Higher-order criteria for nonclassical effects in photon statistics

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Based on inequalities derived from the partial ordering by majorization, some higher-order criteria for the existence of nonclassical effects in photon statistics are introduced. These generalize the well-known concept of two-photon antibunching to many-photon antibunching. The photon-number distribution of the famous two-photon coherent state is used to illustrate these new criteria.

Nonclassical properties of a radiation field such as squeezing and photon antibunching are currently of great interest.[1] The intensive studies in this area thus far have been focused almost exclusively on the lowest-order effects. However, Hong and Mandel[2] recently introduced the concept of higher-order squeezing of a quantum field. Inspired by their work, we introduce in this paper the concept of higher-order antibunching, which is also called sub-Poisson photon-number distribution. The correspondence between antibunching and sub-Poisson distribution has been established by Mandel[3] through the so-called Poisson transform. Therefore, we consider antibunching and the sub-Poissonian distribution as equivalent.

In this paper, we will consider the case of single-mode radiation only.

Let the photon-number distribution of a radiation field be described by $p(n)$ which gives the probability of finding n photons in the field; and let the mth moment of the distribution be defined as $\langle n^m \rangle \equiv \sum_{n=0}^{\infty} n^m p(n)$. Then, in a classical field, we always have

$$\langle n^2 \rangle - \langle n \rangle^2 \geq \langle n \rangle \ . \tag{1}$$

However, in a quantum field, it is possible to see the direction of the inequality sign of (1) reversed. For a coherent state with Poisson distribution, (1) becomes an exact equality; so a coherent state stands on the borderline between classical and nonclassical states. Therefore, in 1979, Mandel[4] introduced a parameter defined as follows:

$$Q \equiv (\langle n^2 \rangle - \langle n \rangle^2 - \langle n \rangle)/\langle n \rangle \ ; \tag{2}$$

$Q<0$ then implies the existence of nonclassical state.

In 1963, Glauber[5] and Sudarshan[6] independently introduced the P representation that gives a quasiprobability distribution in the phase space for a quantum state. It is called a quasiprobability distribution because this P function can assume negative values, which would be nonsense in the classical domain. Therefore our criterion for a nonclassical state is that its P function is not positive semidefinite.

The first step of our work is to establish the following inequality for classical fields:

$$\langle n^{(l+1)} \rangle \langle n^{(m-1)} \rangle \geq \langle n^{(l)} \rangle \langle n^{(m)} \rangle \ , \tag{3}$$

with $l \geq m$, where $\langle n^{(m)} \rangle \equiv \langle n(n-1)\cdots(n-m+1) \rangle$ is a factorial moment. After translating into the P representation, we have

$$\begin{aligned}\langle n^{(l)} \rangle \langle n^{(m)} \rangle &= \int d^2\alpha \pi^{-1} P(\alpha) |\alpha|^{2l} \\ &\quad \times \int d^2\beta \pi^{-1} P(\beta) |\beta|^{2m} \\ &= \tfrac{1}{2} \int \int d^2\alpha \, d^2\beta \pi^{-2} P(\alpha,\beta) \\ &\quad \times (|\alpha|^{2l}|\beta|^{2m} + |\alpha|^{2m}|\beta|^{2l}) \ ,\end{aligned} \tag{4}$$

where $P(\alpha,\beta) \equiv P(\alpha)P(\beta)$ is the symmetric joint probability distribution. We can also put $\langle n^{(l+1)} \rangle \langle n^{(m-1)} \rangle$ in a similar form.

According to the definition and notation of the theory of majorization,[7] for $l \geq m$, we always have

$$(l+1, m-1) \succ (l,m) \ , \tag{5}$$

which means the left-hand side majorizes the right-hand side, which in turn implies

$$|\alpha|^{2l+2}|\beta|^{2m-2} + |\alpha|^{2m-2}|\beta|^{2l+2} \geq |\alpha|^{2l}|\beta|^{2m} + |\alpha|^{2m}|\beta|^{2l} \ . \tag{6}$$

Therefore, as long as $P(\alpha)$ is positive semidefinite, inequality (3) must be true.

Then, if it ever occurs that the direction of the inequality sign of (3) is reversed, it implies that $P(\alpha)$ must assume negative values somewhere in the phase space. Therefore, the criteria for the existence of nonclassical effects can be expressed as

$$\langle n^{(l+1)} \rangle \langle n^{(m-1)} \rangle < \langle n^{(l)} \rangle \langle n^{(m)} \rangle \ ; \tag{7}$$

or, to put it in normalized form,

$$R(l,m) \equiv \langle n^{(l+1)} \rangle \langle n^{(m-1)} \rangle / \langle n^{(l)} \rangle \langle n^{(m)} \rangle - 1 < 0 \ . \tag{8}$$

For the case of two-photon antibunching, in particular, (7) becomes

$$R(1,1) = [(\langle n^2 \rangle - \langle n \rangle) - \langle n \rangle^2]/\langle n \rangle^2 < 0 \ , \tag{9}$$

which is a little different from Mandel's Q parameter given in (2). The definition for the Q parameter has at least two advantages: (1) The numerical value is usually enhanced because the denominator is smaller. (2) For a photon number (Fock) state with $p(n)=\delta_{n,k}$, which is the extreme of nonclassical states, $Q=-1$, a simple constant. Of course, we are hesitant to deviate from established tradition. However, we feel that advantage (1) is somewhat artificial; usually we measure the degree of deviation from equality of two numbers by comparing the difference between the two numbers with one of them as the reference, such as the common definition of percentage error. Advantage (2) is very desirable. Unfortunately, there is no suitable simple expression that can maintain this advantage for higher-order antibunching. So we decided to sacrifice these advantages for the sake of simplicity in expression.

The classical inequality (3) is the most elementary or irreducible type of inequality. We can easily extend it to unlimited number of inequalities. For example, we have

$$\langle n^{(l+2)}\rangle\langle n^{(m-2)}\rangle-\langle n^{(l)}\rangle\langle n^{(m)}\rangle$$
$$=[\langle n^{(l+2)}\rangle\langle n^{(m-2)}\rangle-\langle n^{(l+1)}\rangle\langle n^{(m-1)}\rangle]$$
$$+[\langle n^{(l+1)}\rangle\langle n^{(m-1)}\rangle-\langle n^{(l)}\rangle\langle n^{(m)}\rangle]\geq 0 . \quad (10)$$

Therefore, using the same reasoning, we can extend the criteria for nonclassical effects as follows

$$R_k(l,m)\equiv\langle n^{(l+k)}\rangle\langle n^{(m-k)}\rangle/\langle n^{(l)}\rangle\langle n^{(m)}\rangle-1<0 . \quad (11)$$

The above criteria are still not the most general ones, because they are confined within products of two factorial moments. The concept of majorization can provide inequalities involving products of unlimited number of factorial moments. But we doubt about their practical significances.

As an illustration, we present the many-photon antibunching of the famous two-photon coherent states introduced by Yuen.[8] Using Yuen's original notations, the two-photon coherent states $|\beta\rangle_g$ are eigenstates of the "annihilation" operator b with eigenvalue β; i.e., $b|\beta\rangle_g=\beta|\beta\rangle_g$, where $b\equiv\mu a+\nu a^\dagger$ with $|\mu|^2-|\nu|^2=1$ and a ($a^\dagger$) is the photon annihilation (creation) operator.

The photon-number distribution for a two-photon coherent state is known to be

$$p(n)=\frac{1}{2^n n!}\frac{1}{|\mu|}t^n\exp\left[\frac{2z^*zt-(z^{*2}+z^2)t^2}{1-t^2}\right]\times H_n(z^*)H_n(z) , \quad (12)$$

where $t\equiv|\nu/\mu|$, $z\equiv\beta/\sqrt{2\mu\nu}$, and $H_n(z)$ is the Hermite polynomial.

We first try to determine where antibunching might occur by carrying out a three-dimensional plot of $R(1,1)$ over the complex plane $z=x+iy$ at a fixed value for $t=0.5$, with the positive-value part clipped off, as given in Fig. 1. It is obvious that there exists symmetry between positive and negative x and also between positive and negative y. From Fig. 1 we can see that antibunching occurs in the neighborhood of the x axis. We have also tried the same thing for $R(2,1)$, $R(3,1)$, and $R_2(2,2)$; the situations are all very similar.

We then focus our attention along the positive direction of the x axis. We compare four criteria: $R(1,1)$, $R(2,1)$, $R(3,1)$, and $R_2(2,2)$, corresponding to lines a, b, c, and d, respectively, in Fig. 2 and Fig. 3. We are interest-

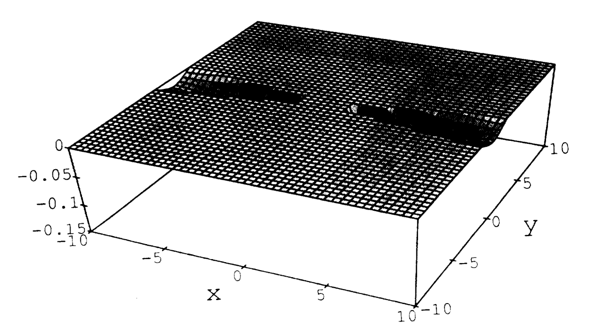

FIG. 1. Three-dimensional plot of $R(1,1)$ (for two-photon coherent state) over the z-complex plane at $t=0.5$ and with the positive-valued part clipped off; the "ditches" along both directions of the x axis are where antibunching occur.

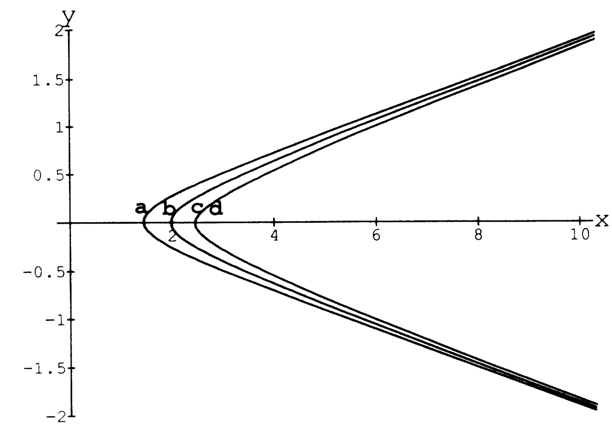

FIG. 2. Magnified "map" (with stretched y axis) of the borderlines of antibunching regions of two-photon coherent state characterized by line a, $R(1,1)$; line b, $R(2,1)$; line c, $R(3,1)$; and line d, $R_2(2,2)$. Note that lines c and d are almost indistinguishable.

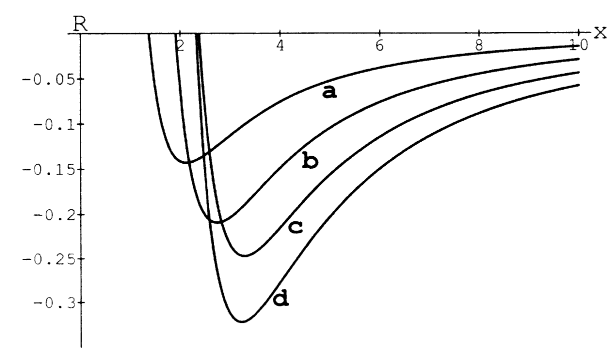

FIG. 3. Depth of the antibunching effects of a two-photon coherent state, occurring along the positive x-axis, indicated by the negative values of line a, $R(1,1)$; line b, $R(2,1)$; line c, $R(3,1)$; and line d, $R_2(2,2)$.

ed in knowing how widespread and how deep into the nonclassical region the various kinds of antibunching occur. Figure 2 shows plots of borderlines for the existence of antibunching; we notice that line *c* and line *d* are almost indistinguishable, and we also notice that lower-order antibunching is always a little more widespread than higher-order one. Figure 3 shows the depth of antibunching; we see that the higher-order ones are deeper than the lower-order ones, the case of $R_2(2,2) \equiv \langle n^{(4)} \rangle / \langle n^{(2)} \rangle^2 - 1$ is especially noteworthy.

From Figs. 2 and 3, we see that the lower-order antibunching is only a little more widespread than the higher-order one; but the latter goes much deeper than the former. Therefore, if a single example can give any reliable clue, we would speculate that, perhaps, the higher-order antibunching is more prominent.

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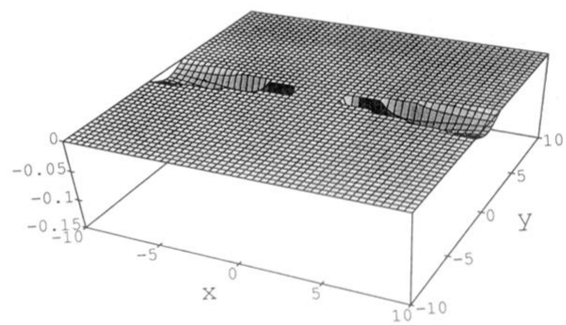

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