Ordering problem in quantum mechanics: Prime quantization and a physical interpretation

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(Received 16 August 1989)

A method for quantizing a classical system, based upon the notion of localization on phase space, is developed. This method, to which is given the name *prime quantization*, is based upon the mathematical theory of positive-operator-valued measures and their relationship to reproducing kernel Hilbert spaces, and to a notion of generalized kinematical variables related to phase space. For a system with $\mathbb{R}^n$ as its configuration space, it has the advantage of being able to connect the well-known problem of ordering in quantum mechanics to the choice of a measuring apparatus in a joint measurement of position and momentum (within the limits of the uncertainty principle). Moreover, for such a system, a choice of ordering in the present context also turns out to be a choice of polarization in the method of geometric quantization.

I. INTRODUCTION

A. Scope of the paper

Two interconnected problems have been studied in this paper. The first concerns the passage from classical to quantum mechanics, based upon a notion of localization on phase space[1,2] and of generalized kinematical variables, and the second, the ordering of operators in quantum mechanics. It is our intention to demonstrate how such a procedure—to which we give the name *prime quantization*[3]—leads in the first instance to a physical interpretation of the ordering of operators and second, within the context of the theory of geometric quantization, to a physical justification for the choice of a polarization.[4,5] We also point out the essential differences between our quantization scheme and those of Mackey,[6] of the geometric school,[5] and the Borel quantization method.[7,8] Likewise, we demonstrate some close affinities between our choice of an ordering (or, what amounts to the same thing, of a reproducing kernel Hilbert space) and the algebraic method for determining a polarization, discussed earlier by Emch.[9] The exposition is first motivated, after a precise statement of the problem, by the well-known example of antinormal ordering in quantum mechanics. Certain mathematical aspects of this model are then isolated, which serve as the point of departure for a general formulation.

In simple terms, the problem of the ordering of operators in quantum mechanics may be stated as follows. Suppose that one has a classical observable. Specifically, this is a real-valued function f of the position variable q and the momentum variable p (assume, to begin with, that the configuration space of the system is $\mathbb{R}^1$, i.e., it moves on the phase space $T^*(\mathbb{R}^1)\cong\mathbb{R}^2$). If f admits a Taylor expansion, one has

$$f(q,p)=\sum_{m,n=0}^{\infty} c_{mn}q^m p^n \ , \tag{1.1}$$

where the c_{mn} are appropriate coefficients which could possibly be zero for $m,n>N$ for some integer N. In a quantized theory, the variables q and p are replaced by essentially self-adjoint unbounded operators Q and P, respectively, on a (separable) Hilbert space $\mathcal{H}$, satisfying the canonical commutation relations (over a common dense domain $\mathcal{D}$)

$$[Q,P]=i, \quad (\hbar=1) \ . \tag{1.2}$$

The question then arises as to what the self-adjoint operator F on $\mathcal{H}$, corresponding to the classical observable f, should be. It seems plausible to quantize f via

$$F=\sum_{m,n=0}^{\infty} c_{mn}Q^m P^n \ . \tag{1.3}$$

But even if we could give a meaning to the infinite sum of unbounded operators appearing in (1.3), and even if F were a self-adjoint operator on a dense domain $\mathcal{D}_F\subset\mathcal{H}$, we would still have to decide on a certain ordering of the noncommuting operators Q and P, when their products appear in (1.3), and this is the subject matter of the present paper. For instance, it is a matter of some arbitrariness as to whether the classical observable qp^2 is to be replaced by the quantum operator $\frac{1}{2}(QP^2+P^2Q)$ or $\frac{1}{4}(QP^2+2PQP+P^2Q)$, or some other self-adjoint combination.

In the literature (see, for example, Refs. 10–12), many different possibilities have been discussed. Some of these are mathematically equivalent; however, in our opinion, not enough physical justification is given for choosing one ordering over another. It is one of our objectives in this paper to show how a large class of orderings may ultimately be related to certain characteristics of the measuring apparatus. In other words, while mathematically arbitrary and in a sense equivalent, these various orderings correspond to different experimental situations, and are to be distinguished on physical grounds.

The discussion up to this point demonstrates that a quantization procedure, applied to a classical system, also provides us with a solution to the ordering problem, and hence we address ourselves, in this paper, to the question of developing such an ordering procedure as follows. Speaking geometrically, suppose that we are given a classical system moving on the manifold M. Its phase space Γ is the cotangent bundle $T^*(M)$, and its classical algebra of observables $\mathcal{A}_{\rm cl}$ is the set of all complex continuous functions on $T^*(M)$, vanishing at infinity. It is well known that $\mathcal{A}_{\rm cl}$ is a commutative C^* algebra (see, for example Ref. 13 for a definition, properties, etc.). The classical observables are then the real functions in this algebra. To quantize $\mathcal{A}_{\rm cl}$, we have to find a (separable) Hilbert space $\mathcal{H}$ and a linear mapping (*the quantization map*),

$$\pi^*:\ \mathcal{A}_{\rm cl}\to\mathcal{L}(\mathcal{H})\ , \tag{1.4}$$

of $\mathcal{A}_{\rm cl}$ into the set $\mathcal{L}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$. Actually, as will be seen later, both the domain and the range of the map π^* have to be enlarged eventually [to include unbounded operators, for example, see (1.2)]. Moreover, on the constant function $f=1$, π^* (possibly extended) ought to act as

$$\pi^*(1)=I\ , \tag{1.5}$$

where I is the identity operator on $\mathcal{H}$. In addition, we require the *nondegeneracy condition* that the C^* algebra generated by the set $\pi^*(\mathcal{A}_{\rm cl})$ be dense in $\mathcal{L}(\mathcal{H})$. To reduce an otherwise vast number of possibilities for the choice of π^*, we make some physical assumptions by introducing the notion of localization on phase space.[1] This then leads to the prescription of "*prime quantization.*"

B. Example of antinormal ordering for systems on $T^*(\mathbb{R}^1)$

It is immediately clear, that the linear map π^* implies, through the passage $f\in\mathcal{A}_{\rm cl}\mapsto\pi^*(f)\in\mathcal{L}(\mathcal{H})$, a certain ordering of the operators Q and P. This interplay between the quantization map and a given ordering of operators is brought out in the following simple example on $T^*(\mathbb{R}^1)$. Consider the so-called antinormal ordering, according to which, for example, the classical function [of the phase-space variables $(q,p)\in T^*(\mathbb{R}^1)$, but written in terms of the complex variable $z=(1/\sqrt{2})(q+ip)$] $f(z,z^*)=z^m z^{*n}$, where m and n are integers, is associated[12] with the quantum mechanical operator $a^m a^{\dagger n}$ (a is the usual annihilation and $a^\dagger$ the creation operator). It can be shown[12] that this ordering is characterized by a one-dimensional projection operator [one for each phase-space point (q,p)], in the Hilbert space of the system

$$2\pi T_A(q,p)=|q,p\rangle\langle q,p|\ . \tag{1.6}$$

Here, $|q,p\rangle$ is a Glauber coherent state[14] (a unit vector) centered at $z=(1/\sqrt{2})(q+ip)$, satisfying the eigenvalue equation

$$a|q,p\rangle=z|q,p\rangle\ . \tag{1.7}$$

Written in terms of the operator function T_A, the antinormal rule of ordering assumes the form

$$f\mapsto F_A=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}T_A(q,p)f(q,p)dq\,dp\ , \tag{1.8}$$

for a given classical observable $f\in\mathcal{A}_{\rm cl}$.

The last relationship can be viewed in two ways. It is first a procedure by which the Q and the P, appearing in the quantum representative F_A of a classical observable such as a monomial $q^m p^n$, say, are antinormally ordered, i.e., one can write $F_A=F_A(Q,P)$, and second, it defines a quantization map π^*_A in the sense of the above discussion, if we set

$$\pi^*_A(f)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}T_A(q,p)f(q,p)dq\,dp\ . \tag{1.9}$$

Indeed, the linearity of π^*_A follows from its integral representation, while from the well-known properties of coherent states[14–16] it follows that for $f=1$,

$$\pi^*_A(1)=I\ . \tag{1.10}$$

Furthermore, as proved in Ref. 17 (see also Ref. 1), the classical position and momentum observables q and p on $T^*(\mathbb{R}^1)$ are mapped by π^*_A to two operators which satisfy the canonical commutation relations

$$[\pi^*_A(q),\pi^*_A(p)]=iI\ . \tag{1.11}$$

Finally, the fact that the C^* algebra generated by $\pi^*_A(\mathcal{A}_{\rm cl})$ is dense in $\mathcal{L}(\mathcal{H})$ is a consequence of Theorem 1 below.

A linear rule of association $f\mapsto F(Q,P)$ also sets up a mapping from the states ρ of the quantum system (i.e., normalized density matrices) to normalized measures μ on the phase space, according to the relation

$$\begin{aligned}\operatorname{tr}[F(Q,P)\rho]&=\operatorname{tr}[\pi^*(f)\rho]\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(q,p)d\mu(q,p)\ .\end{aligned} \tag{1.12}$$

In the case of the antinormal ordering π^*_A, the measure μ_A corresponding to any density matrix ρ turns out to be positive, with

$$\begin{aligned}d\mu_A(q,p)&=\operatorname{tr}[T_A(q,p)\rho]dqdp\\&=\langle q,p|\rho|q,p\rangle\frac{dqdp}{2\pi}\ .\end{aligned} \tag{1.13}$$

The question of which orderings lead to positive measures on phase space had been considered to some extent in Ref. 12 and later much more generally in Ref. 18. A very elegant treatment of this problem in terms of affine maps on the Banach space of all trace class operators has been given in Ref. 19, thereby putting earlier results in an abstract setting. We adopt, in this paper, this latter formulation for the ordering of operators, which simultaneously maps the states to *positive* measures on phase space.

C. Building the mathematical model

The positivity condition imposed upon the measure μ has an interpretation in terms of localization on phase space.[1] In fact, if we assume that the system is localized in Γ, then corresponding to any (Borel) subset Δ of Γ—representing a localization volume—there should exist an observable $a(\Delta)\in\mathcal{L}(\mathcal{H})$ such that $\operatorname{tr}[a(\Delta)\rho]$ would give the probability of finding the quantum system (in the state ρ) localized in the volume Δ of phase space. (Of course, localization in phase space is to be interpreted in an appropriate sense[1] since position and momentum cannot be measured simultaneously with absolute accuracy for a quantum particle. The sense in which the prime quantization procedure automatically resolves this question is discussed more fully in Sec. IV.) If we insist on the probability interpretation of $\mathrm{tr}[a(\Delta)\rho]$, a comparison with (1.12) shows that we must have

$$\mathrm{tr}[a(\Delta)\rho]=\int_{\Gamma}\chi_{\Delta}(\zeta)d\mu(\zeta)=\mu(\Delta)\ , \tag{1.14}$$

$$\mu(\Gamma)=I\ , \tag{1.15}$$

where $\zeta\in\Gamma$ and χ_{Δ} is the characteristic function of the set Δ. Thus μ must, in fact, be a probability measure.

The phase space Γ, whether considered as the spectrum of $\mathcal{A}_{\mathrm{cl}}$, or as a cotangent bundle $T^*(M)$, can be equipped with a natural positive measure ν—the basic measure in the case of $\mathcal{A}_{\mathrm{cl}}$ (Ref. 13) or the volume form on $T^*(M)$—which has support on the whole of Γ. If in addition we assume that the number of quantum particles that can be accommodated in a unit phase space cell is finite (so that we may normalize to one particle per phase space cell of volume $\hbar$), then we can prove[1,18] the existence of a positive operator valued function

$$T:\ \Gamma\to\mathcal{L}(\mathcal{H})\ , \tag{1.16}$$

for which

$$a(\Delta)=\int_{\Delta}T(\zeta)d\nu(\zeta)\ . \tag{1.17}$$

Hence the measure μ in (1.14) becomes

$$d\mu(\zeta)=\mathrm{tr}[T(\zeta)\rho]d\nu(\zeta)\ . \tag{1.18}$$

Comparing this with (1.13), the localization operators in (1.17) for the antinormal ordering (on $\mathcal{H}$) are

$$a_A(\Delta)=\frac{1}{2\pi}\int_{\Delta}|q,p\rangle\langle q,p|dq\,dp\ . \tag{1.19}$$

The operators $a(\Delta)$ are clearly positive for all Δ, and, in fact, they define a *positive operator valued* (POV) measure having the properties

$$a(\phi)=0,\quad \phi=\text{null set} \tag{1.20a}$$

$$a(\Gamma)=I\ , \tag{1.20b}$$

$$a\left[\bigcup_{i\in J}\Delta_i\right]=\sum_{i\in J}a(\Delta_i),\quad \Delta_i\cap\Delta_j=\phi\ \text{ for } i\neq j\ . \tag{1.20c}$$

The sum in (1.20c) converges weakly, and J is a discrete index set. In addition, Eq. (1.17) expresses the fact that the POV measure a has the *bounded operator density* T. Finally, note that in terms of the POV measure a, the quantization map π^* in (1.4) becomes

$$\pi^*(f)=\int_{\Gamma}f(\zeta)da(\zeta)=\int_{\Gamma}T(\zeta)f(\zeta)d\nu(\zeta)\ , \tag{1.21}$$

which is clearly perceived as being the generalization of the antinormal rule of ordering (1.8).

The mapping π^* may be given an alternative, and in some sense a more useful, mathematical description as follows. Let $\mathcal{T}(\mathcal{H})$ be the Banach space (under the trace norm)[13] of all trace class operators on $\mathcal{H}$. It is well known[13] that $\mathcal{L}(\mathcal{H})$, equipped with the strong operator topology, is the Banach space dual of $\mathcal{T}(\mathcal{H})$. Consider also the Banach space $L^1(\Gamma,\nu)$ and its dual $L^{\infty}(\Gamma,\nu)$. From (1.18) it is clear that the measure μ has a density which is an element of $L^1(\Gamma,\nu)$. Hence, for a fixed POV measure a, the relations (1.17) and (1.18) imply a mapping π from $\mathcal{T}(\mathcal{H})^+$ to $L^1(\Gamma,\nu)^+$, which consequently can be continued[18,19] to a bounded linear map,[20] the dual of which is π^* (and hence the notation),

$$\pi:\ \mathcal{T}(\mathcal{H})\to L^1(\Gamma,\nu)\ . \tag{1.22}$$

The dual,

$$\pi^*:\ L^{\infty}(\Gamma,\nu)\to\mathcal{L}(\mathcal{H})\ , \tag{1.23}$$

is then the bounded linear map which in (1.21) defines an ordering of operators. Of course, as mentioned subsequent to Eq. (1.4), in specific cases it may be desirable, or indeed necessary, to enlarge both the domain and the range of π^*.

II. TWO RELATED MATHEMATICAL NOTIONS

The ordering of operators, when seen in the light of our discussion so far, can be crystallized (as will be shown in Sec. III) in two mathematical notions—the Naimark extension of a POV measure and a reproducing kernel Hilbert space. For the type of POV measures with which we are concerned, the two notions are interrelated (see Theorems A1 and A2 in the Appendix), and together form the heart of our quantization procedure (Sec. III). The effect of introducing the reproducing kernel Hilbert space amounts to choosing a coordinatization in the Hilbert space of quantum-mechanical states in the sense that certain preferred classical observables, namely, generalized position and momentum, which appear [on $T^*(\mathbb{R}^n)$, or for example, if $T^*(M)$ has a global chart] as multiplication by q and p, respectively, are picked up to go over to the quantum operators. For $T^*(\mathbb{R}^n)$, these latter satisfy the canonical commutation relations [see Eq. (1.11)]. We remark that up to this point, the Hilbert space $\mathcal{H}$ on which the map π^* in (1.4) and (1.21), or even the map π^*_A in (1.9), has been defined is still quite arbitrary. It was only assumed to be an abstract Hilbert space, admitting the normalized POV measure a, and has not, for example, been realized as a concrete space of functions. The property that the POV measure a admits the bounded positive density T [see Eq. (1.17)], actually allows us to construct such a realization for $\mathcal{H}$ in terms of certain functions on the phase space Γ, achieving thereby the above-mentioned coordinatization. (Note that, in this section, Γ could be any smooth manifold and not necessarily one with the structure of a phase space.)

A. Extension theorem

Let us recall a theorem due to Naimark (see, for example, Ref. 21), according to which any normalized POV measure a on an abstract Hilbert space $\mathcal{H}$ can be extended to a projection-valued (PV) measure $\tilde{P}$ on a larger Hilbert space $\tilde{\mathcal{H}}$, in a certain minimal sense. More specifically, let $\mathcal{B}(\Gamma)$ be the set of all Borel sets of Γ. Then, according to this theorem, there exists (i) a Hilbert space $\tilde{\mathcal{H}}$ on which there is defined a PV measure $\tilde{P}(\Delta)$, $\Delta \in \mathcal{B}(\Gamma)$; (ii) a subspace $\hat{\mathcal{H}}$ of $\tilde{\mathcal{H}}$, with corresponding projection operator $\mathbb{P}$,

$$\mathbb{P}\tilde{\mathcal{H}} = \hat{\mathcal{H}} \; ; \tag{2.1}$$

(iii) a unitary mapping W: $\mathcal{H} \rightarrow \hat{\mathcal{H}}$, such that

$$\mathbb{P}\tilde{P}(\Delta)\mathbb{P} = Wa(\Delta)W^{-1}, \quad \Delta \in \mathcal{B}(\Gamma) \; . \tag{2.2}$$

Additionally, $\tilde{\mathcal{H}}$ may be chosen to be minimal in the sense that every other extended space $\tilde{\mathcal{H}}'$ having properties (2.1) and (2.2) contains a subspace which is unitarily equivalent to $\tilde{\mathcal{H}}$. This minimal pair $\{\tilde{\mathcal{H}}, \tilde{P}\}$ is therefore uniquely (up to isomorphism) determined by $\mathcal{H}$ and a. The set of vectors

$$\hat{\mathcal{S}} = \{\tilde{P}(\Delta)\hat{\phi} \,|\, \Delta \in \mathcal{B}(\Gamma), \hat{\phi} \in \hat{\mathcal{H}} = W\mathcal{H}\} \tag{2.3}$$

is total in $\hat{\mathcal{H}}$ (i.e., its linear span is dense in $\hat{\mathcal{H}}$).

In the example of the POV measure a_A in (1.19), the extended space $\tilde{\mathcal{H}}$ is isomorphic[17] to $L^2(\mathbb{R}^2, dqdp)$, on which the operators $\tilde{P}(\Delta)$ take the form

$$[\tilde{P}(\Delta)\tilde{\Psi}](q,p) = \chi_\Delta(q,p)\tilde{\Psi}(q,p) \; , \tag{2.4}$$

$\tilde{\Psi} \in L^2(\mathbb{R}^2, dqdp)$. Using the subscript A to denote the various operators $\mathbb{P}_A, W_A$, etc. in the case of antinormal ordering, one can prove[1,16,22,23] that for $\phi \in \mathcal{H}$, its image Ψ in $\hat{\mathcal{H}} \equiv \mathcal{H}_A = \mathbb{P}_A \tilde{\mathcal{H}} = \mathbb{P}_A L^2(\mathbb{R}^2, dqdp)$ is obtained as

$$\Psi(q,p) \equiv (W_A\phi)(q,p) = \frac{1}{(2\pi)^{1/2}} \langle q,p | \phi \rangle \; . \tag{2.5}$$

Additionally, on $\mathcal{H}_A$ one has the POV measure $\hat{a}_A$, as the image of a_A,

$$\hat{a}_A(\Delta) = W_A a_A(\Delta) W_A^{-1} = \mathbb{P}_A \tilde{P}(\Delta) \mathbb{P}_A, \quad \Delta \in \mathcal{B}(\Gamma) \; , \tag{2.6}$$

while for the identity operator I on $\mathcal{H}$,

$$W_A I W_A^{-1} = \mathbb{P}_A \; . \tag{2.7}$$

The fact that here the extended space $\tilde{\mathcal{H}}$, and hence the subspace $\hat{\mathcal{H}}_A$ turn out to be spaces of phase-space functions (recall that $|q,p\rangle$ is a vector in an abstract Hilbert space $\mathcal{H}$) is not at all fortuitous, as will be seen in a little while. What is also worth noticing is that the functions $\Psi_A \in \hat{\mathcal{H}}_A$ are all bounded[22] (indeed even continuous) since by (2.5),

$$|\Psi(q,p)| \leq \frac{1}{(2\pi)^{1/2}} \|\phi\| \; . \tag{2.8}$$

On the other hand, given an arbitrary POV measure a and its extension $\tilde{P}$, we may use the latter to give an alternative expression for the quantization in (1.21), as follows. For a classical observable $f \in \mathcal{A}_{\text{cl}}$, let

$$\tilde{P}(f) = \int_\Gamma f(\zeta) d\tilde{P}(\zeta) \tag{2.9}$$

whenever the integral on the right is (weakly) defined. From (1.21) and (2.2) it is then clear that the quantization map π^* may be rewritten as

$$f \mapsto \pi^*(f) = \mathbb{P}\tilde{P}(f)\mathbb{P} \; . \tag{2.10}$$

[It is understood here that the operators $\pi^*(f)$ act on the Hilbert space $\hat{\mathcal{H}} = W\mathcal{H}$, more properly, combining (1.21) and (2.2) we should have written $\mathbb{P}\tilde{P}(f)\mathbb{P} = W\pi^*(f)W^{-1}$.]

B. Coordinatization in $\hat{\mathcal{H}}$

Having transferred the problem of constructing the quantization map π^* from the abstract Hilbert space $\mathcal{H}$ to the subspace $\hat{\mathcal{H}}$ of the canonically extended space $\tilde{\mathcal{H}} = L^2(\Gamma, d\nu)$, we see next how the fact that the POV measure a admits a positive density may be used to introduce a coordinatization in $\hat{\mathcal{H}}$. The key concept here is that of a reproducing kernel.

To explore this, let us go back to our example of the antinormal ordering, and note that from (2.5) and (2.8) it follows that the *evaluation map* $E_{q,p}^A$: $\mathcal{H}_A \rightarrow \mathbb{C}$ (where $\mathbb{C}$ is the set of all complex numbers), at the point $(q,p) \in \Gamma$, namely,

$$E_{q,p}^A(\Psi) = \Psi(q,p) \; , \tag{2.11}$$

is continuous.[22,24] This fact allows us to construct the reproducing kernel K_A,

$$K_A(q,p;q',p') \equiv E_{q,p}^A (E_{q',p'}^A)^* = \frac{1}{2\pi} \langle q,p | q',p' \rangle \; , \tag{2.12}$$

where $(E_{q,p}^A)^*$: $\mathbb{C} \rightarrow \mathcal{H}_A$ is the adjoint of the map $E_{q,p}^A$. The kernel K_A is, in fact, the integral kernel of the projection $\mathbb{P}_A$ in (2.7), i.e., for $\tilde{\Psi} \in L^2(\mathbb{R}^2, dqdp)$,

$$(\mathbb{P}_A \tilde{\Psi})(q,p) = \int_{\mathbb{R}^2} K_A(q,p;q',p') \tilde{\Psi}(q',p') dq'dp' \; . \tag{2.13}$$

The operators $E_{q,p}^A$ and $(E_{q,p}^A)^*$ may be used to write the operators $\hat{a}_A(\Delta)$ in (2.6). Indeed, from (2.4), (2.6), and (2.11) one easily derives that

$$\hat{a}_A(\Delta) = \int_\Delta (E_{q,p}^A)^* E_{q,p}^A dqdp \; , \tag{2.14}$$

and comparing with (1.17) and (1.19),

$$T_A(q,p) = (E_{q,p}^A)^* E_{q,p}^A = \frac{1}{2\pi} |q,p\rangle\langle q,p| \tag{2.15}$$

is the density for $\hat{a}_A$ in this representation. All the relationships in this example carry over to the general case, which we now examine. For this we need to define more accurately the mathematical concept of a reproducing kernel. The definition we give here is somewhat different from that found usually in the literature;[16,24,25] however, as pointed out in Ref. 22 it allows us to establish a deep connection between POV measures and reproducing kernel Hilbert spaces (see the Appendix).

As before, let Γ be the phase space of a classical system, ν a Borel measure on it, and suppose that to each $\zeta \in \Gamma$ we associate a Hilbert space $\mathcal{K}_\zeta$ in such a way that the direct integral Hilbert space (see, for example, Ref. 13)

$$\tilde{\mathcal{H}} = \int_\Gamma^\oplus \mathcal{K}_\zeta d\nu(\zeta) \tag{2.16}$$

exists. For $\zeta, \zeta' \in \Gamma$, let $\mathcal{L}(\mathcal{K}_\zeta, \mathcal{K}_{\zeta'})$ be the set of all bounded linear maps from $\mathcal{K}_\zeta$ to $\mathcal{K}_{\zeta'}$. For any $A\in\mathcal{L}(\mathcal{K}_\zeta,\mathcal{K}_{\zeta'})$, let $A^*\in\mathcal{L}(\mathcal{K}_{\zeta'},\mathcal{K}_\zeta)$ be its adjoint in the sense that

$$\langle v|Au\rangle_{\zeta'}=\langle A^*v|u\rangle_\zeta \ , \tag{2.17}$$

for all $u\in\mathcal{K}_\zeta$, $v\in\mathcal{K}_{\zeta'}$, and $\langle\ |\ \rangle_\zeta$ denoting the scalar product in $\mathcal{K}_\zeta$.

Definition 1. A reproducing kernel K on $\tilde{\mathcal{H}}$ is a mapping $(\zeta,\zeta')\in\Gamma\times\Gamma\mapsto K(\zeta,\zeta')\in\mathcal{L}(\mathcal{K}_{\zeta'},\mathcal{K}_\zeta)$, such that (i)

$$K(\zeta,\zeta')=K(\zeta',\zeta)^*,(\zeta,\zeta')\in\Gamma\times\Gamma \ ; \tag{2.18}$$

(ii) $K(\zeta,\zeta)$ is strictly positive definite as an operator in $\mathcal{L}(\mathcal{K}_\zeta)$, $\forall\zeta\in\Gamma$; (iii) the integral operator $\mathbb{P}_K$ on $\tilde{\mathcal{H}}$, such that

$$(\mathbb{P}_K\tilde{\Psi})(\zeta)=\int_\Gamma K(\zeta,\zeta')\tilde{\Psi}(\zeta')d\nu(\zeta') \ , \tag{2.19}$$

for all $\tilde{\Psi}\in\tilde{\mathcal{H}}$, exists and is bounded; (iv) for all $u\in\mathcal{K}_\zeta$ and $v\in\mathcal{K}_{\zeta''}$,

$$\int_\Gamma\langle u|K(\zeta,\zeta')K(\zeta',\zeta'')v\rangle_\zeta d\nu(\zeta')=\langle u|K(\zeta,\zeta'')v\rangle_\zeta \tag{2.20}$$

for all $(\zeta,\zeta'')\in\Gamma\times\Gamma$.

In the Appendix we discuss the relationship between reproducing kernel Hilbert spaces and POV measures admitting bounded positive densities. As shown by Theorems A1 and A2, this relationship is one to one. In this connection, the following results are pertinent to our discussion here.

(i) The operator $\mathbb{P}_K$ in (2.19) is a projection operator onto a subspace $\mathcal{H}_K$ of $\tilde{\mathcal{H}}$. Hence we refer to $\mathcal{H}_K$ as a *reproducing kernel Hilbert* space with kernel K.

(ii) On $\mathcal{H}_K$, the kernel K defines an evaluation map E_ζ^K: $\mathcal{H}_K\to\mathcal{K}_\zeta$,

$$E_\zeta^K(\Psi)=\Psi(\zeta)=\int_\Gamma K(\zeta,\zeta')\Psi(\zeta')d\nu(\zeta'),\quad \Psi\in\mathcal{H}_K \ . \tag{2.21}$$

[The measure theoretic technicality concerning the fact that $\Psi(\zeta)$ may actually be defined for *all* $\zeta\in\Gamma$ is discussed in the Appendix.]

(iii) In terms of E_ζ^K and its adjoint $(E_\zeta^K)^*$: $\mathcal{K}_\zeta\to\mathcal{H}_K$, the reproducing kernel is

$$K(\zeta,\zeta')=E_\zeta^K(E_{\zeta'}^K)^* \ , \tag{2.22}$$

and there is a canonically associated POV measure

$$a_K(\Delta)=\int_\Delta(E_\zeta^K)^*E_\zeta^K d\nu(\zeta),\quad \Delta\in\mathcal{B}(\Gamma) \tag{2.23}$$

on $\mathcal{H}_K$, such that $\tilde{\mathcal{H}}$ is the minimally extended space in the sense of Naimark.

(iv) Given an abstract Hilbert space $\mathcal{H}$ and a normalized POV measure a on it, which admits a positive ν density, we may find a unitarily equivalent [see Eqs. (A11)–(A13)] reproducing kernel Hilbert space $\mathcal{H}_K$, with canonically associated POV measure a_K. It is instructive to see how the latter is obtained, generalizing the construction in the example of antinormal ordering, i.e., Eqs. (2.5)–(2.7). Since each $T(\zeta)$ in (1.17) is a bounded positive operator, the square root $T(\zeta)^{1/2}$ exists. Let $\mathcal{N}_\zeta$ be its null space

$$\mathcal{N}_\zeta=\{\phi\in\mathcal{H}|T(\zeta)^{1/2}\phi=0\} \ . \tag{2.24}$$

Then $\mathcal{K}_\zeta$ is obtained by closing the quotient space $\mathcal{H}/\mathcal{N}_\zeta$ with respect to the scalar product

$$\langle[\phi]_\zeta|[\Psi]_\zeta\rangle_\zeta=\langle\phi|T(\zeta)\Psi\rangle_{\mathcal{H}} \ , \tag{2.25}$$

where $[\phi]_\zeta$ and $[\Psi]_\zeta$ are the equivalence classes in $\mathcal{H}/\mathcal{N}_\zeta$ of ϕ and Ψ, respectively. The spaces $\mathcal{K}_\zeta$ are then used to form $\tilde{\mathcal{H}}$ and the unitary map W_K embedding $\mathcal{H}$ onto a reproducing kernel subspace $\mathcal{H}_K\subset\tilde{\mathcal{H}}$ is

$$(W_K\phi)(\zeta)\equiv\Psi_K(\zeta)=[\phi]_\zeta \ . \tag{2.26}$$

(For technical details, see Ref. 22.)

(v) There exists an *overcomplete* set of vectors in $\mathcal{H}_K$ [i.e., a set whose linear span is dense in $\mathcal{H}_K$ and whose members satisfy the resolution of the identity condition (2.33) below]

$$\mathcal{S}_K=\{\xi_\zeta^v\in\mathcal{H}_K|\xi_\zeta^v(\zeta')=K(\zeta',\zeta)v,\zeta,\zeta'\in\Gamma,v\in\mathcal{K}_\zeta,\|v\|_\zeta=1\} \ . \tag{2.27}$$

If $d(\zeta)$ is the dimension [concerning the ζ dependence of $d(\zeta)$, see remark in the Appendix] of $\mathcal{K}(\zeta)$, $\{v_\zeta^i\}_{i=1}^{d(\zeta)}$ an orthonormal basis in $\mathcal{K}(\zeta)$, and ξ_ζ^i are the vectors in $\mathcal{S}_K$ such that

$$\xi_\zeta^i(\zeta')=K(\zeta',\zeta)v_\zeta^i \ , \tag{2.28}$$

then

$$K(\zeta,\zeta')=\sum_{i=1}^{d(\zeta)}\sum_{j=1}^{d(\zeta')}|v_\zeta^i\rangle K_{ij}(\zeta,\zeta')\langle v_{\zeta'}^j| \ , \tag{2.29}$$

$$K_{ij}(\zeta,\zeta')=\langle\xi_\zeta^i|\xi_{\zeta'}^j\rangle_{\mathcal{H}_K} \ , \tag{2.30}$$

and

$$T_K(\zeta)=\sum_{i=1}^{d(\zeta)}|\xi_\zeta^i\rangle\langle\xi_\zeta^i| \ . \tag{2.31}$$

(vi) In the special case where $\mathcal{K}_\zeta=\mathbb{C}$ for all ζ [so that $\tilde{\mathcal{H}}$ is isomorphic to $L^2(\Gamma,\nu)$], the operator $T_K(\zeta)$ in (2.31) becomes a scalar multiple of a one-dimensional projection operator, corresponding to a vector ξ_ζ, depending on K, and furthermore, the set of vectors $\xi_\zeta,\zeta\in\Gamma$ is the overcomplete set $\mathcal{S}_K$ in this case. The expressions for $K(\zeta,\zeta')$, $a_K(\Delta)$, etc. take [see Eqs. (2.11)–(2.15)] the forms

$$K(\zeta,\zeta')=\langle\xi_\zeta|\xi_{\zeta'}\rangle \ , \tag{2.32}$$

$$a_K(\Delta)=\int_\Delta T_K(\zeta)d\nu(\zeta),\quad T_K(\zeta)=|\xi_\zeta\rangle\langle\xi_\zeta| \ , \tag{2.33}$$

$$E_\zeta^K(\Psi)=\Psi(\zeta)=\langle\xi_\zeta|\Psi\rangle \ , \tag{2.34}$$

these expressions holding for all $\zeta\in\Gamma$ and $\Psi\in\mathcal{H}_K$. If A is any bounded operator on $\mathcal{H}_K$, then its kernel $A(\zeta,\zeta')$ [see Eqs. (A17)–(A19)] is simply

$$A(\zeta,\zeta')=\langle\xi_\zeta|A\xi_{\zeta'}\rangle \ . \tag{2.35}$$

This completes the coordinatization of $\mathcal{H}$. Indeed, in the isomorphic space $\mathcal{H}_K$, all vectors Ψ are functions of ζ and operators A are given via kernels which are functions on $\Gamma\times\Gamma$.

III. PRIME QUANTIZATION AND ORDERING

In this section we employ the results of the previous section to formulate the prime quantization procedure and discuss the related ordering. We also construct nontrivial examples to illustrate the theory in the case of $\Gamma=\mathbb{R}^3$, where we introduce kinematical variables via the extended Galilean group and discuss the related orderings.

A. Quantization map

First, we embed $\mathcal{A}_{\rm cl}$ into an algebra of operators on $L^2(\Gamma,\nu)$, using the PV measure in (A8). Thus $f\in\mathcal{A}_{\rm cl}$ is mapped to the operator $\tilde{P}(f)$ of multiplication on $L^2(\Gamma,\nu)$,

$$[\tilde{P}(f)\Psi](\zeta)=f(\zeta)\Psi(\zeta),\quad \Psi\in L^2(\Gamma,\nu)\ . \tag{3.1}$$

In a standard manner, the domain of the homomorphism $\tilde{P}$ may be extended to the set $L^\infty(\Gamma,\nu)$ of all bounded ν-measurable functions on Γ and its range to a commutative von Neumann algebra. In that case, for any ν-measurable function $f\in L^\infty(\Gamma,\nu)$,

$$\tilde{P}(f)=\int_\Gamma f(\zeta)d\tilde{P}(\zeta)\ , \tag{3.2}$$

where the integration is with respect to the PV measure $\tilde{P}$ defined in the Appendix, Eq. (A8). [Note that in this instance $\mathcal{H}_\zeta=\mathbb{C}$, for all $\zeta\in\Gamma$ in the direct integral (2.16), so that $\tilde{\mathcal{H}}\cong L^2(\Gamma,\nu)$.]

Definition 2. A prime quantization (hence, an ordering of operators) of the classical algebra $\mathcal{A}_{\rm cl}$ is a positive linear (quantization) map

$$\pi^*:\ L^\infty(\Gamma,\nu)\rightarrow\mathcal{L}(\mathcal{H})\ , \tag{3.3}$$

satisfying (i) the C^* algebra generated by the set,

$$\pi^*(\mathcal{A}_{\rm cl})=\{\pi^*(f)|f\in\mathcal{A}_{\rm cl}\}\ , \tag{3.4}$$

is weakly dense in $\mathcal{L}(\mathcal{H})$; (ii) $\mathcal{H}=\mathcal{H}_K$ is a reproducing kernel Hilbert subspace of $L^2(\Gamma,\nu)$. In other words, the projection operator $\mathbb{P}$ should have kernel K: $\Gamma\times\Gamma\rightarrow\mathbb{C}$, which is separately continuous in each variable, and

$$\pi^*(f)=\mathbb{P}\tilde{P}(f)\mathbb{P},\quad f\in L^\infty(\Gamma,\nu) \tag{3.5}$$

where, as indicated, π^* depends on the kernel K.

In view of the results of Sec. II, it follows that any prime quantization gives rise to a POV measure a_K [see (2.33)] on the Borel sets of the phase space Γ, for which

$$\pi^*(f)=\int_\Gamma f(\zeta)da_K(\zeta)=\int_\Gamma f(\zeta)(E_\zeta^K)^*E_\zeta^K d\nu(\zeta)\ . \tag{3.6}$$

Moreover, $\mathcal{H}$ and a_K determine $L^2(\Gamma,\nu)$ uniquely in the sense of Theorem A1. The unit constant function $f(\zeta)=1$, $\zeta\in\Gamma$ is mapped to the identity operator on $\mathcal{H}$ since, clearly,

$$\pi^*(1)=a_K(\Gamma)=\mathbb{P}\ . \tag{3.7}$$

The condition which ensures that our quantization map π^* be nondegenerate, in the sense that the C^* algebra generated by $\pi^*(\mathcal{A}_{\rm cl})$ be dense in $\mathcal{L}(\mathcal{H})$, is brought out in the next theorem. The condition is not sufficient, but is good enough for our purposes.

Theorem 1. If the phase space Γ, considered as a topological space, has no discrete part, then the prime quantization map π^* is nondegenerate.

Proof. Since the kernel K is fixed, we drop the index K. Let $[\pi^*(\mathcal{A}_{\rm cl})]''$ be the von Neumann algebra (see Ref. 13) generated by the set $\mathcal{M}$, which we take to be the C^* algebra generated by $\pi^*(\mathcal{A}_{\rm cl})$. Then, by von Neumann's density theorem[13] $\mathcal{M}$ is weakly dense in $[\pi^*(\mathcal{A}_{\rm cl})]''$. We show that $[\pi^*(\mathcal{A}_{\rm cl})]''=\mathcal{L}(\mathcal{H})$, by proving that the commutant $[\pi^*(\mathcal{A}_{\rm cl})]'$ of $[\pi^*(\mathcal{A}_{\rm cl})]''$ is $\{\mathbb{C}I\}$, I being the identity operator on $\mathcal{H}$. Consider the operators

$$a(\Delta)=\int_\Gamma\chi_\Delta(\zeta)da(\zeta)=\pi^*(\chi_\Delta)\ , \tag{3.8}$$

for all $\Delta\in\mathcal{B}(\Gamma)$. Clearly, $a(\Delta)\in[\pi^*(\mathcal{A}_{\rm cl})]''$, and let $A\in[\pi^*(\mathcal{A}_{\rm cl})]'$ be self-adjoint

$$a(\Delta)A=Aa(\Delta),\quad A=A^*\ . \tag{3.9}$$

From (3.6) it then follows that

$$E_\zeta^*E_\zeta A=AE_\zeta^*E_\zeta \tag{3.10}$$

for ν almost all $\zeta\in\Gamma$. If $\phi\in\mathcal{H}$ is arbitrary, then (3.10) implies that for all $\zeta'\in\Gamma$,

$$(E_\zeta^*E_\zeta A\phi)(\zeta')=(AE_\zeta^*E_\zeta\phi)(\zeta')\ ,$$

whence, by (2.21)

$$E_{\zeta'}E_\zeta^*E_\zeta A\phi=E_{\zeta'}AE_\zeta^*E_\zeta\phi\ , \tag{3.11}$$

for all $\zeta'\in\Gamma$, and for ν almost all ζ. Using (2.22), (A17), and (3.7), we get

$$\begin{aligned}\int_\Gamma K(\zeta',\zeta)E_\zeta AE_{\zeta''}^*E_{\zeta''}\phi d\nu(\zeta'')\\ =\int_\Gamma A(\zeta',\zeta)E_\zeta E_{\zeta''}^*E_{\zeta''}\phi d\nu(\zeta'')\\ =>\int_\Gamma K(\zeta',\zeta)A(\zeta,\zeta'')\phi(\zeta'')d\nu(\zeta'')=A(\zeta',\zeta)\phi(\zeta)\ .\end{aligned} \tag{3.12}$$

Since (3.12) holds for all $\phi\in\mathcal{H}$, we get

$$\int A(\zeta,\zeta')\phi(\zeta'')d\nu(\zeta'')=\frac{A(\zeta',\zeta)}{K(\zeta',\zeta)}\phi(\zeta) \tag{3.13a}$$

for $K(\zeta',\zeta)\neq0$, and

$$A(\zeta',\zeta)=0,\quad \text{for } K(\zeta',\zeta)=0\ . \tag{3.13b}$$

These two equations hold for all $\zeta'\in\Gamma$ and for ν almost all $\zeta\in\Gamma$. But since the left-hand side of (3.13a) is independent of ζ', we must have

$$\frac{A(\zeta',\zeta)}{K(\zeta',\zeta)}=f(\zeta)\quad \text{for } K(\zeta',\zeta)\neq0 \tag{3.14}$$

where f is a complex valued function of ζ, and since $K(\zeta',\zeta)$ is separately continuous in ζ and ζ', so also is

$A(\zeta',\zeta)$ (by Theorem A3). Thus $f(\zeta)$ may be defined for all ζ, and similarly (3.13b) holds for all $(\zeta',\zeta)\in\Gamma\times\Gamma$. Thus we may write

$$A(\zeta',\zeta)=g(\zeta)K(\zeta',\zeta)\ , \tag{3.15}$$

where

$$g(\zeta)=\begin{cases} f(\zeta), & \text{for } K(\zeta',\zeta)\neq 0 \\ 0, & \text{for } K(\zeta',\zeta)=0\ . \end{cases} \tag{3.16}$$

Combining (3.15) with (3.12) and (A18),

$$(A\phi)(\zeta)=g(\zeta)\phi(\zeta) \tag{3.17}$$

for all $\zeta\in\Gamma$ and $\phi\in\mathcal{H}$. By (2.34), this relation may be rewritten as

$$\langle \xi_\zeta | A\phi\rangle = g(\zeta)\langle \xi_\zeta|\phi\rangle\ ,$$

whence one gets the eigenvalue equation

$$A\xi_\zeta=g(\zeta)\xi_\zeta\ . \tag{3.18}$$

Since A is bounded and self-adjoint, and ξ_ζ is an eigenvector for each ζ, $g(\zeta)$ must lie in the discrete spectrum of A. On the other hand, $g(\zeta)$ is a continuous function by (3.15) and (3.16). Hence, since Γ does not contain a discrete part, therefore $g(\zeta)=\lambda$, a constant, for all $\zeta\in\Gamma$. Thus

$$A=\lambda I\ , \tag{3.19}$$

and since $[\pi^*(\mathcal{A}_{\text{cl}})]'$ is a von Neumann algebra, it is generated by all its self-adjoint elements, implying that for any $A\in[\pi^*(\mathcal{A}_{\text{cl}})]'$, (3.19) holds for some $\lambda\in\mathbb{C}$. This proves the theorem. Q.E.D.

As a further general comment on the quantization map π^*, we mention that to each density matrix ρ on $\mathcal{H}_K$, there exists a probability measure

$$d\mu(\zeta)=\text{tr}[(E_\zeta^K)^*E_\zeta^K\rho]d\nu(\zeta)\ , \tag{3.20}$$

as required by (1.18), and satisfying

$$\text{tr}[\pi^*(f)\rho]=\int_\Gamma f(\zeta)d\mu(\zeta)\ . \tag{3.21}$$

In view of Theorem 1, all quantum-mechanical expectation values can be written as limits of classical expectation values of the type given by the right-hand side of (3.21). Additionally, for an arbitrary Borel set $\Delta\subset\Gamma$,

$$\mu(\Delta)=\text{tr}[a_K(\Delta)\rho] \tag{3.22}$$

[compare with (1.14)], and hence the operators $a_K(\Delta)$ play the role of localization operators, if we interpret $\mu(\Delta)$ as being the probability that the quantum system in state ρ is localized in the phase-space volume Δ. This underscores the feature of the prime quantization program whereby localization on phase space Γ is exploited. Our results remain true for any manifold M. We have identified M with Γ because any choice of the kernel depends on a physical interpretation of some operators—preferably, generalized position and momentum—as kinematical variables on Γ. In the next section we shall discuss this point more from a physical point of view.

B. Free particle in $\mathbb{R}^3$

We now construct a kernel such that, in the case of free particles on $T^*(\mathbb{R}^3)$, we pick kinematical variables and recover by the prime quantization procedure the canonical commutation relations, in the sense of Eq. (1.11). Actually, we shall do much further, and prove that there do indeed exist interesting ordering procedures in this case, other than the antinormal ordering, which fall within the scope of Definition 2. We emphasize again, that apart from being given a classical algebra $\mathcal{A}_{\text{cl}}$, this method of quantization requires the existence of a natural measure ν on the classical phase space, as well as subspaces of the Hilbert space $L^2(\Gamma,\nu)$, which admit reproducing kernels (i.e., subspaces which in many realistic examples consist of continuous functions). Moreover, it is the subspace $\mathcal{H}_K$ which carries the quantized system. If these conditions are to be fulfilled, one would have to choose an appropriate kernel and as mentioned above, this can be done by defining kinematical variables, for example, through symmetry arguments or via a kinematical group. In the case of a free particle in $T^*(\mathbb{R}^3)\cong\mathbb{R}^6$, the kinematical group is the extended Galilei group $\tilde{G}$, and $\mathcal{H}_K$ should carry an irreducible representation of $\tilde{G}$. This forces the kernel to be $\tilde{G}$ covariant.

Consider now the classical C^* algebra $C_\infty(\Gamma)$ of all continuous functions on Γ, which vanish at infinity. For the measure ν on Γ, we take the Lebesgue measure $d\mathbf{q}d\mathbf{p}$. Note that if Θ is the phase subgroup and T the subgroup of time translations of $\tilde{G}$, then

$$\Gamma\cong\tilde{G}/\Theta\otimes T\otimes\text{SO}(3)\ , \tag{3.23}$$

as a homogeneous space, and $d\mathbf{q}d\mathbf{p}$ is the corresponding invariant measure[23] on Γ. Consider the Hilbert space

$$\tilde{\mathcal{H}}=L^2(\mathbb{R}^6,d\mathbf{q}d\mathbf{p})\ . \tag{3.24}$$

We construct now a subspace $\mathcal{H}_{e,l}$ of $\tilde{\mathcal{H}}$, which admits a reproducing kernel $K_{e,l}$. Let e be a square integrable, rotationally invariant function on $\mathbb{R}^3$, normalized to unity, i.e.,

$$\int_{\mathbb{R}^3}|e(\mathbf{k})|^2d\mathbf{k}=1\ , \tag{3.25}$$

$$e(R\mathbf{k})=e(\mathbf{k}),\quad R\in\text{SO}(3)\ , \tag{3.26}$$

the latter equation being true for almost all $\mathbf{k}\in\mathbb{R}^3$. For $l=0,1,2,3,\ldots$, define

$$K_{e,l}(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}')=\frac{2l+1}{(2\pi)^3}\int_{\mathbb{R}^3}\exp[i\mathbf{k}\cdot(\mathbf{q}-\mathbf{q}')]\times\mathcal{P}_l\left[\frac{(\mathbf{k}-\mathbf{p})\cdot(\mathbf{k}-\mathbf{p}')}{\|\mathbf{k}-\mathbf{p}\|\|\mathbf{k}-\mathbf{p}'\|}\right]\times\overline{e(\mathbf{k}-\mathbf{p})}e(\mathbf{k}-\mathbf{p}')d\mathbf{k}\ , \tag{3.27}$$

where $\mathcal{P}_l$ is a Legendre polynomial of order l. It is then possible to show[23] that $K_{e,l}$ is $\tilde{G}$ covariant and satisfies all the properties (2.18)–(2.20) of a reproducing kernel. Let $\mathbb{P}_{e,l}$ be the corresponding projection operator

$$(\mathbb{P}_{e,l}\tilde{\Psi})(\mathbf{q},\mathbf{p})=\int_{\mathbb{R}^6}K_{e,l}(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}')\tilde{\Psi}(\mathbf{q}',\mathbf{p}')d\mathbf{q}'d\mathbf{p}' , \quad (3.28)$$

$\tilde{\Psi}\in\tilde{\mathcal{H}}$, and $\mathcal{H}_{e,l}$ the subspace

$$\mathcal{H}_{K_{e,l}}\equiv\mathcal{H}_{e,l}=\mathbb{P}_{e,l}L^2(\mathbb{R}^6,d\mathbf{q}d\mathbf{p}) . \quad (3.29)$$

Then, from the nature of the kernel $K_{e,l}$ in (3.27), it is clear that the projected functions in $\mathcal{H}_{e,l}$,

$$\Psi_{e,l}=\mathbb{P}_{e,l}\tilde{\Psi},\quad \tilde{\Psi}\in\tilde{\mathcal{H}} , \quad (3.30)$$

are continuous in the variables $(\mathbf{q},\mathbf{p})$. Moreover, $\mathcal{H}_{e,l}$ carries a unitary irreducible representation[23] of the extended Galilei group $\tilde{G}$ (corresponding to a particle of mass m and spin l). For the particular case of $l=0$, and

$$e(\mathbf{k})=\pi^{-3/4}\exp(-\mathbf{k}^2/2) , \quad (3.31)$$

we recover the antinormal ordering mentioned previously. However, our main aim is to elucidate the different ordering possibilities, when e does not necessarily have the form (3.31). Moreover, we also want to study the physical significance of the function e, and the consequent connection between the experimental set up and the choice of ordering.

As mentioned in Sec. I [see Eqs. (1.5) and (1.11)], in this model, the quantized versions [via (3.5)] of the classical position and momentum observables $\mathbf{q}$ and $\mathbf{p}$ obey the canonical commutation relations. Indeed, let

$$f^i(\mathbf{q},\mathbf{p})=q^i,\quad g^i(\mathbf{q},\mathbf{p})=p^i,\quad i=1,2,3 . \quad (3.32)$$

Then, a straightforward calculation using (3.1), (3.5), (3.27), and (3.28) shows that

$$Q^i\equiv\mathbb{P}_{e,l}\tilde{P}(f^i)\mathbb{P}_{e,l}=q^i+i\frac{\partial}{\partial q^i} \quad (3.33a)$$

$$P^i\equiv\mathbb{P}_{e,l}\tilde{P}(g^i)\mathbb{P}_{e,l}=-i\frac{\partial}{\partial q^i} , \quad (3.33b)$$

so that

$$[Q^i,P^j]=i\delta_{ij}I \quad (3.34)$$

on a stable dense domain $\mathcal{D}\subset\mathcal{H}_{e,l}$. Moreover, using (A8) and the fact that the POV measure $a_{e,l}$ canonically associated to $K_{e,l}$ [see (2.23)] is given by

$$a_{e,l}(\Delta)=\mathbb{P}_{e,l}\tilde{P}(\Delta)\mathbb{P}_{e,l} , \quad (3.35)$$

we find that, for $\tilde{\Psi}\in\tilde{\mathcal{H}}$,

$$[a_{e,l}(\Delta)\tilde{\Psi}](\mathbf{q},\mathbf{p})=\int_{\mathbb{R}^6}K^{\Delta}_{e,l}(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}')\tilde{\Psi}\times(\mathbf{q}',\mathbf{p}')d\mathbf{q}d\mathbf{p} , \quad (3.36)$$

where

$$K^{\Delta}_{e,l}(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}')=\int_{\Delta}K_{e,l}(\mathbf{q},\mathbf{p};\mathbf{q}'',\mathbf{p}'')\times K_{e,l}(\mathbf{q}'',\mathbf{p}'';\mathbf{q}',\mathbf{p}')d\mathbf{q}''d\mathbf{p}'' . \quad (3.37)$$

In view of Eqs. (3.33) we may use the notation $F_{e,l}(\mathbf{Q},\mathbf{P})$ for the quantized form of the classical observable $f(\mathbf{q},\mathbf{p})$ in complete analogy with (1.8), i.e.,

$$F_{e,l}=\mathbb{P}_{e,l}\tilde{P}(f)\mathbb{P}_{e,l} . \quad (3.38)$$

Moreover, from Eqs. (3.32) and (3.33) we see that $F_{e,l}$ yields an ordering of the operators $\mathbf{Q}$ and $\mathbf{P}$ in the quantization of the classical function f.

It is important to note that the ordering is Galilean covariant in a sense which we now make explicit. As mentioned before, $\mathcal{H}_{e,l}$ carries an irreducible representation of $\tilde{G}$, corresponding to a particle of mass m and spin l. Indeed, as shown in Ref. 23 the representation in question is given by the unitary operators $U(g)$, $g=(\theta,b,\mathbf{a},\mathbf{v},R)\in\tilde{G}$, on $\tilde{\mathcal{H}}$,

$$[U(g)\tilde{\Psi}](\mathbf{q},\mathbf{p})=\exp\{i[\theta+(P^2/2m)b+m\mathbf{v}\cdot(\mathbf{q}-\mathbf{a})]\}\times\tilde{\Psi}(R^{-1}(\mathbf{q}-\mathbf{a}),R^{-1}(\mathbf{p}-m\mathbf{v})) , \quad (3.39)$$

where θ is the phase translation, b the time translation, $\mathbf{a}$ the space translation, $\mathbf{v}$ the velocity boost, R the spatial rotation, and

$$P^2=-\nabla_q^2 . \quad (3.40)$$

This representation is highly reducible, and each subspace of $\tilde{\mathcal{H}}$ of the type $\mathcal{H}_{e,l}$ carries an irreducible subrepresentation $U_{e,l}$ of U. Moreover, $\tilde{\mathcal{H}}$ decomposes into an infinite direct sum of such subspaces. If we restrict ourselves to the isochronous subgroup $\tilde{G}'$ of $\tilde{G}$, which does not contain time translations, then it can be shown[23] that

$$U_{e,l}(g)a_{e,l}(\Delta)U_{e,l}(g)^*=a_{e,l}(g[\Delta]) , \quad (3.41)$$

$g[\Delta]$ being the translation of the set $\Delta\in\mathcal{B}(\Gamma)$ by g, where the action of $g\in\tilde{G}'$ on $(\mathbf{q},\mathbf{p})\in\Gamma$ $(=\mathbb{R}^6)$ is given by

$$g(\mathbf{q},\mathbf{p})=(R\mathbf{q}+\mathbf{a},R\mathbf{p}+m\mathbf{v}) . \quad (3.42)$$

For the classical observable f, if $g[f]$ is the function

$$g[f](\mathbf{q},\mathbf{p})=f(g^{-1}(\mathbf{q},\mathbf{p})) , \quad (3.43)$$

Eq. (3.38) is seen to imply that

$$U_{e,l}(g)F_{e,l}[U_{e,l}(g)]^*=\mathbb{P}_{e,l}\tilde{P}(g[f])\mathbb{P}_{e,l} \quad (3.44)$$

for all $g\in\tilde{G}'$. Thus the Galilean-transformed classical observable $g[f]$ corresponds to the Galilean-transformed quantum observable $U_{e,l}(g)F_{e,l}[U_{e,l}(g)]^*$, establishing the Galilean covariance of the ordering procedure (3.38), and hence of $\pi^*_{e,l}$.

As a final result in this section, we combine the discussion above together with the decomposition of $\tilde{\mathcal{H}}$ given in Theorem 2.1 of Ref. 23 to arrive at a complete description of *all* possible ordering procedures in $\tilde{\mathcal{H}}$. Denote by μ the measure

$$d\mu(r)=4\pi r^2dr,\quad r\in\mathbb{R}^+ , \quad (3.45)$$

and let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal set of vectors in $L^2(\mathbb{R}^+,\mu)$. Then, considering e_n as a function of the type e in (3.25) and (3.26), let us construct for each n and l $(=0,1,2,\ldots)$ the subspace $\mathcal{H}_{n,l}$ of $\tilde{\mathcal{H}}$ via (3.27)–(3.30), and let $K_{n,l}$ be the corresponding reproducing kernel.

Theorem 2. **Each e_n defines a Galilean covariant ordering of operators on $\tilde{\mathcal{H}}=L^2(\mathbb{R}^6,d\mathbf{q}d\mathbf{p})$, by which every bounded measurable function f on $\Gamma=\mathbb{R}^6$, for a particle of mass m and integral spin l, is mapped to a quantum operator $F_{n,l}\in\mathcal{L}(\mathcal{H}_{n,l})$ defined by the integral kernel**

$$(F_{n,l}\Psi_{n,l})(\mathbf{q},\mathbf{p})=\int_{\mathbb{R}^6} f(\mathbf{q}',\mathbf{p}')F_{n,l}(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}')\Psi_{n,l}(\mathbf{q}',\mathbf{p}')d\mathbf{q}'d\mathbf{p}' , \tag{3.46}$$

$$F_{n,l}(\mathbf{q},\mathbf{p};\mathbf{q}',\mathbf{p}')=\left[\frac{2l+1}{(2\pi)^3}\right]^2\int_{\mathbb{R}^6} d\mathbf{k}\,d\mathbf{k}'\exp[i\mathbf{k}\cdot(\mathbf{q}-\mathbf{q}'')+i\mathbf{k}'\cdot(\mathbf{q}''-\mathbf{q}')]\overline{e_n(\mathbf{k}-\mathbf{p})}e_n(\mathbf{k}-\mathbf{p}'')\overline{e_n(\mathbf{k}'-\mathbf{p}'')}$$
$$\times e_n(\mathbf{k}'-\mathbf{p}')\mathcal{P}_l(\mathbf{p}\cdot\mathbf{p}''/\|\mathbf{p}\|\,\|\mathbf{p}''\|)\mathcal{P}_l(\mathbf{p}''\cdot\mathbf{p}'/\|\mathbf{p}''\|\,\|\mathbf{p}'\|) . \tag{3.47}$$

The Hilbert space $\tilde{\mathcal{H}}$ decomposes into a direct sum

$$\tilde{\mathcal{H}}=\oplus\sum_{n=1}^{\infty}\sum_{l=0}^{\infty}\mathcal{H}_{n,l} \tag{3.48}$$

of the subspaces $\mathcal{H}_{n,l}$, each one of which carries a $(2l+1)$-fold reducible representation of the canonical commutation relations

$$[Q^i,P^j]=i\delta_{ij}I,\quad i,j=1,2,3 \tag{3.49}$$

corresponding to a particle of mass m and spin l. Conversely, every Galilean covariant ordering procedure on $\mathcal{H}$ for a quantum system of mass m and spin $l=0,1,2,\ldots$, determines a unique vector e in $L^2(\mathbb{R}^+,\mu)$ which fixes the corresponding reproducing kernel subspace $\mathcal{H}_{e,l}\subset\tilde{\mathcal{H}}$.

For a proof, note that the relations (3.46) and (3.47) follow from (3.27), (3.28), and (3.38). The decomposition in (3.48) and the canonical commutation relations (CCR) in (3.49) follow from Theorem 2.1 in Ref. 23. For the converse, using arguments on evaluation maps[22,24] one can establish the existence[23] of a unique resolution generator ξ in each reproducing kernel Hilbert subspace of $\tilde{\mathcal{H}}$. This generator, in its turn, determines the unique vector e in $L^2(\mathbb{R}^+,\mu)$ as a consequence of Theorem 3.4 in Ref. 23. In ending this section, we remark that since Γ can also be identified topologically as the quotient space[26]

$$\Gamma=\mathcal{P}_+^{\uparrow}/T\otimes \mathrm{SO}(3) \tag{3.50}$$

of the Poincare group $\mathcal{P}_+^{\uparrow}$ by the subgroup of time translations T and rotations SO(3), it is possible to find[27] kernels and reproducing kernel subspaces $\mathcal{H}_{e,l}$ of $\tilde{\mathcal{H}}$ which carry irreducible representations of $\mathcal{P}_+^{\uparrow}$, corresponding to particles with mass $m>0$ and spin $l=0,1,2,\ldots$. The mapping [see (3.5) and (3.38)]

$$\pi_{e,l}^*(f)=\mathbb{P}_{e,l}\tilde{P}(f)\mathbb{P}_{e,l},\quad \mathcal{H}_{e,l}=\mathbb{P}_{e,l}\mathcal{H} , \tag{3.51}$$

then defines a quantization of a free particle with relativistic kinematics.

Analogous results could also be obtained for systems with nonintegral spins, but then the starting Hilbert space in (3.24) would have to be replaced by

$$\tilde{\mathcal{H}}=\mathbb{C}^2\otimes L^2(\mathbb{R}^6,d\mathbf{q}d\mathbf{p}) , \tag{3.52}$$

and similarly, $U(g)$ in (3.39) would have to be replaced by a correspondingly different representation.

IV. PHYSICAL INTERPRETATION

We refer to the free particle on $T^*(\mathbb{R}^3)$ of the previous section. The vector $e\in L^2(\mathbb{R}^+,\mu)$, which, according to Theorem 2, characterizes an ordering, is amenable to a physical interpretation, which we develop in this section. It will turn out that the vector e may be related to the uncertainty in a joint measurement of position and momentum. Consequently, the ordering is ultimately dependent upon the measuring apparatus. Experimental setups which allow a determination of e have been discussed in Refs. 28 and 29. Moreover, as described in detail in these two references, the use of POV measures is the only satisfactory way to handle mathematically the joint measurement of incompatible observables in quantum mechanics.

Consider the representation $U_{e,l}$ of the $\tilde{G}$ on $\mathcal{H}_{e,l}$ given by (3.39) (actually by its restriction to $\mathcal{H}_{e,l}$). For simplicity, take the spin-zero case ($l=0$). Since $\Psi\in\mathcal{H}_{e,0}$ relates to a particle on $T^*(\mathbb{R}^3)$, we would like to interpret $\|\Psi(\mathbf{q},\mathbf{p})\|^2$ as the probability density of finding the particle localized at the phase space point $(\mathbf{q},\mathbf{p})$. However, since the uncertainty relation given by the standard quantization method does not allow simultaneous accurate measurements of $\mathbf{q}$ and $\mathbf{p}$, therefore, if $\|\Psi(\mathbf{q},\mathbf{p})\|^2$ is to have a physical meaning at all, the point $(\mathbf{q},\mathbf{p})$, as it appears in $\|\Psi(\mathbf{q},\mathbf{p})\|^2$, should entail the existence of two confidence functions[18] $\chi_{\mathbf{q}}$ and $\hat{\chi}_{\mathbf{p}}$ centered at $\mathbf{q}$ and $\mathbf{p}$, respectively. In that case $\chi_{\mathbf{q}}(\mathbf{q}')$ would give the probability density for the particle to be actually at the point $\mathbf{q}'$, when it is experimentally observed to be at $\mathbf{q}$. Thus $\|\Psi(\mathbf{q},\mathbf{p})\|^2$ is to be interpreted as the joint probability density for finding the particle at the "smeared point $(\mathbf{q},\mathbf{p})$." For a reasonably accurate localization of the particle in configuration space, $\chi_{\mathbf{q}}$ would have to be a function with support mainly in a small neighborhood around q. A similar interpretation would be given for $\hat{\chi}_{\mathbf{p}}$. Moreover, if $\sigma_i(\chi_o)$ [respectively, $\sigma_i(\hat{\chi}_o)$] is the standard deviation of q_i (respectively, p_i) with respect to χ_o (respectively to $\hat{\chi}_o$), we expect that

$$\sigma_i(\chi_o)\sigma_j(\hat{\chi}_o)\geq\tfrac{1}{2},\quad i,j=1,2,3 . \tag{4.1}$$

It was proved, first in Ref. 18, and later shown in various different physical contexts,[2,28] that this indeed is the case. Moreover, it can be shown that covariance under space translations implies that

$$\chi_{\mathbf{q}}(\mathbf{q}')=\chi_o(\mathbf{q}'-\mathbf{q}) , \tag{4.2a}$$

$$\hat{\chi}_{\mathbf{p}}(\mathbf{p}')=\hat{\chi}_o(\mathbf{p}'-\mathbf{p}) . \tag{4.2b}$$

The interesting point is that the reproducing kernel is uniquely fixed by the confidence functions through

$$\chi_o(\mathbf{q})=|e(\mathbf{q})|^2 , \tag{4.3a}$$

$$\hat{\chi}_o(\mathbf{p})=|\tilde{e}(\mathbf{p})|^2 , \tag{4.3b}$$

$\tilde{e}$ being the Fourier transform of e. One can check that the situation where (4.1) holds with equality is precisely the case of the antinormal ordering,[18] so that e has the form in (3.31). Thus the antinormal ordering is optimal, in the sense that it envisages maximum accuracy in a theory based upon localization in phase space. We hasten to add, however, that the confidence functions χ_{q} and $\hat{\chi}_{\mathrm{p}}$, while representing uncertainties in a joint measurement of **q** and **p**, do not arise as a result of any "uncontrollable disturbance" of the measured system by the measuring apparatus. In other words, the vector e, and hence the ordering convention, truly reflects a property of the measurement procedure. Realistic instruments for performing such measurements have been considered in Refs. 28 and 29, where it is also shown how joint measurements of position and momentum lead to a determination of e.

V. QUANTIZATION SCHEMES COMPARED

It is instructive to study the similarities, or otherwise, of the prime quantization scheme as outlined here, and the various other schemes, such as, for example, the geometric,[5,7,30] the Mackey,[6] and the Borel[3,8] quantization schemes which have been proposed in the past. In the Mackey scheme, one utilizes the theory of imprimitivity for locally compact groups to define localization on a manifold M. Various generators of the kinematic variables then yield the quantum-mechanical operators of generalized position, momentum, etc., as well as their correct commutation relations. In the prime quantization scheme, localization is on the phase space $\Gamma=T^*(M)$, and is in itself defined independently of any group action or symmetry properties related to $T^*(M)$. But such properties have to be invoked later in order to choose the physically interesting quantizations, for example, by choosing kinematical observables, such as was done in Ref. 8, where the model gave observables equivalent to a set $U(\Gamma)$, by a generalized system of covariance. Hence a prime quantization leads to a pair $[U(\Gamma),\pi^*(\mathcal{A}_{\mathrm{cl}}(\Gamma))]$ and is applicable to any $T^*(M)$ (Ref. 31) and any spin.[1]

The relationship between the present scheme and geometric quantization appears most clearly at the stage where, in the latter approach, one makes a *choice of polarization.* Geometric quantization proceeds in two stages. In the first, or prequantization stage, a map $\mathfrak{P}$ is found from the classical algebra $\mathcal{A}_{\mathrm{cl}}$ to a set of first-order differential operators on Γ,

$$\mathfrak{P}(f)=-i\nabla_{X_f}+f,\quad f\in\mathcal{A}_{\mathrm{cl}}\ , \tag{5.1}$$

where ∇ denotes the covariant derivative on the manifold Γ and X_f is the vector field canonically associated to f by the symplectic form on $\Gamma=T^*(M)$. The map $\mathfrak{P}$ satisfies the requirements [as operators on $L^2(\Gamma,\nu)$]

$$\mathfrak{P}(\alpha f+\beta g)=\alpha\mathfrak{P}(f)+\beta\mathfrak{P}(g)\ , \tag{5.2}$$

$$\mathfrak{P}(1)=I_\Gamma\ , \tag{5.3}$$

$$\mathfrak{P}(\{f,g\})=-i[\mathfrak{P}(f),\mathfrak{P}(g)]\ , \tag{5.4}$$

for all $f,g\in\mathcal{A}_{\mathrm{cl}}$ and all $\alpha,\beta\in\mathbb{C}$. In (5.3), 1 is the constant function with value $1\in\mathbb{R}$, I_Γ is the identity operator on $L^2(\Gamma,\nu)$; in (5.4) $\{f,g\}$ is the Poisson bracket of f and g and $[\mathfrak{P}(f),\mathfrak{P}(g)]$ the commutator bracket of $\mathfrak{P}(f)$ and $\mathfrak{P}(g)$. Considering now the case $T^*(\mathbb{R}^3)$ and taking for f and g, the corresponding position and momentum observables, respectively, in (3.32), one gets

$$\mathfrak{P}(q^i)=q^i+i\frac{\partial}{\partial p^i}\ , \tag{5.5a}$$

$$\mathfrak{P}(p^i)=-i\frac{\partial}{\partial q^i},\quad i=1,2,3 \tag{5.5b}$$

and the CCR

$$\mathfrak{P}(\{q^i,p^j\})=-i[\mathfrak{P}(q^i),\mathfrak{P}(p^j)]=\delta_{ij}\ . \tag{5.6}$$

Equations (5.5) and (5.6) ought to be compared with Eqs. (3.33) and (3.34); but there is an important difference between the two sets. Indeed, (3.33) and (3.34), obtained by prime quantization, form an irreducible set on the reproducing kernel Hilbert space $\mathcal{H}_{e,l}$, while the set (5.5) and (5.6), obtained by prequantization, is a highly reducible set on $L^2(\Gamma,\nu)$. In the geometric quantization scheme, one has, therefore, to implement at this stage a choice of polarization to obtain from (5.5) and (5.6) an irreducible representation of the CCR. Thus our choice of a reproducing kernel Hilbert space corresponds to the choice of a polarization. In physical terms, therefore, the choice of polarization is ultimately a choice of an ordering of operators. This last point, and its relationship to the modular structure of the Weyl algebra of the CCR has been worked out in detail in a simple example in Ref. 32.

ACKNOWLEDGMENTS

Part of this work was done while one of the authors (S.T.A.) was visiting the Arnold Sommerfeld-Institut of the Technische Universität Clausthal, and he acknowledges support from the Alexander von Humboldt-Stiftung, Federal Republic of Germany. At other times he benefited from extremely fruitful discussions with Professor Gérard G. Emch of the University of Florida, Gainesville. This work was also partly supported by the Natural Sciences and Engineering Research Council of Canada.

APPENDIX

We collect here some results on reproducing kernels and their relationship to POV measures. The general references for the material here are Refs. 22 and 24. Consider the direct integral Hilbert space introduced in Eq. (2.16) and the associated measurable field[13] $\zeta\mapsto\mathcal{H}_\zeta$ of Hilbert spaces. In general, for $\zeta\neq\zeta'$, $\mathcal{H}_\zeta$ and $\mathcal{H}_{\zeta'}$ could have different dimensions; however, we assume that all the spaces $\mathcal{H}_\zeta,\zeta\in\Gamma$, are separable. For physical reasons we may assume, although this is not necessary mathematically, that for all ζ, $\mathcal{H}_\zeta$ is isomorphic to a fixed Hilbert space $\mathcal{H}$. Let $\prod_{\zeta\in\Gamma}\mathcal{H}_\zeta$ be the Cartesian product of these spaces, equipped with the natural vector space structure (no topology assumed) arising from the spaces $\mathcal{H}_\zeta$, and let H be a fixed subspace of the direct integral space $\tilde{\mathcal{H}}$ in (2.16).

Definition A1. A ν selection σ for $H \subset \tilde{\mathcal{H}}$ is a linear map

$$\sigma:\ H \to \prod_{\zeta \in \Gamma} \mathcal{K}_\zeta \ , \tag{A1}$$

which associates to each ν-equivalence class $[f]$ of functions in H, a function $\zeta \mapsto \sigma([f])(\zeta)$ in $\prod_{\zeta\in\Gamma}\mathcal{K}_\zeta$, for which

$$[f]=[\sigma([f])] \ . \tag{A2}$$

The significance of the existence of a ν selection on a subspace $H \subset \tilde{\mathcal{H}}$ is the following. The elements of $\tilde{\mathcal{H}}$ are equivalence classes of ν-measurable functions $f \in \prod_{\zeta\in\Gamma}\mathcal{K}_\zeta$. If $[f]$ is the equivalence class of f, then two functions f_1 and f_2 in $[f]$ differ only perhaps on a set of measure zero. In each equivalence class $[f]$ we may choose a representative function f, but, in general, we may not be able to choose it in such a way as to render the mapping $[f] \to f$ *linear* for the *whole* of $\tilde{\mathcal{H}}$. However, there may still be subspaces H of $\tilde{\mathcal{H}}$ over which this mapping is linear. Such subspaces are then actually Hilbert spaces of *functions* $\sigma([f])$, as opposed to *equivalence classes* of functions $[f]$. The existence of ν selections is precisely what characterizes the reproducing kernel Hilbert spaces $\mathcal{H}_K \subset \tilde{\mathcal{H}}$.

Lemma A1. There exists a ν selection σ on $\mathcal{H}_K$ such that, for each $\zeta \in \Gamma$, the linear mapping (*evaluation map*) E_ζ^K: $\mathcal{H}_K \to \mathcal{K}_\zeta$, defined by it,

$$E_\zeta^K(\Psi)=\sigma([\Psi])(\zeta), \quad \Psi \in \mathcal{H}_K \ , \tag{A3}$$

is continuous and has dense range in $\mathcal{K}_\zeta$. Furthermore,

$$\sigma([\Psi])(\zeta)=\int_\Gamma K(\zeta,\zeta')\Psi(\zeta')d\nu(\zeta') \tag{A4}$$

and

$$K(\zeta,\zeta')=E_\zeta^K(E_{\zeta'}^K)^* \ . \tag{A5}$$

As a consequence of this lemma, the elements $\Psi \in \mathcal{H}_K$ can be considered to be actual functions on Γ, since we may use the function $\zeta \mapsto \sigma([\Psi])(\zeta)$ for the vector $\Psi \in \mathcal{H}_K$. Hence it is unnecessary to consider equivalence classes $[\Psi]$ for the elements of $\mathcal{H}_K$. Using this convention we may write (A4) simply as

$$\Psi(\zeta)=\int_\Gamma K(\zeta,\zeta')\Psi(\zeta')d\nu(\zeta'), \quad \zeta \in \Gamma \ , \tag{A6}$$

which is the usual "reproducing" property. The continuity of the evaluation map leads to the norm estimate, for all $\zeta \in \Gamma$, $\Psi_K \in \mathcal{H}_K$,

$$\|\Psi(\zeta)\|_{\mathcal{K}_\zeta} \le \|\Psi\|_{\mathcal{H}_K}\|K(\zeta,\zeta)\|^{1/2} \ , \tag{A7}$$

where for $K(\zeta,\zeta)$ the operator norm on $\mathcal{K}_\zeta$ is meant.

If all the Hilbert spaces $\mathcal{K}_\zeta$ are isomorphic to a single Hilbert space $\mathcal{K}$, so that $K(\zeta,\zeta') \in \mathcal{L}(\mathcal{K})$ for all $(\zeta,\zeta') \in \Gamma\times\Gamma$ and $\tilde{\mathcal{H}}$ is isomorphic to $\mathcal{K}\otimes L^2(\Gamma,\nu)$, and if the function $(\zeta,\zeta') \mapsto K(\zeta,\zeta')$ is assumed to be separately continuous in the norm topology of $\mathcal{L}(\mathcal{K})$, then (A7) may be used to deduce the continuity of the functions Ψ [more precisely, of $\sigma([\Psi])$]. This is the situation that prevails for K_A in (2.12), where $\mathcal{K}_\zeta=\mathbb{C}$.

Each reproducing kernel Hilbert space $\mathcal{H}_K$ admits a canonical POV measure, whose Naimark extension lives on $\tilde{\mathcal{H}}$. Moreover, this POV measure has a bounded ν density in the sense of Eq. (1.17). Conversely, given an abstract Hilbert space $\mathcal{H}$ with a normalized POV measure a having a ν density, there exists a reproducing kernel Hilbert space which is isomorphic to it. Specifically, we have the following result (given here without proof).

Theorem A1. Let $\hat{\mathcal{H}}$ be an arbitrary subspace of $\tilde{\mathcal{H}}$. Then $\hat{\mathcal{H}}$ is a reproducing kernel Hilbert space, i.e., $\hat{\mathcal{H}}=\mathcal{H}_K$, if and only if for each $\zeta \in \Gamma$ there exists a continuous linear evaluation map E_ζ^K: $\mathcal{H}_K \to \mathcal{K}_\zeta$. In this case the PV measure $\tilde{P}$ and the space $\tilde{\mathcal{H}}$ with

$$[\tilde{P}(\Delta)\Psi](\zeta)=\chi_\Delta(\zeta)\Psi(\zeta), \quad \Psi \in \tilde{\mathcal{H}} \tag{A8}$$

form the unique minimal extension in the sense of Naimark, of the canonically associated POV measure a_K and the Hilbert space $\mathcal{H}_K$, where

$$a_K(\Delta)=\int_\Delta T_K(\zeta)d\nu(\zeta) \ , \tag{A9}$$

$$T_K(\zeta)=(E_\zeta^K)^*E_\zeta^K \ , \tag{A10}$$

and $\Delta \in \mathcal{B}(\Gamma)$, the kernel K being given by $K(\zeta,\zeta')=E_\zeta^K(E_{\zeta'}^K)^*$.

Theorem A2. Let $\mathcal{H}$ be a separable Hilbert space, on which there exists a normalized POV measure a, defined on the Borel sets of a locally compact space Γ, and admitting a bounded ν density T. Then, there exists a direct integral Hilbert space $\tilde{\mathcal{H}}$ [of the type in Eq. (2.16)], a reproducing kernel Hilbert space $\mathcal{H}_K \subset \tilde{\mathcal{H}}$, with canonically associated POV measure a_K as in (A9) and (A10), and a unitary map W_K such that

$$W_K\mathcal{H}=\mathcal{H}_K \ , \tag{A11}$$

$$W_K a(\Delta)W_K^{-1}=a_K(\Delta) \ , \tag{A12}$$

$$W_K T(\zeta)W_K^{-1}=(E_\zeta^K)^*E_\zeta^K \ , \tag{A13}$$

for all $\Delta \in \mathcal{B}(\Gamma)$ and $\zeta \in \Gamma$.

Lemma A2. The following norm estimates hold:

$$\|K(\zeta,\zeta')\|=\|T(\zeta)^{1/2}T(\zeta')^{1/2}\| \ , \tag{A14}$$

$$\|E_\zeta^K\|=\|(E_\zeta^K)^*\|=\|K(\zeta,\zeta)\|^{1/2} = \|T(\zeta)\|^{1/2} \ , \tag{A15}$$

for all $(\zeta,\zeta') \in \Gamma\times\Gamma$, the norms being calculated for the appropriate spaces in each case.

As a final remark, we note that since (A5) can be written as

$$K(\zeta,\zeta')=E_\zeta^K\mathbb{P}_K(E_{\zeta'}^K)^* \ , \tag{A16}$$

we can define, for an arbitrary bounded operator $A \in \mathcal{L}(\mathcal{H}_K)$, a kernel function $(\zeta,\zeta') \to A(\zeta,\zeta') \in \mathcal{L}(\mathcal{K}_\zeta,\mathcal{K}_{\zeta'})$ as

$$A(\zeta,\zeta')=E_\zeta^K A_K(E_{\zeta'}^K)^* \ . \tag{A17}$$

This function is measurable, and in terms of it,

$$(A\Psi)(\zeta)=\int_\Gamma A(\zeta,\zeta')\Psi(\zeta')d\nu(\zeta') \ , \tag{A18}$$

for all $\zeta\in\Gamma$ and $\Psi\in\mathcal{H}_K$. Also, for all $(\zeta,\zeta')\in\Gamma\times\Gamma$,

$$\|A(\zeta,\zeta')\|\leq\|K(\zeta,\zeta)\|^{1/2}\|K(\zeta',\zeta')\|^{1/2}\|A\|_{\mathcal{H}_K}. \quad \text{(A19)}$$

As mentioned before, the case where $\mathcal{K}_\zeta=\mathcal{K}$ (some fixed Hilbert space), for all $\zeta\in\Gamma$ is of special interest to us. In this case $\tilde{\mathcal{H}}$ is isomorphic to $\mathcal{K}\otimes L^2(\Gamma,\nu)$, and $K(\zeta,\zeta')$ and $A(\zeta,\zeta')$ are bounded operators in $\mathcal{L}(\mathcal{K})$.

We can, as a consequence of Lemma A2 and Eq. (A19) derive the following result.

Theorem A3. If the mapping $(\zeta,\zeta')\mapsto K(\zeta,\zeta')$ is separately continuous, in the norm topology of $\mathcal{L}(\mathcal{K})$, then (i) for each $\Psi\in\mathcal{H}_K$, the function $\zeta\mapsto\Psi(\zeta)$, with values in $\mathcal{K}$, is continuous in the norm of $\mathcal{K}$; and (ii) $A(\zeta,\zeta')$ is separately continuous in the norm of $\mathcal{L}(\mathcal{K})$, for each bounded operator $A\in\mathcal{L}(\mathcal{H}_K)$.

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