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Stability of ballooning modes in a rotating plasma

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An equation describing ballooning modes localized to a magnetic field line embedded in a rotating plasma is derived and analyzed. On a typical ballooning growth time these modes may be completely stabilized by shear in the flow, in a way which appears to be simple to achieve experimentally. Their long-time behavior is more complicated.

Ballooning modes[1,2] are the pressure-driven instabilities which are considered to be the most dangerous for confined plasmas, and are believed to impose the most severe upper limit on the plasma pressure for a given strength of the confining magnetic field. It is highly desirable to stabilize these modes and in this article we suggest that this aim may be achieved, at least for a sufficiently long time, by introducing a sheared toroidal flow into the equilibrium state, of the form $\mathbf{u}=R\Omega(\psi)\hat{\zeta}$. (Poloidal flows tend to damp out fairly rapidly.[3]) Here ζ is the toroidal angle in the axisymmetric device, $\hat{\zeta}$ is the toroidal unit vector, θ is the poloidal angle (along the short way around the torus), and R is the distance from the symmetry axis. ψ labels magnetic flux surfaces such that the magnetic field is $\mathbf{B}=\nabla\psi\times\nabla(\zeta-q\theta)$, where $q=q(\psi)$, and $\mathbf{u}$, p, and ρ are the plasma velocity, pressure, and density, respectively. An equation for the equilibrium state can be found in Ref. 4, where we use the relation $p=S(\psi)\rho^{\gamma}$, $\gamma=\frac{5}{3}$, and $S(\psi)$ is given.

It was previously observed[5,6] that the usual techniques of deriving the ballooning-mode equations break down if the toroidal flow is not a rigid rotation. Thus, progress here requires an improved understanding of the ballooning-mode phenomenon. The view we follow[5,7] is that the ballooning "mode" is actually a wave packet propagating, or standing, along its ray (a magnetic field line) in the sense of geometrical optics. The standard geometrical optics description[7,8] is to represent the dependent variable $\boldsymbol{\xi}(\mathbf{x},t)$ as

$$\boldsymbol{\xi}=\exp(i\chi/\epsilon)(\boldsymbol{\xi}^0+\epsilon\boldsymbol{\xi}^1+\cdots)\,, \qquad (1)$$

where $\epsilon\ll 1$ is a measure of the fast variation of the phase, and $\chi(\mathbf{x},t)$ and $\boldsymbol{\xi}^j(\mathbf{x},t)$ are functions of order 1. Substitution of Eq. (1) in the given set of equations yields, in leading order, an equation for χ solved along rays, and a set of "transport equations" for the wave amplitude which determine the $\boldsymbol{\xi}^j$ along the same rays. Now, ansatz (1) is often used to generate the ballooning-mode equation for plasmas without equilibrium flows,[9] with $\chi=\chi(\zeta-q\theta,\psi)$ being a time-independent function which is constant along field lines, and the $\boldsymbol{\xi}^j$ having a harmonic dependence of $\exp(i\omega t)$ on time. Indeed, the ballooning equation is merely the equation for $\boldsymbol{\xi}^0$ along a field line, the ray, which is determined from the phase equation $\mathbf{B}\cdot\nabla\chi=0$. Thus, the geometrical optics interpretation of ballooning modes is consistent with earlier works on the subject.

A distinguishing feature of the ballooning mode is that it actually consists of four coupled waves, the Alfven and slow magnetosonic waves of magnetohydrodynamics, each with two branches of waves traveling either in the positive or negative direction of the field line, their ray. Uncoupled, such waves are stable and were described in Refs. 5 and 7. The reason for the coupling in the ballooning case is that the spatial direction of the most rapid variation of the waves considered, the direction of $\nabla\chi$, is perpendicular to $\mathbf{B}$. The phase velocity of all four waves in this direction is the same, namely zero,[10] which implies a "degeneracy" of the waves and the coupling of their amplitude equations along their common ray.[7,8]

With the presence of flow in the equilibrium state, all wave velocities are shifted by the fluid velocity[10] such that each wave travels along its own ray which no longer parallels the field line, and thus appears to be uncoupled to the other waves. For example, the two Alfven waves travel in the nonantiparallel directions of $\mathbf{u}+\mathbf{B}/\sqrt{\rho}$ and $\mathbf{u}-\mathbf{B}/\sqrt{\rho}$. If $\Omega=$const, a rigid rotation, it was realized in Ref. 5 that from a coordinate frame rotating toroidally with the flow frequency Ω, the waves are still seen as traveling along field lines, they are coupled together and the ballooning modes are present. We now extend this result to the case $\Omega=\Omega(\psi)$, the sheared flow. In principle, we can shift to a moving frame, which here means moving to a cumbersome sheared coordinate system which measures $\tilde{\zeta}=\zeta-\Omega(\psi)t$, and carry out the geometrical optics procedure with the time-independent eikonal $\chi=\chi(\tilde{\zeta}-q\theta,\psi)$. Instead, we use the fact that the ray phenomenon itself does not depend on the coordinate system used, only its representation does. We thus remain in the laboratory frame, and express the eikonal in the coordinates of this frame. Before proceeding, we call attention to the following fundamental fact. A finite wave frequency ω in a rotating frame is Doppler-shifted to $\omega+n\Omega$ in the rest frame,[5] where n is the toroidal mode number of the wave. The ballooning mode corresponds[1,2] to $n\to\infty$. Thus, the presence of flow implies a need to resolve the spectral point of $\omega=\infty$ in the rest frame. As will be seen, this point involves solutions with a nonexponential dependence on the time t.

To proceed, we use the same equations and notations as in Sec. III of Ref. 4. The linearized equation of motion is

$$\rho\boldsymbol{\xi}_{tt}+2\rho\mathbf{u}\cdot\nabla\boldsymbol{\xi}_t+\mathbf{F}(\boldsymbol{\xi})+\mathbf{G}(p_*)=0\,, \qquad (2)$$

where ξ is the Lagrangian displacement, and p_* is the perturbed total pressure (kinetic plus magnetic),

$$F(\xi)=\rho\mathbf{u}\cdot\nabla(\mathbf{u}\cdot\nabla\xi-\xi\cdot\nabla\mathbf{u})+\rho(\mathbf{u}\cdot\nabla\xi)\cdot\nabla\mathbf{u}+(\mathbf{B}\cdot\nabla\text{ terms})\,, \quad (3)$$

$$G(p_*)=\nabla p_*+(\mathbf{B}\cdot\nabla\text{ terms})\,, \quad (4)$$

$$p_*=\mathbf{B}\cdot(\mathbf{B}\cdot\nabla\xi-\xi\cdot\nabla\mathbf{B})-\xi\cdot\nabla p-(\gamma p+B^2)\text{div}\xi\,. \quad (5)$$

The $\mathbf{B}\cdot\nabla$ terms are terms involving derivatives of the appropriate perturbed variables only along field lines, as well as undifferentiated terms. It is noted that the operator $\rho\mathbf{u}\cdot\nabla$ is anti-Hermitian, while F and G are Hermitian.[4] We now make ansatz (1) in the laboratory frame, where $\chi=\chi(\zeta-q\theta-\Omega t,\psi)$, and ϵ may be thought of as the inverse toroidal mode number. p_* has a similar expansion. Note that $\zeta-\Omega t$ is merely the toroidal angle $\tilde{\zeta}$ in the rotating frame. We also note that $(\partial/\partial t+\mathbf{u}\cdot\nabla)\chi=0$, $\mathbf{B}\cdot\nabla\chi=0$, so the first three terms in Eq. (2) that involve derivatives of the form $(\partial/\partial t+\mathbf{u}\cdot\nabla)^2$ and $\mathbf{B}\cdot\nabla$, do not produce terms larger than $O(1)$. Equation (5) implies to $O(1/\epsilon)$ that $\xi^0\cdot\nabla\chi=0$, and Eq. (4) yields $p_*^0\nabla\chi=0$, thus $p_*^0=0$. To $O(1)$ we now have $G(p_*)=ip_*^1\nabla\chi$, thus the two components of Eq. (2) perpendicular to $\nabla\chi$ now read

$$P\left[\rho\frac{\partial^2}{\partial t^2}+2\rho\mathbf{u}\cdot\nabla\frac{\partial}{\partial t}+F\right]P\xi=0\,. \quad (6)$$

Here we have dropped the superscript from ξ^0, and P is a projection operator which annihilates the component along $\nabla\chi$,

$$P=I-\frac{1}{|\nabla\chi|^2}\nabla\chi\nabla\chi\,, \quad (7)$$

where I is the identity operator and the last term is a dyadic. Note that P is Hermitian, $P^*=P$, and Eq. (6) describes two equations for the two components of ξ normal to $\nabla\chi$. Such a ξ satisfies $\xi=P\xi$, and the last P in Eq. (6) was explicitly displayed in order to point out that the new operators, such as PFP, have the same symmetry properties as the original ones.

Equation (6) contains spatial derivatives of ξ only of the form $\mathbf{u}\cdot\nabla$ or $\mathbf{B}\cdot\nabla$, thus it may be viewed as restricted to a magnetic flux surface. For any such surface $\psi=\psi_0$, it is now convenient to move to a rigidly rotating frame with frequency $\Omega(\psi_0)$. This eliminates all $\mathbf{u}\cdot\nabla$ derivatives and yields,

$$P\left[\rho\frac{\partial^2}{\partial t^2}+2\rho\Omega\frac{\partial}{\partial t}\hat{\mathbf{z}}\times+\tilde{F}\right]P\xi=0\,, \quad (8)$$

where the middle term involving $\hat{\mathbf{z}}\times\xi_t$ expresses the Coriolis force, and $\tilde{F}(\xi)=F(\xi)-2\rho\Omega\mathbf{u}\cdot\nabla\xi_{,\zeta}+\rho\Omega^2\xi_{,\zeta\zeta}$. Here $\xi_{,\zeta}$ denotes the derivative of ξ with respect to ζ of the cylindrical coordinates R,ζ,z, where the unit vectors remain undifferentiated. [For $\xi\sim\exp(in\zeta)$, $\xi_{,\zeta}=in\xi$.] From Ref. 4 it is seen that all spatial derivatives, other than $\mathbf{B}\cdot\nabla$, disappear and $\tilde{F}$ can be made positive definite, a property which will be used to generate a stability criterion. We note that Eq. (8), the ballooning-mode equation, depends *explicitly* on time through P. Indeed, $\nabla\chi=\chi_a[\nabla(\zeta-q\theta)-t\nabla\Omega]+\chi_\psi\nabla\psi$, where $\sigma=\zeta-q\theta-\Omega t$. Moreover, in the present rotating frame, $a\rightarrow\zeta-q\theta$ which is constant along field lines. Thus χ_a and χ_ψ are *constant* in Eq. (8), and we may define $\lambda=\chi_\psi/\chi_a$ such that $\nabla\chi=\nabla(\zeta-q\theta)-t\nabla\Omega+\lambda\nabla\psi$. The constant λ determines the direction of polarization of the wave. For each field line, Eq. (8) yields a one-parameter family of ballooning modes, for $-\infty<\lambda<\infty$. This parameter is equivalent to the parameter y_0 of Ref. 2 which is used as the origin of their "quasimode," and similarly could be eliminated by a translation shift in θ if $dq/d\psi\neq0$ at $\psi=\psi_0$, which is assumed here.

In order to determine the time dependence of solutions of Eq. (8), we recall again that all the coefficients have their value at $\psi=\psi_0$. We now write

$$\nabla\chi=\nabla\zeta-q\nabla\theta+\lambda\nabla\psi-(\theta+t\dot{\Omega}/\dot{q})\nabla q\,, \quad (9)$$

where the dot denotes $d/d\psi$, and we consider θ, the poloidal angle, to be the parameter along the field line, such that $\mathbf{B}\cdot\nabla=(\mathbf{B}\cdot\nabla\theta)\partial/\partial\theta$. The coefficients in Eq. (8), except for the secular term in $\nabla\chi$, the last term in Eq. (9), have a period of 2π in θ, representing the periodic structure of the equilibrium state as seen when moving along a field line. We now transform Eq. (8) to a coordinate frame drifting along the field line by defining $\tilde{\theta}=\theta+t\dot{\Omega}/\dot{q}$, where again $\dot{\Omega}$ and $\dot{q}$ are the value of the functions at $\psi=\psi_0$, such that in $\nabla\chi$ we have $\nabla\theta\rightarrow\nabla\tilde{\theta}$. Note also that $\partial/\partial\theta\rightarrow\partial/\partial\tilde{\theta}$, $\partial/\partial t\rightarrow\partial/\partial t+(\dot{\Omega}/\dot{q})\partial/\partial\tilde{\theta}$. Now t does not appear secularly since it is absorbed in $\tilde{\theta}$ in Eq. (9). Instead, the coefficients in the ballooning equation are functions of $\tilde{\theta}-t\dot{\Omega}/\dot{q}$ and are *periodic* in time with period $T=2\pi\dot{q}/\dot{\Omega}$.

An evolution equation with periodic coefficients in time gives rise to solutions analogous to those described by Floquet's theory for ordinary differential equations.[11] That is, solutions are sums (or integrals) of elementary solutions of the form $\xi(\tilde{\theta},t)=e^{i\nu t}\eta(\tilde{\theta},t)$, where, for fixed $\tilde{\theta}$, η is periodic in t with period T, while ν may be real or complex. To see it, we define the solution operator (the propagator) $U(t)$, such that $U(t)\xi(\tilde{\theta},0)=\xi(\tilde{\theta},t)$, and observe the property due to periodicity of the coefficients; $U(t+T)=U(t)U(T)$. Defining the ν spectrum by $U(T)\xi_0=e^{i\nu T}\xi_0$, we can verify directly that $\eta(\tilde{\theta},t)=e^{-i\nu t}U(t)\xi_0(\tilde{\theta})$ is periodic in t. This result means that, just as in the well-known Mathieu equation,[11] solutions may be parametrically unstable on the time scale of $t/T\gg1$, although instability may be avoided by the right choice of equilibrium profiles. To summarize, the time dependence of a ballooning mode in the presence of shear flow, up to a rotation of the particular flux surface with the flow frequency, is

$$\xi(\theta,t)=e^{i\nu t}\eta(\theta+2\pi t/T,t),\ \ T=2\pi\dot{q}/\dot{\Omega}\,. \quad (10)$$

The ballooning-mode structure is modulated periodically in time (the last t variable of η), and also drifts in space along a field line at a poloidal angular speed of $2\pi/T$, unlike the no-flow case. This drift is responsible for possible parametric destabilization when the mode interacts with the periodic structure of the underlying magnetic field. Such a behavior was recently verified numerically.[12] We note that the boundedness in time of η amounts to a stabilization of the ballooning mode by the flow shear. The possible exponential increase described by v is due to a parametric instability rather than to a ballooning instability (which we denote by ω).

In order to proceed with a more detailed investigation of the behavior in time of η we use the following approximation. We observe that Ω is typically rather small, as the flow velocity is at most of the order of the sound speed which scales with the (low) plasma pressure. This will be made more precise shortly, when we discuss the large-aspect-ratio scaling. Without transforming to $\tilde{\theta}$, and using $\nabla\chi$ in Eq. (9), we can represent $P=P(\delta t)$, where $\delta = a|\nabla\Omega| \ll 1$. Defining $\tau=\delta t$ and replacing $\partial/\partial t$ by $\delta\partial/\partial\tau$, Eq. (8) attains the form of a singular perturbation to which we apply the standard WKB method. Using the ansatz $\xi=\exp[i\phi(\tau)/\delta](\xi_0+\delta\xi_1+\cdots)$, where $\xi_j=\xi_j(x,\tau)$ but ϕ depends on τ only, we find to $O(1)$,

$$P(-\rho\omega^2+2i\omega\rho\Omega\hat{z}\times+\tilde{F})P\xi_0=0\,, \tag{11}$$

where $\omega(\tau)=d\phi/d\tau$, $P=P(\tau)$, and $\xi_0=P\xi_0$. The slow time τ only enters as a parameter in Eq. (11), which now looks as if a dependence $\xi\sim\exp(i\omega t)$ was assumed in Eq. (8). We note that our asymptotic expansion is valid for $\tau=O(1)$, or t up to order T of Eq. (10). This range of T is contained in the periodic part of ξ in Eq. (10). Thus instability with a complex ω only means that the mode amplitude grows initially (up to time T) at an exponential rate, before decreasing again. (On a longer time scale it may become parametrically unstable.) Again this is consistent with the calculation in Ref. 12. It is interesting to notice that the time dependence may be eliminated from P by changing $\lambda - t\,d\Omega/d\psi \rightarrow \lambda$. (This again indicates that the mode drifts along the field line.)

The relevant ballooning stability criterion for $t=O(T)$ is that every eigenvalue ω of Eq. (11) is real. Since we require stability for all λ, we might as well consider Eq. (8) without the frozen $t\nabla\Omega$ term in $P(\tau)$. The flow shear now enters only through $\tilde{F}$. A sufficient condition for stability[5] is for $P\tilde{F}P$ to be positive semidefinite. Expressing $P\xi_0 = X\mathbf{N}+Z\mathbf{B}/\rho$, where $\mathbf{N}=\nabla\chi\times\mathbf{B}/B^2$, we find, similar to the calculation in Ref. 5,

$$\delta W \equiv \int P\hat{F}P\xi_0\cdot\xi_0\frac{dl}{B} = \int a_1(Z'+a_2X)^2\frac{dl}{B} + \int\{|\mathbf{N}|^2X'^2 - X^2[2(\boldsymbol{\kappa}\cdot\mathbf{N})(\mathbf{J}\times\mathbf{B}\cdot\mathbf{N}) - R(\mathbf{N}\cdot\hat{\mathbf{R}})\mathbf{N}\cdot\nabla(\rho\Omega^2) + (\rho R\Omega^2)^2(\mathbf{N}\cdot\hat{\mathbf{R}})^2/\gamma p]\}\frac{dl}{B}\,. \tag{12}$$

Here the integral is taken along the infinitely long field line, a_1 and a_2 depend on equilibrium quantities, $a_1>0$, and the energy δW is minimized by taking $Z'=-a_2X$. This leaves a quadratic form in the single variable X. In Eq. (12) $\hat{\mathbf{R}}$ is the radial unit vector, a prime denotes $\mathbf{B}\cdot\nabla$, $\mathbf{J}=\nabla\times\mathbf{B}$, and $\boldsymbol{\kappa}$ is the curvature of the magnetic field, $\boldsymbol{\kappa}=\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}$, where $\hat{\mathbf{b}}=\mathbf{B}/B$. In the absence of flow the potentially destabilizing effect comes from the familiar term[1,2] $(\boldsymbol{\kappa}\cdot\mathbf{N})(\nabla p\cdot\mathbf{N})$.

The contribution of the flow to the stability criterion is most clearly seen when using a large-aspect-ratio expansion,[5] where $\epsilon=a/R_0$, and R_0 is the major radius of the torus. Instead of nondimensionalizing the equation, we take a and the toroidal field to be of order 1, $R=O(1/\epsilon)$, $p=O(\epsilon)$, $\psi=O(1)$, and $\Omega=O(\epsilon^{3/2})$. This implies that the Mach number is $O(1)$. As in Ref. 5 one finds for an instability $\omega=O(\epsilon)$, and the middle term in Eq. (11) drops out as too small. Thus, the positivity of δW is *necessary and sufficient* for ballooning stability on the time scale of $t=O(T)$. We point out that the validity of the asymptotic expansion in small Ω leading to this result holds, as noted following Eq. (11), for t up to $O(\epsilon^{-3/2})$. A typical ballooning-mode growth time is $O(\omega^{-1})\sim\epsilon^{-1} \ll \epsilon^{-3/2}$. Thus, our criterion is indeed relevant to the modes under consideration. Continuing with the asymptotics, we have to order ϵ^2,

$$\delta W = \int\{|\mathbf{N}|^2X'^2 - X^2[2(\boldsymbol{\kappa}\cdot\mathbf{N})(\nabla p\cdot\mathbf{N}) - R(\mathbf{N}\cdot\hat{\mathbf{R}})\mathbf{N}\cdot\nabla(\rho\Omega^2)]\}\frac{dl}{B}\,. \tag{13}$$

To leading order $\boldsymbol{\kappa}=-\hat{\mathbf{R}}/R$, and R in Eq. (13) may be taken to be the constant R_0. The terms in the brackets of this equation are then equal to $2(\boldsymbol{\kappa}\cdot\mathbf{N})\mathbf{N}\cdot\nabla(p+\rho R_0^2\Omega^2/2)$. For further simplicity, we write Eq. (13) in a more familiar form[13] appropriate for the limit of $\epsilon \ll 1$ and cylindrical flux surfaces,

$$\delta W = \int_{-\infty}^{\infty}\left[[1+(\theta\hat{s}+\hat{\lambda})^2]\left(\frac{dX}{d\theta}\right)^2 - G[\cos\theta+(\theta\hat{s}+\hat{\lambda})\sin\theta]X^2\right]d\theta\,. \tag{14}$$

Here $G=2R_0(q/B_0)^2(p+\rho R_0^2\Omega^2/2)'$, $\hat{s}=rq'/q$, where r is the poloidal radial variable and the prime denotes d/dr. B_0 is the (constant) toroidal field, and the parameter $\hat{\lambda}$ is related to the previous λ such that we still need to consider all $-\infty<\hat{\lambda}<\infty$. The two other parameters of the problem are the magnetic shear $\hat{s}$ and the pressure gradient G. We find that the effect of the flow on ballooning stability, aside from changing the equilibrium state, is to modify the kinetic pressure by adding to it the flow pressure, such that the gradient of the total pressure is the force that may act against the magnetic curvature. In particular, we may *completely stabilize* the ballooning modes during their typical growth time by generating an Ω which increases fast enough from the center of the plasma to the outer flux surfaces, as has been anticipated recently.[14] This may be achieved by injecting neutral beams tangentially to the plasma boundary.

Finally, we point out that the introduction of sheared flow may not necessarily give rise to the Kelvin-Helmholtz instability. Such a side effect does not appear unavoidable since it is known that the presence of a magnetic field parallel to the flow, a condition which approximately holds in a tokamak, acts to suppress the instability.[15] Thus, by eliminating the ballooning instability, the use of sheared flow may offer the way to increase the plasma pressure to a much higher level[16] known as the "second stability regime."

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