SU(2) and SU(1,1) phase states

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Phase operators and phase states are introduced in the Hilbert space H_{2j+1} associated with the SU(2) group. The phase operators obey the SU(2) algebra and play a dual role to the standard angular-momentum operators. A finite Weyl group plays a fundamental role in those ideas. In the SU(1,1) case the exponential of the phase operators is nonunitary, and the phase states form an overcomplete set which is used to formulate an analytic representation.

I. INTRODUCTION

Phase operators and phase states have been studied in the harmonic-oscillator context in Ref. 1. Here we extend these ideas in the context of SU(2) and SU(1,1). Both of these groups play an important role in many quantum-optics problems, e.g., in the study of squeezed states, parametric amplifiers, frequency converters, interferometers, etc.

The difficulties with hermiticity that phase operators have in the harmonic oscillator disappear in the context of the compact SU(2) group. In Sec. II we consider the standard $(2j+1)$-dimensional Hilbert space associated with the usual angular-momentum operators J_z, J_+, and J_-. We show that there are three phase operators θ_z, θ_+, and θ_- which obey the SU(2) algebra and play a dual role in relation to the J_z, J_+, and J_-. We also show that the phase states $|\theta;jn\rangle$ form an orthonormal basis and play a dual role in the standard angular-momentum basis for which we use the notation $|J;jm\rangle$. An arbitrary state can be expressed in terms of the $|J;jm\rangle$ basis or in terms of the $|\theta;jn\rangle$ basis, and this leads correspondingly to the J and θ representations; the two are related through a finite Fourier transform. An essential role in those arguments is played by a finite Weyl group similar to the one studied in Ref. 2 in a general context. Here we use this group in our own context and explain its importance for the angular-momentum Hilbert space.

In Sec. III we study a group of unitary phase transforms which contains as a subgroup the finite Weyl group. This generalizes considerably the phase transformations which have been used in Ref. 3 in a different context and which extended the work of Ref. 4. The present framework, which is based on a finite-dimensional Hilbert space, is much more suitable for the exposition and generalization of these ideas.

In Sec. IV we study phase states and phase operators in the context of the noncompact SU(1,1) group. The associated Hilbert space is now infinite dimensional. We express the SU(1,1) generators in terms of a "radial" operator K_r and the "exponential of the phase" operators E_+ and E_-. The E_+ and E_- are nonunitary and they are similar to the ones used in the harmonic-oscillator case.[1] The eigenstates of E_- for the harmonic-oscillator case have been studied in Ref. 5 and have been called phase states. Here we extend these arguments to the SU(1,1) case. We show that our phase states form an overcomplete set of states and that they can be used to define an "analytic representation" in which each state $|f\rangle$ is represented by a function $f(z)$ which is analytic in the unit disc $|z|<1$. Analytic representations are very useful because they can exploit the very powerful theory of analytic functions. Bargmann[6] has developed an analytic representation for the harmonic-oscillator Hilbert space using the overcomplete basis of coherent states. Here we study an analytic representation for our Hilbert space using the overcomplete basis of phase states. We conclude in Sec. V with a discussion of our results and comments concerning applications to specific quantum-optics models.

II. SU(2) PHASE STATES

Let J_z, $J_+=J_x+iJ_y$, and $J_-=J_x-iJ_y$ be the usual generators of the SU(2) algebra.

$$[J_z,J_\pm]=\pm J_\pm,\quad [J_+,J_-]=2J_z\ . \tag{1}$$

We shall introduce later the θ_z, θ_+, and θ_- operators which will also obey the SU(2) algebra. For this reason we use the notation $|J;jm\rangle$ for the usual basis,

$$\begin{aligned}
&J^2|J;jm\rangle=j(j+1)|J;jm\rangle\ ,\\
&J_z|J;jm\rangle=m|J;jm\rangle\ ,\\
&J_+|J;jm\rangle=[j(j+1)-m(m+1)]^{1/2}|J;jm+1\rangle\ ,\\
&J_-|J;jm\rangle=[j(j+1)-m(m-1)]^{1/2}|J;jm-1\rangle\ ,
\end{aligned}\tag{2}$$

and we shall use the notation $|\theta;jn\rangle$ for the analogous basis with respect to the θ operators. The j takes integer values (and we shall refer to them as the Bose sector) or half-integer values (Fermi sector). In order to simplify the notation, we allow m to take all the integer (or half-integer) values modulo $2j+1$, (e.g., $|J;jj+1\rangle=|J;j-j\rangle$, etc.).

The $|J;jm\rangle$ form an orthonormal basis in a $(2j+1)$-dimensional Hilbert space that we call H_{2j+1},

$$\begin{aligned}
&\sum_{m=-j}^{j}|J;jm\rangle\langle J;jm|=\mathbb{1}\ ,\\
&\langle J;jm|J;jn\rangle=\delta_{mn}\ .
\end{aligned}\tag{3}$$

In our notation the symbol δ_{mn} is equal to 1 if m is equal to n modulo $2j+1$. We shall study in this paper various interesting properties of the space H_{2j+1} (with fixed j) and of the operators acting upon it. The Casimir operator is (Schur's lemma)

$$J^2=J_z^2+\tfrac{1}{2}(J_+J_-+J_-J_+)=j(j+1)\mathbb{1} \ . \tag{4}$$

We replace the "Cartesian" operators J_+ and J_- with the "radial" operator J_r and the "exponential of the phase" operator E

$$J_r=(J_+J_-)^{1/2} \ , \tag{5}$$

$$E=\sum_{m=-j}^{j}|J;jm+1\rangle\langle J;jm| \ . \tag{6}$$

Unlike the harmonic-oscillator case,[1] the E is here a unitary operator,

$$E^{-1}=E^{+}=\sum_{m=-j}^{j}|J;jm\rangle\langle J;jm+1| \ . \tag{7}$$

We can now prove that

$$\begin{aligned} &J_+=J_rE \ , \\ &J_-=E^{+}J_r \ , \\ &[J_r,J_z]=0 \ , \\ &J_r|J;jm\rangle=[j(j+1)-m(m-1)]^{1/2}|J;jm\rangle \ , \\ &J_r=[j(j+1)\mathbb{1}-J_z^2+J_z]^{1/2} \ . \end{aligned} \tag{8}$$

An easy way of proving (8) and other similar relations is to express all the operators in the $|J;jm\rangle\langle J;jn|$ basis,

$$\begin{aligned} &J_+=\sum_{m=-j}^{j-1}[j(j+1)-m(m+1)]^{1/2}|J;jm+1\rangle\langle J;jm| \ , \\ &J_-=\sum_{m=-j}^{j-1}[j(j+1)-m(m+1)]^{1/2}|J;jm\rangle\langle J;jm+1| \ , \end{aligned} \tag{9}$$

$$J_z=\sum_{m=-j}^{j}m|J;jm\rangle\langle J;jm| \ .$$

It is clear that the J_r is a Hermitian operator which commutes with J_z,J^2. The "cosine" and "sine" operators are defined as

$$\begin{aligned} &C=\tfrac{1}{2}(E+E^{-1}), \quad S=\frac{1}{2i}(E-E^{-1}) \ , \\ &C^2+S^2=1, \quad [C,S]=0 \end{aligned} \tag{10}$$

and are Hermitian operators.

In order to find the eigenstates of E we consider separately the Bose sector ($j=1,2,\ldots$) and the Fermi sector ($j=\tfrac{1}{2},\tfrac{3}{2},\ldots$). In the Bose sector the eigenstates of E are

$$|\theta;jn\rangle=(2j+1)^{-1/2}\sum_{m=-j}^{j}\exp\left[i\frac{2\pi}{2j+1}nm\right]|J;jm\rangle \ , \tag{11}$$

where n is an integer modulo $(2j+1)$. We have used the notation $|\theta;jn\rangle$ because we will see that they are eigenstates of the operators θ^2 and θ_z, which are introduced later.

In the Fermi sector the eigenstates of E are

$$|\theta;jn\rangle=(2j+1)^{-1/2}\sum_{m=-j}^{j}\exp\left[i\frac{2\pi}{2j+1}(n+\tfrac{1}{2})(m+\tfrac{1}{2})\right]|J;jm\rangle \ , \tag{12}$$

in the Fermi sector.

where n is a half-integer modulo $(2j+1)$. The fact that the n of Eq. (11) and the $(n+\tfrac{1}{2})$ of Eq. (12) take integer values is essential for proof that these states are eigenstates of E. The $\tfrac{1}{2}$ of the $(m+\tfrac{1}{2})$ in Eq. (12) is not so essential in the sense that it only gives an overall phase factor to the state (12). On the other hand, it simplifies the notation and makes it easier to see the duality between the J and θ operators later. We refer to the states (11) and (12) as phase states. We can show

$$\begin{aligned} &E|\theta;jn\rangle=\exp(-i\theta_n)|\theta;jn\rangle \ , \\ &C|\theta;jn\rangle=\cos\theta_n|\theta;jn\rangle \ , \\ &S|\theta;jn\rangle=\sin(-\theta_n)|\theta;jn\rangle \ , \end{aligned} \tag{13}$$

where

$$\theta_n=\frac{2\pi n}{2j+1} \tag{14}$$

in the Bose sector and

$$\theta_n=\frac{2\pi(n+\tfrac{1}{2})}{2j+1} \tag{15}$$

in the Fermi sector.

A useful equality that is needed in the proof of a lot of the relations given below is

$$(2j+1)^{-1}\sum_{n=-j}^{j}\exp[i\theta_n(m-m')]=\delta_{mm'} \ . \tag{16}$$

The phase states of Eqs. (11) and (12) form an orthonormal basis in H_{2j+1},

$$\begin{aligned} &\langle\theta;jn|\theta;jm\rangle=\delta_{nm} \ , \\ &\sum_{n=-j}^{j}|\theta;jn\rangle\langle\theta;jn|=\mathbb{1} \ . \end{aligned} \tag{17}$$

It is now easy to invert Eqs. (11) and (12) to get

$$|J;jm\rangle=(2j+1)^{-1/2}\sum_{n=-j}^{j}\exp\left[-i\frac{2\pi nm}{2j+1}\right]|\theta;jn\rangle \ , \tag{18}$$

$$|j;jm\rangle=(2j+1)^{-1/2}\times\sum_{n=-j}^{j}\exp\left[-i\frac{2\pi}{2j+1}(n+1/2)(m+1/2)\right]\times|\theta;jn\rangle\ , \qquad (19)$$

in the Bose and Fermi sectors, correspondingly.

We next give the following relations for the phase states:

$$\begin{aligned}&\langle C\rangle=\langle\theta;jn|C|\theta;jn\rangle=\cos\theta_n\ ,\\&\Delta C^2=\langle C^2\rangle-\langle C\rangle^2=0,\quad \langle J_z\rangle=0\ ,\\&\Delta J_z^2=(2j+1)^{-1}\sum_{m=-j}^{j}m^2\ ,\\&\langle J_r\rangle=(2j+1)^{-1}\sum_{m=-j}^{j}[j(j+1)-m(m-1)]^{1/2}\ ,\\&\langle J_x\rangle=\cos\theta_n\langle J_r\rangle,\quad \langle J_y\rangle=-\sin\theta_n\langle J_r\rangle\ .\end{aligned}\qquad (20)$$

These relations provide an insight into the nature of the phase states and should be compared and contrasted to the following relations for the states $|J;jm\rangle$:

$$\begin{aligned}&\langle C\rangle=\langle J;jm|C|J;jm\rangle=0\ ,\\&\Delta C^2=\langle C^2\rangle-\langle C\rangle^2=\tfrac{1}{2},\quad \langle J_z\rangle=m,\quad \Delta J_z^2=0\ ,\\&\langle J_r\rangle=[J(J+1)-m(m-1)]^{1/2},\quad \langle J_x\rangle=0\ ,\\&\langle J_y\rangle=0\ .\end{aligned}\qquad (21)$$

We see that the states $|J;jm\rangle$ have a definite value of J_z ($\Delta J_z=0$) and an indefinite value of the angle ($\Delta C\neq 0$); the states $|\theta;jn\rangle$ have a definite value of the angle ($\Delta C=0$) and an indefinite value of J_z ($\Delta J_z\neq 0$).

It is clear from (13) that

$$E=\sum_{n=-j}^{j}\exp(-i\theta_n)|\theta;jn\rangle\langle\theta;jn|\ . \qquad (22)$$

The phase operator θ_z can be introduced as

$$\theta_z=\sum_{n=-j}^{j}n|\theta;jn\rangle\langle\theta;jn|\ , \qquad (23)$$

and clearly has the $|\theta;jn\rangle$ as eigenstates. The relation between E and θ_z is

$$E=\exp\left[-i\frac{2\pi}{2j+1}\theta_z\right]\ , \qquad (24)$$

$$E=\exp\left[-i\frac{\pi}{2j+1}\right]\exp\left[-i\frac{2\pi}{2j+1}\theta_z\right]\ , \qquad (25)$$

in the Bose and Fermi sectors, respectively.

We introduce the operator F which is dual to the operator E as

$$F=\exp\left[i\frac{2\pi}{2j+1}J_z\right]\ , \qquad (26)$$

$$F=\exp\left[i\frac{\pi}{2j+1}\right]\exp\left[i\frac{2\pi}{2j+1}J_z\right]\ , \qquad (27)$$

in the Bose and Fermi sectors, respectively. It is easily proved that

$$E^{2j+1}=1\ , \qquad (28)$$

$$F^{2j+1}=1\ . \qquad (29)$$

The unitary operators

$$\left\{E^kF^l\exp\left[i\frac{2\pi m}{2j+1}\right]\middle|k,l,m\in Z_{2j+1}\right\}\ , \qquad (30)$$

where Z_{2j+1} is the set of integers modulo $(2j+1)$, form a discrete, finite Weyl group similar to the one that has been studied in Ref. 2 in a different context. Indeed, using (6), (13), (11), (26), (12), and (27) we prove

$$\begin{aligned}&E^k|J;jm\rangle=|J;jm+k\rangle\ ,\\&E^k|\theta;jn\rangle=\exp(-ik\theta_n)|\theta;jn\rangle\ ,\end{aligned}\qquad (31)$$

$$\begin{aligned}&E^kJ_zE^{-k}=J_z-k\mathbb{1}\ ,\\&E^k\theta_zE^{-k}=\theta_z\ ,\end{aligned}\qquad (32)$$

$$\begin{aligned}&F^l|J;jm\rangle=\exp(il\theta_m)|J;jm\rangle\ ,\\&F^l|\theta;jn\rangle=|\theta;jn+l\rangle\ ,\end{aligned}\qquad (33)$$

$$\begin{aligned}&F^lJ_zF^{-l}=J_z\ ,\\&F^l\theta_zF^{-l}=\theta_z-l\mathbb{1}\ ,\end{aligned}\qquad (34)$$

where θ_n and θ_m are given in Eqs. (14) and (15) for the Bose and Fermi sectors, respectively. Using (31)–(34) we easily prove

$$F^lE^k=E^kF^l\exp\left[i\frac{2\pi}{2j+1}lk\right]. \qquad (35)$$

If the above group were continuous, infinitesimal k and l in Eq. (35) would lead to $[J_z,\theta_z]=i\mathbb{1}$. This is, however, not the case, and the $[J_z,\theta_z]=i\mathbb{1}$ cannot be inferred. At the end of this section, we will calculate the $[J_z,\theta_z]$ and show explicitly that it is not equal to $i\mathbb{1}$.

In analogy to Eq. (8) we introduce the operators

$$\begin{aligned}&\theta_+=\theta_rF\ ,\\&\theta_-=F^+\theta_r\ ,\\&\theta_r=[j(j+1)\mathbb{1}-\theta_z^2+\theta_z]^{1/2}\ .\end{aligned}\qquad (36)$$

Using (36) and (34) we can prove

$$\begin{aligned}&\theta_+|\theta;jn\rangle=[j(j+1)-n(n+1)]^{1/2}|\theta;jn+1\rangle\ ,\\&\theta_-|\theta;jn\rangle=[j(j+1)-n(n-1)]^{1/2}|\theta;jn-1\rangle\ ,\end{aligned}\qquad (37)$$

which we use to prove

$$\begin{aligned}&[\theta_+,\theta_-]=2\theta_z\ ,\\&[\theta_z,\theta_\pm]=\pm\theta_\pm\ .\end{aligned}\qquad (38)$$

We see that the θ operators obey the SU(2) algebra. The corresponding Casimir operator is

$$\theta^2=\theta_z^2+\tfrac{1}{2}(\theta_+\theta_-+\theta_-\theta_+)=j(j+1)\mathbb{1} . \quad (39)$$

It is clear that there exists a duality between the θ operators and their eigenstates and the J operators and their eigenstates.

Let $|S\rangle$ be a state in the Hilbert space H_{2j+1}. In the "J_z representation" this state is represented by the $(2j+1)$ complex numbers $a_m=\langle J;jm|S\rangle$; in the "$\theta_z$ representation" it is represented by the $(2j+1)$ complex numbers $b_n=\langle\theta;jn|s\rangle$. The two representations are related through the Fourier transform,

$$a_m=(2j+1)^{-1/2}\sum_{n=-j}^{j}b_n\exp\left[i\frac{2\pi nm}{2j+1}\right] , \quad (40)$$

$$a_m=(2j+1)^{-1/2}\sum_{n=-j}^{j}b_n\exp\left[i\frac{2\pi}{2j+1}(n+\tfrac{1}{2})(m+\tfrac{1}{2})\right] , \quad (41)$$

in the case of the Bose and Fermi sectors, respectively.

As an example of the J-θ duality we present the SU(2) coherent states and their dual counterparts. The SU(2) coherent states can be introduced by one of the following equivalent definitions:[7,8] (i)

$$|J;\psi\phi\rangle=\exp(aJ_+-a^*J_-)|J;j-j\rangle ,$$
$$a=-\tfrac{1}{2}\psi e^{-i\phi},\quad 0\le\psi\le\pi,\quad 0\le\phi\le 2\pi \quad (42)$$

(ii)

$$J_n|J;\psi\phi\rangle=-j|J;\psi\phi\rangle ,$$
$$n=(\sin\psi\cos\phi,\sin\psi\sin\phi,\cos\psi) ,$$
$$J=(J_x,J_y,J_z) ,$$
$$J_n=nJ \quad (43)$$

or (iii)

$$|J;z\rangle=(1+|z|^2)^{-j}\sum_m\left[\frac{(2j)!}{(j+m)!(j-m)!}\right]^{1/2}\times z^{j+m}|J;jm\rangle , \quad (44)$$
$$z=-\tan(\tfrac{1}{2}\psi)e^{-i\phi} .$$

It is clear from (43) that just as the $|J;j-j\rangle$ is the eigenstate of J_z with eigenvalue $-j$, the $|J;\psi\phi\rangle$ is the eigenstate of J_n with eigenvalue $-j$. And of course for $\psi=0$ we have $|J;\psi=0,\phi\rangle=|J;j-j\rangle$.

We call the above states J-coherent states, and it is clear that we can define analogous θ-coherent states as follows: (i)

$$|\theta;\psi\phi\rangle=\exp(a\theta_+-a^*\theta_-)|\theta;j-j\rangle ,$$
$$a=-\tfrac{1}{2}\psi e^{-i\phi},\quad 0\le\psi\le\pi,\quad 0\le\phi\le 2\pi \quad (45)$$

(ii)

$$\theta_n|\theta;\psi\phi\rangle=-j|\theta;\psi\phi\rangle ,$$
$$n=(\sin\psi\cos\phi,\sin\psi\sin\phi,\cos\psi) ,$$
$$\theta=(\theta_x,\theta_y,\theta_z)$$
$$\theta_n=n\theta , \quad (46)$$

and (iii)

$$|\theta;z\rangle=(1+|z|^2)^{-j}\sum_m\left[\frac{(2j)!}{(j+m)!(j-m)!}\right]^{1/2}z^{j+m}|\theta;jm\rangle , \quad (47)$$
$$z=-\tan(\tfrac{1}{2}\psi)e^{-i\phi} .$$

We next calculate the matrix elements $\langle J;jm|\theta_z|J;jn\rangle$. Using (23), (11), and (12) we get

$$\begin{aligned}\theta_{mn}&=\langle J;jm|\theta_z|J;jn\rangle\\&=\sum_{k=-j}^{j}k\langle J;jm|\theta;jk\rangle\langle\theta;jk|J;jn\rangle\\&=(2j+1)^{-1}\sum_{k=-j}^{j}k\exp[i\theta_k(m-n)]\\&=\begin{cases}0 \text{ if } m=n,\\ r_{mn}^{j+1+w}(r_{mn}-1)^{-1} \text{ if } m\ne n ,\\ r_{mn}=\exp\left[i\frac{2\pi}{2j+1}(m-n)\right] ,\end{cases}\end{aligned} \quad (48)$$

where θ_k has been defined in (14) and (15) and w is 0 and $\frac{1}{2}$ in the Bose and Fermi sectors, respectively. We see that all the diagonal elements are equal to zero. This should be compared and contrasted with the $\langle J;jm|J_z|J;jn\rangle$ where all the nondiagonal elements are equal to zero. We can now easily calculate the matrix elements of the operator $[J_z,\theta_z]$,

$$\langle J;jm|[J_z,\theta_z]|J;jn\rangle=(m-n)\langle J;jm|\theta_z|J;jn\rangle . \quad (49)$$

This confirms that the $[J_z,\theta_z]$ is not equal to $i\mathbb{1}$. The θ operators and the phase states for $j=\frac{1}{2},1$ are given in a matrix form in Appendix A.

III. UNITARY PHASE TRANSFORMATIONS

In this section we introduce more general operators than the E,F. In the sum of Eq. (6) we introduce a phase factor to each of the terms and get more general operators. In a different context, something similar has been done in Ref. 3, as a generalization of the work of Ref. 4. The present context, which uses a finite-dimensional Hilbert space, is much more suitable for the extension of these ideas.

We consider the $(2j+1)$ unitary operators

$$U_k(\Gamma)=\sum_{m=-j}^{j}\exp(i\gamma_{m+k,m})|J;jm+k\rangle\langle J;jm| , \quad (50)$$

where $k=-j,\dots,j$ and $\Gamma=\{\gamma_{m+k,m}|m=-j,\dots,j\}$ is a set of $(2j+1)$ real numbers. We can easily prove

$$U_k(\Gamma)U_k^\dagger(\Gamma)=U_k^\dagger(\Gamma)U_k(\Gamma)=\mathbb{1} , \quad (51)$$

$$U_k(B)U_l(\Gamma)=U_{k+l}(S=B+\Gamma) , \quad (52)$$

where $S=B+\Gamma$ is the set

$$S=\{S_{m+k+l,m}=\beta_{m+k+l,m+l}+\gamma_{m+l,m}\,|\,m=-j,\ldots,j\}\ . \tag{53}$$

We recall that in our notation the $m+l$, $m+k+l$, $k+l$, etc., are defined modulo $(2j+1)$. The $\{U_k(\Gamma)\}$ form a group. Indeed, we see from (52) that the product of two elements of this set belongs to the set, and it is easy to prove that

$$\begin{aligned} &U_{k=0}(\Gamma=0)=\mathbb{1}\ ,\\ &U_k^{-1}(\Gamma)=U_k^{\dagger}(\Gamma)\ ,\\ &U_k(B)[U_l(\Gamma)U_m(\Delta)]=[U_k(B)U_l(\Gamma)]U_m(\Delta)\ . \end{aligned} \tag{54}$$

Note that the operators (30) are special cases of the operators (50). Indeed,

$$U_{k=0}(\Gamma\{\gamma_{m,m}=l\theta_m\})=F^l\ , \tag{55}$$

$$U_k(\Gamma=0)=E^k\ . \tag{56}$$

The Weyl group (30) is a subgroup of the more general group (50).

We next give some examples of these unitary transformations. The state

$$|S\rangle=\sum_{m=-j}^{J}S_m|J;jm\rangle \tag{57}$$

is transformed into

$$U_k(\Gamma)|S\rangle=\sum_{m=-j}^{J}S_m\exp(i\gamma_{m+k,m})|J;jm+k\rangle\ . \tag{58}$$

A general operator

$$A=\sum_{m,n}a_{mn}|J;jm\rangle\langle J;jn| \tag{59}$$

is transformed into

$$U_k(\Gamma)AU_k^{\dagger}(\Gamma)=\sum_{m,n}a_{mn}\exp[i(\gamma_{m+k,m}-\gamma_{n+k,n})]\times|J;jm+k\rangle\langle J;jn+k|\ . \tag{60}$$

Using (60) we show that the J_z is transformed as

$$U_k(\Gamma)J_zU_k^{\dagger}(\Gamma)=J_z-k\mathbb{1}\ , \tag{61}$$

and the θ_z is transformed as

$$U_k(\Gamma)\theta_zU_k^{\dagger}(\Gamma)=\sum_{m,n}\theta_{mn}\exp[i(\gamma_{m+k,m}-\gamma_{n+k,n})]\times|J;jm+k\rangle\langle J;jn+k|\ , \tag{62}$$

where θ_{mn} has been given in (48). The dual operators to (50) are introduced as

$$V_k(\Gamma)=\sum_{m=-j}^{J}\exp(i\gamma_{m+k,m})|\theta;jm+k\rangle\langle\theta;jm|\ . \tag{63}$$

They also form a group which contains the Weyl group (30) as a subgroup,

$$V_{k=0}(\Gamma=\{\gamma_{m,m}=k\theta_m\})=E^k, \tag{64}$$

$$V_l(\Gamma=0)=F^l\ . \tag{65}$$

The operators $U_k(\Gamma)$ do not commute with the operators $V_l(\Delta)$. There is no simple expression for the

$$U_k(\Gamma)V_l(\Delta)U_k^{-1}(\Gamma)V_l^{-1}(\Delta);$$

only in the special case of the operators E^k and F^l do we have the result (35). Next we discuss two applications of the operators $U_k(\Gamma)$ and $V_l(\Delta)$.

(i) We consider all the states $|S'\rangle=U_{k=0}(\Gamma)|S\rangle$ of Eq. (58) with $k=0$, fixed $|S\rangle$, and variable Γ. As in Ref. 3 we also consider random uniformly distributed Γ, in which case we get a mixed state described by the density matrix

$$\rho=\sum_{m=-j}^{J}|S_m|^2|J;jm\rangle\langle J;jm|\ . \tag{66}$$

Measurements of J_z cannot distinguish those states. Indeed, for any function $f(J_z)$, we can prove, using (61), that the $\langle S'|f(J_z)|S'\rangle$ is independent of Γ and is equal to $\mathrm{Tr}[\rho f(J_z)]$ for the ρ of Eq. (66). This is something we should have in mind when we use the results of quantum measurements.

(ii) Evolution operators $\exp(iHt)$ with Hamiltonians H which are complicated nonlinear functions $H(J_z)$ of J_z are special cases of the operators $U_{k=0}(\Gamma)$,

$$\exp[iH(J_z)t]=\exp\left[it\sum_m H(m)|J;jm\rangle\langle J;jm|\right]\ . \tag{67}$$

Using (58) we easily see that the evolution in time of a state like (57) is given by

$$|S;t\rangle=\sum_{m=-j}^{J}S_m\exp[itH(m)]|J;jm\rangle\ . \tag{68}$$

More generally, we believe that the operators $U_k(\Gamma)$ and $V_l(\Delta)$ can be useful in the study of the evolution of systems with nonlinear Hamiltonians and in the understanding of current quantum-optics problems like the collapse and revival of quantum coherence, the propagation in nonlinear materials, etc.

IV. SU(1,1) PHASE STATES

Let K_0, K_+, and K_- be operators which satisfy the SU(1,1) algebra

$$[K_0,K_\pm]=\pm K_\pm,\quad [K_-,K_+]=2K_0\ . \tag{69}$$

The various representations of SU(1,1) have been studied in Refs. 8 and 9. Here we consider the so-called discrete series of representations and introduce the usual basis

$$\begin{aligned} &K^2|k;\mu\rangle=k(k-1)|k,\mu\rangle,\quad K_0|k,\mu\rangle=\mu|k;\mu\rangle\ ,\\ &K_+|k;\mu\rangle=[\mu(\mu+1)-k(k-1)]^{1/2}|k;\mu+1\rangle\ ,\\ &K_-|k;\mu\rangle=[\mu(\mu-1)-k(k-1)]^{1/2}|k;\mu-1\rangle\ ,\\ &k=\tfrac{1}{2},1,\tfrac{3}{2},\ldots,\quad \mu=k+N\ , \end{aligned} \tag{70}$$

where N is a non-negative integer. The set

$$\{|k;\mu=k+N\rangle|k\text{:fixed};N=0,1,2,\ldots\}\tag{71}$$

forms an orthonormal basis in an infinite-dimensional Hilbert space that we call H_k

$$\sum_{N=0}^{\infty}|k;k+N\rangle\langle k;k+N|=\mathbb{1}\ ,$$
$$\langle k;k+N|k;k+M\rangle=\delta_{NM}\ ,\tag{72}$$

In this paper we study some aspects of the Hilbert space H_k (with fixed k). The Casimir operator is (by Schur's lemma)

$$K^2=K_0^2-\tfrac{1}{2}(K_+K_-+K_-K_+)=k(k-1)\mathbb{1}\ .\tag{73}$$

We now replace the "Cartesian operators" K_+ and K_- with the "polar" operators

$$K_r=(K_+K_-)^{1/2}\ ,$$
$$E_+=\sum_{N=0}^{\infty}|k;k+N+1\rangle\langle k;k+N|\ ,\tag{74}$$
$$E_-=E_+^{\dagger}=\sum_{N=0}^{\infty}|k;k+N\rangle\langle k;k+N+1|\ .$$

The K_r is the "radial" operator and the E_+ and E_- are the "exponential of the phase" operators. We can prove that

$$K_+=K_rE_+,\quad K_-=E_-K_r,\quad E_+E_-=1-|k;k\rangle\langle k;k|\ ,$$
$$E_-E_+=1,\quad K_r^2=K_0^2-K_0-k(k-1)\mathbb{1}\ ,\quad [K_r,K_0]=0\ .\tag{75}$$

An easy way of proving Eqs. (75) and other similar relations is to express all the operators in the $|k;\mu\rangle\langle k,\nu|$ basis

$$K_0=\sum_{\mu}\mu|k;\mu\rangle\langle k,\mu|\ ,$$
$$K_+=\sum_{\mu}[\mu(\mu+1)-k(k-1)]^{1/2}|k;\mu+1\rangle\langle k;\mu|\ ,$$
$$K_-=\sum_{\mu}[\mu(\mu+1)-k(k-1)]^{1/2}|k;\mu\rangle\langle k;\mu+1|\ ,\tag{76}$$
$$K_r=\sum_{\mu}[\mu(\mu-1)-k(k-1)]^{1/2}|k;\mu\rangle\langle k;\mu|\ ,$$
$$\mu=k,k+1,\ldots\ .$$

We see from (75) that as in the harmonic-oscillator case,[1] the E_+ and E_- are not unitary operators. We should like to emphasize, however, that the E_+ is an isometric operator, i.e., it preserves the scalar product. Indeed, the $E_-E_+=1$ easily leads to the result $\langle S_1|E_+^{\dagger}E_+|S_2\rangle=\langle S_1|S_2\rangle$ for any states $|S_1\rangle,|S_2\rangle$. In finite-dimensional Hilbert spaces the concept of isometry is equivalent to the concept of unitarity. In infinite-dimensional Hilbert spaces the former is a weaker concept and the E_+ is an example of an isometric but nonunitary transformation.[10] A unitary transformation maps a Hilbert space onto itself. An isometric but nonunitary transformation maps a Hilbert space onto one of its subspaces; for example, the E_+ maps H_k onto $H_k-\{|k;k\rangle\}$. The eigenkets of the operator E_- are

$$|z\rangle=(1-|z|^2)^{1/2}\sum_{N=0}^{\infty}z^N|k;k+N\rangle,\quad |z|<1$$
$$E_-|z\rangle=z|z\rangle\ .\tag{77}$$

We refer to them as phase states. Similar states have been studied in Ref. 5 in the harmonic-oscillator context. For $k=\frac{1}{2}$ the states (77) are identical to the SU(1,1) coherent states;[8] but for $k=1,\frac{3}{2},\ldots,$ they are different from the SU(1,1) coherent states. The requirement $|z|<1$ is essential for the normalizability of the states $|z\rangle$. There are no eigenkets of the operator E_+. The states $|z\rangle$ form an overcomplete set. This can be easily seen if we consider all the states $||z|e^{i\phi}\rangle$ with fixed $|z|<1$ and $0\le\phi<2\pi$; multiplication of both sides of Eq. (77) by $e^{-iN\phi}$ and integration lead to expressions for all the $|k;k+N\rangle$ in terms of the $||z|e^{i\phi}\rangle$. A stronger result concerning the overcompleteness of the $|z\rangle$ states is given in Appendix B.

Let $|f\rangle$ be an arbitrary (normalized) state in the Hilbert space H_k,

$$|f\rangle=\sum_{N=0}^{\infty}f_N|k;k+N\rangle;\quad \sum_{N=0}^{\infty}|f_N|^2=1\ .\tag{78}$$

We define the "analytic representation" by mapping the state $|f\rangle$ into

$$f(z)=(1-|z|^2)^{-1/2}\langle z^*|f\rangle=\sum_{N=0}^{\infty}f_Nz^N\ .\tag{79}$$

The sum in (79) converges for $|z|<1$. The factor $(1-|z|^2)^{-1/2}$ cancels the nonanalytic factor $(1-|z|^2)^{1/2}$ of Eq. (77) and the function $f(z)$ is analytic in the unit disc $D=\{|z|<1\}$. We give two examples: the state $|k;k+N\rangle$ is represented by the function z^N and the phase state $|z_0\rangle$ by the function $(1-zz_0)^{-1}$.

The fact that $\sum_N|f_N|^2=1$ implies that the limit

$$\lim_{\rho\to 1}f(z=\rho e^{i\theta})=\sum_{N=0}^{\infty}f_Ne^{iN\theta}=f(\theta)\tag{80}$$

exists and it is called the "boundary function." The integral

$$\int_0^{2\pi}|f(\theta)|^2d\theta=\sum_{N=0}^{\infty}|f_N|^2=1\tag{81}$$

also exists, and for wave functions like (78) which are normalized to 1, is equal to 1. The functions $f(z)$ that have those properties belong to the so-called Hardy space $H^2(D)$. Hardy spaces have some very powerful properties and for this reason they have been studied extensively by mathematicians (e.g., Ref. 11). We next point out two basic properties of the Hardy spaces.

(i) Using the analyticity property we easily prove that the boundary function $f(\theta)$ uniquely determines the $f(z)$ in the whole disc

$$f(z=re^{i\theta})=\int_0^{2\pi}\frac{f(\theta)}{1-re^{i\theta}}d\theta\ .\tag{82}$$

(ii) In the Fourier expansion (80) of the boundary function $f(\theta)$, the N takes only non-negative values. This is associated with the fact that the $f(z)$ of Eq. (79) is an analytic function. This should be compared and contrasted with the Fourier expansion of a general periodic function $\phi(\theta)$ in $L^2[0;2\pi]$,

$$\phi(\theta)=\sum_{N=-\infty}^{\infty}\phi_N e^{iN\theta}\,, \tag{83}$$

where N takes both positive and negative values. The function $\phi(\theta)$ of (83) is not a boundary function in the Hardy space $H_2(D)$. The so-called "analytic projection" of $\phi(\theta)$ defined as[11]

$$\phi_1(\theta)=\sum_{N=0}^{\infty}\phi_N e^{iN\theta} \tag{84}$$

is a boundary function in $H_2(D)$. An alternative way of expressing this property is that the boundary function $f(\theta)$ must obey the

$$\int_0^{2\pi}f(\theta)e^{iN\theta}d\theta=0,\quad N=1,2,\ldots\,. \tag{85}$$

The importance of this property for phase operators has been pointed out in Ref. 12.

It is clear that the function $f(z)$ can be used to describe the general state $|f\rangle$ of Eq. (78). The scalar product of two such states is

$$\langle f|g\rangle=\int\frac{d\theta}{2\pi}f^*(\theta)g(\theta)=\sum_{N=0}^{\infty}f_N^*g_N\,. \tag{86}$$

The function $f(z)$ is in some sense the analog of the θ representation of the SU(2) case, which we studied in Sec. II. The θ basis there was orthonormal and it could be easily used. The $|z\rangle$ basis here is overcomplete but it leads to an analytic representation which could be useful for phase studies. For example, the phase operator E_+ can be easily represented as multiplication by z,

$$(1-|z|^2)^{1/2}\langle z^*|E_+|f\rangle=zf(z)\,. \tag{87}$$

The operator E_- is represented as

$$(1-|z|^2)^{1/2}\langle z^*|E_-|f\rangle=z^{-1}[f(z)-f_0]\,. \tag{88}$$

The subtraction of $f_0=f(0)$ ensures that the result is an analytic function.

An interesting property of the phase states (77), is that the distribution

$$P_N=|\langle k;k+N|z\rangle|^2\equiv|z|^{2N}(1-|z|^2) \tag{89}$$

is identical to a thermal distribution with temperature $T=(-\ln|z|^2)^{-1}$ (in units of $k_B=\hbar=1$). Note, however, that the state (77) is a pure state; the density matrix $|z\rangle\langle z|$ contains the diagonal elements associated with the distribution (89) and also nonzero off-diagonal elements. Only if the phase $\phi=\arg(z)$ becomes random with a uniform distribution between (0.2π), do the off-diagonal elements become zero and we have a mixed state described by the thermal density matrix,

$$\rho=\int_0^{2\pi}\frac{d\theta}{2\pi}|z\rangle\langle z| =(1-|z|^2)\sum_{N=0}^{\infty}|z|^{2N}|k;k+N\rangle\langle k;k+N|\,. \tag{90}$$

We conclude this section by briefly introducing operators analogous to those of Sec. III. Let

$$U_m(\Gamma)=\sum_{N=0}^{\infty}\exp(i\gamma_{N+M,N})|k;k+N+M\rangle\langle k;k+N|\,, \tag{91}$$

where $\Gamma=\{\gamma_{N+M,N}|N=0,1,\ldots\}$ is a sequence of real numbers. For $M=0$ the above operator is unitary; for $M=1,2,\ldots$, it is isometric but nonunitary. For $\Gamma=0$ we get

$$U_0(\Gamma=0)=\mathbb{1},\quad U_M(\Gamma=0)=E_+^M\,. \tag{92}$$

These operators could be used in arguments similar to those of Sec. III and Refs. 3 and 4. For example, the $U_0(\Gamma)$ could be used on the SU(1,1) coherent states[8]

$$|w\rangle=\exp(wK_+)(1-|w|^2)^{K_0}\exp(-w^*K_-)|k;k\rangle =(1-|w|^2)^k\sum_{N=0}^{\infty}\left[\frac{\Gamma(N+2k)}{\Gamma(2k)N!}\right]^{1/2}w^N|k;k+N\rangle \tag{93}$$

where $|w|<1$ to give the "generalized" SU(1,1) coherent states

$$U_0(\Gamma)|w\rangle=(1-|w|^2)^k\sum_{N=0}^{\infty}\left[\frac{\Gamma(N+2k)}{\Gamma(2k)N!}\right]^{1/2}\times W^N\exp(i\gamma_{N,N})|k;k+N\rangle\,. \tag{94}$$

They are "ordinary" SU(1,1) coherent states with respect to the operators [use Eq. (76)]

$$\begin{aligned}
K_0'&=U_0(\Gamma)K_0U_0^\dagger(\Gamma)=K_0\,,\\
K_+'&=U_0(\Gamma)K_+U_0^\dagger(\Gamma)=\sum_\mu[\mu(\mu+1)-k(k-1)]^{1/2}\exp[i(\gamma_{N+1,N+1}-\gamma_{N,N})]|k;k+N+1\rangle\langle k;k+N|\,,\\
K_-'&=U_0(\Gamma)K_-U_0^\dagger(\Gamma)=\sum_\mu[\mu(\mu+1)-k(k-1)]^{1/2}\exp[i(\gamma_{N,N}-\gamma_{N+1,N+1})]|k;k+N\rangle\langle k;k+N+1|\,,\\
\mu&=k+N\,,
\end{aligned} \tag{95}$$

in the sense that

$$U_0(\Gamma)|w\rangle=\exp(wK'_+)(1-|w|^2)^{K'_0}\exp(-w^*K'_-)|k,k\rangle\ . \tag{96}$$

This generalizes into SU(1,1) the ideas of Refs. 3 and 4.

V. CONCLUSIONS

We have considered phase operators and phase states in the angular-momentum Hilbert space H_{2j+1}. The phase operators θ_z, θ_+, and θ_- obey the SU(2) algebra and play a dual rule in relation to the angular-momentum operators J_z, J_+, and J_-. The phase states $|\theta;jn\rangle$ form an orthonormal set which is dual in relation to the $|J;jm\rangle$. The $|\theta;jn\rangle$ are states with a definite value of the angle and an indefinite value of J_z (i.e., $\Delta C=0$, $\Delta\theta_z=0$, and $\Delta J_z\neq 0$); the $|J;jm\rangle$ are states with a definite value of J_z and an indefinite value of the angle (i.e., $\Delta J_z=0$, $\Delta C\neq 0$, and $\Delta\theta_z\neq 0$). The finite Weyl group (30) plays an important role in these arguments. The J-θ duality leads to the SU(2) θ-coherent states of Eqs. (45)–(47).

We have also considered the SU(1,1) group and expressed its generators in terms of the "radial" operator K_r and the "exponential of the phase" operators E_+ and E_-. The E_+ and E_- are not unitary and the phase states (77) form an overcomplete set of states. These states have been used to introduce the analytic representation of Eq. (79).

The emphasis in this paper has been on the fundamental ideas but it is clear that this formalism can be applied to many quantum-optics problems. The two-mode Hamiltonian

$$H=\omega_1 a_1^\dagger a_1+\omega_2 a_2^\dagger a_2+\lambda a_1 a_2^\dagger+\lambda^* a_1^\dagger a_2 \tag{97}$$

that has been used in connection with frequency converters,[13] interferometers,[14] etc., can be studied with the Schwinger representation of SU(2),[15]

$$\begin{aligned} &J_+=a_1^\dagger a_2,\quad J_-=a_1 a_2^\dagger\ ,\quad J_z=\tfrac{1}{2}(a_1^\dagger a_1-a_2^\dagger a_2)\ ,\\ &J^2=[\tfrac{1}{2}(a_1^\dagger a_1+a_2^\dagger a_2)][\tfrac{1}{2}(a_1^\dagger a_1+a_2^\dagger a_2)+1]\ . \end{aligned} \tag{98}$$

The number eigenstates $|N_1=j+m, N_2=j-m\rangle$ correspond to the states $|J;jm\rangle$, and all the relations of Secs. II and III can be applied to this model.

The SU(1,1) group has been used in connection with parametric amplifiers,[13] interferometers,[14] squeezing,[16] etc. It has been explained in Ref. 16 that for the two-mode Hamiltonian

$$H=\omega_1 a_1^\dagger a_1+\omega_2 a_2^\dagger a_2+\lambda a_1 a_2+\lambda^* a_1^\dagger a^\dagger \tag{99}$$

the number eigenstates $|N_1=N+2k-1, N_2=N\rangle$ correspond to the states $|k;k+N\rangle$ and therefore the relations of Sec. IV can be applied to this model.

An interesting approach to the harmonic-oscillator phase states has been developed in Ref. 17 where a cutoff-number eigenstate $|S\rangle$ has been used in order to avoid the difficulties associated with the unitarity of E_+,E_-; all the results are calculated as a function of S and the limit $S\to\infty$ is considered at the end. In the same spirit, it would be interesting to investigate the harmonic-oscillator case as the limit $j\to\infty$ of our SU(2) model. Concepts like the Inonu-Wigner group contraction[18] might be helpful in this direction.

Phase operators have been used in various optics problems such as the analysis of squeezed states,[19] the analysis of phase measurements,[20] etc., and we hope that the ideas of this paper will be useful in pursuing further those studies.

APPENDIX A

We present here the θ operators in a matrix form for the simple cases $j=\frac{1}{2}$ and 1. The calculation is based on (48) and (37) and it is lengthy but straightforward. Using the standard basis

$$|J;\tfrac{1}{2}\tfrac{1}{2}\rangle=\begin{bmatrix}1\\0\end{bmatrix}\ ,\qquad |J;\tfrac{1}{2}-\tfrac{1}{2}\rangle=\begin{bmatrix}0\\1\end{bmatrix}\ , \tag{A1}$$

we express the θ operators as

$$\theta_z=\frac{1}{2}\begin{bmatrix}0&-1\\-1&0\end{bmatrix}\ ,\qquad \theta_+=\frac{1}{2}\begin{bmatrix}-1&-1\\1&1\end{bmatrix}\ ,\qquad \theta_-=\frac{1}{2}\begin{bmatrix}-1&1\\-1&1\end{bmatrix} \tag{A2}$$

and the phase states as

$$|\theta;\tfrac{1}{2}\tfrac{1}{2}\rangle=2^{-1/2}\begin{bmatrix}-1\\1\end{bmatrix},\quad |\theta;\tfrac{1}{2}-\tfrac{1}{2}\rangle=2^{-1/2}\begin{bmatrix}1\\1\end{bmatrix}\ . \tag{A3}$$

For the $j=1$ case we again use the standard basis

$$|J;1\,1\rangle=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad |J;1\,0\rangle=\begin{bmatrix}0\\1\\0\end{bmatrix},\quad |J;1-1\rangle=\begin{bmatrix}0\\0\\1\end{bmatrix} \tag{A4}$$

and express the θ operators as

$$\theta_z=3^{-1/2}\begin{bmatrix}0&i&-i\\-i&0&i\\i&-i&0\end{bmatrix}\ ,$$

$$\theta_+=\frac{\sqrt{2}}{3}\begin{bmatrix}2\lambda&\lambda+1&\lambda+\lambda^*\\1+\lambda&2&1+\lambda^*\\\lambda^*+\lambda&\lambda^*+1&2\lambda^*\end{bmatrix}\ ,\qquad \lambda=\exp\left[i\frac{2\pi}{3}\right]\ ,\qquad \theta_-=\theta_+^\dagger \tag{A5}$$

and the phase states as

$$|\theta;1\,1\rangle = 3^{-1/2}\begin{bmatrix}\lambda\\1\\\lambda^*\end{bmatrix},$$

$$|\theta;1\,0\rangle = 3^{-1/2}\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \text{(A6)}$$

$$|\theta;1-1\rangle = 3^{-1/2}\begin{bmatrix}\lambda^*\\1\\\lambda\end{bmatrix}.$$

APPENDIX B

In this appendix we prove the powerful statement that if z_N is a convergent series to some point z_0 of the unit disc $D=\{|z|<1\}$, then the set $\{|z_N\rangle\}$ of phase states is overcomplete. The proof is similar to that of Ref. 21 for ordinary coherent states and it is based on the theory of zeros of analytic functions.

If a state $|f\rangle$ is orthogonal to some phase state $|z_0\rangle$, then $f(z_0)=0$, i.e., the point z_0 is a zero of the analytic function $f(z)$. We know that, apart from the trivial case where $f(z)=0$ everywhere in the unit disc D, the zeros of the function $f(z)$ are "isolated" points. This means that there is no state $|f\rangle$ which is orthogonal to all $\{|z_N\rangle\}$ and this proves that this set is at least complete. In fact, it is overcomplete because the same argument is also valid even if we omit a finite number of terms from the sequence z_N.

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