1$) of position measurements still better than the von Neumann model which corresponds to $d=1$.

In order to clarify conclusions (iii) and (iv), we shall determine, from Eqs. (8)–(10), all $\underline{Z}$ that clears condition (23).

The case $|\underline{Z}|=0$: We have $d=1$, $K\tau=1/\gamma$, and

$$\underline{Z} = \frac{1}{K\tau} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \tag{24}$$

Thus this case is reduced to the von Neumann model.

This case $|\underline{Z}|>0$: We have $-3<d<1$, $\cos DK\tau = (1+d)/2$, and

$$\underline{Z} = \frac{D}{2\sin DK\tau} \begin{pmatrix} 1-d & 2(d-1) \\ 2 & d-1 \end{pmatrix} . \tag{25}$$

Thus all models with $|d|<1$ are included in this periodic case. In particular, the $(1,-2,2)$ model corresponds to the case $d=0$.

The case $|\underline{Z}|<0$: We have $1<d$, $\cosh EK\tau = (1+d)/2$, and

$$\underline{Z} = \frac{E}{2\sinh EK\tau} \begin{pmatrix} 1-d & 2(d-1) \\ 2 & d-1 \end{pmatrix} . \tag{26}$$

Since $\sigma(\psi_1)=\Delta X<d\Delta X=\varepsilon(\psi_2)$ for all ψ_1, ψ_2, all models in this case satisfy the SQL and hence they are less interesting than the other cases.

Thus all values of d with $-3<d$ is obtained from the coupling of Eq. (2). In particular, the periodicity of the solution for the case $|d|<1$ is an interesting conclusion. In this connection, Bondurant[12] proposed an improvement of the interferometric position measurement with a Kerr cell and a feedback loop and realized a position measurement similar to our periodic case ($|d|<0$). Indeed, for the case $|d|<1$, we have $d^2\hat{x}(t)/dt^2 = -K^2|\underline{Z}|\hat{x}(t)$. Thus our model appears to have a feedback force proportional to $\hat{x}(t)$, which is achieved by the optimal adjustment of the Kerr cell in the Bondurant model[12] if we neglect the detection noise.

In view of the recent development of the generation of squeezed states[13] with $\Delta X \ll \hbar/\sqrt{2}$, we can conclude that the effort to realize the measurement similar to the models with $|d| \ll 1$ will achieve much more precise measurements than the presently supposed standard quantum limit.

I thank H. P. Yuen and his colleagues at Northwestern University for their hospitality and helpful discussions during my sabbatical year at which time this work was undertaken. I am indebted to Y. Yamamoto and G. Björk for calling my attention to Ref. 12.

*On leave from the Department of Mathematics, College of General Education, Nagoya University, Nagoya 464, Japan.

[1]J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton Univ. Press, Princeton, NJ, 1955), pp. 442–445.
[2]C. M. Caves, Phys. Rev. Lett. **54**, 2465 (1985).
[3]C. M. Caves, Phys. Rev. D **33**, 1643 (1986); **35**, 1815 (1987).
[4]C. M. Caves and G. J. Milburn, Phys. Rev. A **36**, 5543 (1987).
[5]See, e.g., V. B. Braginsky and Yu. I. Vorontsov, Ups. Fiz. Nauk **114**, 41 (1974) [Sov. Phys. Usp. **17**, 644 (1975)]; C. M. Caves, K. S. Throne, R. W. P. Drever, V. D. Sandberg, and M. Zimmermann, Rev. Mod. Phys. **52**, 341 (1980).
[6]M. Ozawa, Phys. Rev. Lett. **60**, 385 (1988).
[7]M. Ozawa, in *Squeezed and Nonclassical Light,* edited by P. Tombesi and E. R. Pike (Plenum, New York, 1989), p. 263.
[8]H. P. Yuen, Phys. Rev. Lett. **51**, 719 (1983).
[9]See, for a general approach, A. Barchielli, Phys. Rev. D **32**, 347 (1985). The couplings discussed in this article are excluded from his choice of interactions [cf. Eq. (3.11) of his paper].
[10]See, for a rigorous discussion, Eq. (62) in Ref. 7 and the discussions there. See also, Eq. (2.8) in Ref. 4.
[11]See Ref. 6 and, for rigorous discussion, Theorem 3 of Ref. 7.
[12]R. S. Bondurant, Phys. Rev. A **34**, 3927 (1986).
[13]See, e.g., J. Opt. Soc. Am. B **4**, No. 10 (1987).