RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 41, NUMBER 2 15 JANUARY 1990

Multiplicative noise and homoclinic crossing: Chaos

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(Received 7 September 1989)

We consider here the effect of noise on homoclinic crossing. It is shown, by means of a stochastic Melnikov function, that the noise may, on the average, suppress and, in the case considered here, induce homoclinic crossing.

There recently have been a number of interesting numerical studies of the interplay of both multiplicative and Langevin noise with chaotic behavior on attractors, utilizing discrete and continuous models.[1-4] The principal effect of noise is to destroy the periodic points embedded in the positive region of the Lyapunov exponent and to modify the period doubling route to chaos. This has also been seen in an accousto-optical device exhibiting bistability.[5] More recently,[6] "early" chaos, i.e., a shift in the onset of chaos induced by multiplicative noise in the logistic map, has been apparently found below the noise-free threshold. We would call this "noise-induced chaos." Carlson has investigated, in general, the shift map on a Cantor set in the presence of thermal noise.[7,8] He has shown that the sequences corresponding to the homoclinic points of the Cantor set may be removed by the noise.

It is our purpose here to investigate analytically such an effect on the onset of homoclinic crossing[9] in weakly dissipative systems. In a more extensive treatment,[10] we have utilized a generalization of the Melnikov function[11,12] to stochastic processes to test the effect of weak Langevin noise on homoclinic crossing. Assuming initially that the average value of the position and velocity $(\langle x(0)\rangle,\langle \dot{x}(0)\rangle)$ are the separatrix values $(x_s(0),\dot{x}_s(0))$ we have shown, using the Wentzel-Kramers-Brillouin (WKB) approximation (and numerically), that the effect of the noise was to *suppress* homoclinic crossing on the average and raise the homoclinic threshold. The system must be driven harder by the external deterministic force in the presence of Langevin noise, to induce the multiple phase-space crossings of the forward and reverse manifolds near the unstable fixed point. Here, we will see that this is not necessarily the case when $(\langle x(0)\rangle,\langle \dot{x}(0)\rangle)$ are no longer the separatrix values.

Consider a stochastic nonlinear oscillator with nonlinearity $f(x)$:

$$\dot{x}=v\,,\qquad \dot{v}+\beta f(x)+\zeta(t)f(x)=-kv+Qg(t,t_0)\,. \tag{1}$$

Here $k\equiv\tau^{-1}$ is the damping constant and $g(t,t_0)$ the external (usually sinusoidal) driving term, the oscillator being assumed to have unit mass. In the multiplicative noise term, we will take the realizations of the noise $\dot{w}=\zeta(t)$ to be white with zero mean:

$$\langle\zeta(t)\rangle=m=0\,,\quad \langle\zeta(t)\zeta(t+\tau)\rangle=\sigma^2\delta(\tau)\,.$$

For $\sigma^2=0$, it is well known that such driven oscillators exhibit homoclinic chaos, the Duffing oscillator being a classic example.[7,12] The Melnikov function[11] is the only simple test of the onset of homoclinic crossing and the subsequent multiple crossings of the forward and reverse phase-space manifolds.

In Ref. 10, we introduced a stochastic Melnikov function Δ_ζ which, in the framework of a *weak-noise* approximation, is given by

$$\Delta_\zeta(t)=-k\int_{-\infty}^{\infty}\dot{x}^2(t)dt+Q\int_{-\infty}^{\infty}\dot{x}(t)g(t,t_0)dt\equiv\Delta_s(t_0)+\Delta'_\zeta(t_0)\,, \tag{2a}$$

$$\Delta_s(t_0)=-k\int_{-\infty}^{\infty}\dot{x}_s^2(t)dt+Q\int_{-\infty}^{\infty}\dot{x}_s(t)g(t,t_0)dt\,, \tag{2b}$$

$$\Delta'_\zeta(t_0)=-2k\int_{-\infty}^{\infty}\dot{x}_s(t)\dot{X}(t)dt+Q\int_{-\infty}^{\infty}\dot{X}(t)g(t,t_0)dt-k\int_{-\infty}^{\infty}\dot{X}^2(t)dt\,, \tag{2c}$$

where $x_s(t)$ and $\dot{x}_s(t)$ are the separatrix solutions to $\ddot{x}+\beta f(x)=0$, and $(X(t),\dot{X}(t)=V(t))$ are small deviations introduced by the noise. Following the procedure of Ref. 10 we find that these deviations are solutions of the weak-noise Langevin equation for the time-dependent Ornstein-Uhlenbeck process:

$$\dot{X}=V\,,\qquad \dot{V}-\omega^2(t)X=F_\zeta(t)\,, \tag{3}$$

where

$$\omega^2(t)=-\left[\frac{d}{dx}f(x)\right]_{x_s(t)}\,,$$

$$F_\zeta(t)=-f(x_s(t))\zeta(t)\,.$$

Here, $\Delta_s(t_0)$ is the deterministic dynamics Melnikov function which for the Duffing oscillator is[12]

$$\Delta_s(t_0)=-4k/3+2^{1/2}\pi\Omega Q\sin(\Omega t_0)\,\mathrm{sech}(\pi\Omega/2)\,. \tag{4}$$

For this case we have[10]

$$\beta f(x) \equiv -x+x^3 ,$$

$$x_s(t) \equiv 2^{1/2} \operatorname{sech} t ,$$

$$\omega^2(t) \equiv 1-6\operatorname{sech}^2 t ,$$

and a sinusoidal driving term, $Q\sin[\Omega(t-t_0)]$, has been introduced.

A word should be said about the generalization. Under each realization of the ensemble of weak noise, $\Delta_\zeta^\prime$ introduces a shift in the point of homoclinic tangency, $\Delta_s(t_0^\prime)=0(\Delta_s(t_0)\gtrless 0, t_0\neq t_0^\prime)$. This is much like the deterministic effect of the term depending on the damping in Eq. (4). Under many realizations of the ensemble, there is an *average* shift $\langle\Delta_\zeta\rangle$. We must emphasize that rigorously, there is no stochastic analog to the Birkoff-Smale theorem[7] for nonlinear driven stochastic differential equations; however, we are here doing weak-noise perturbation theory and we expect the Cantor set structure not to be qualitatively modified by the noise. After tangency, $\Delta_\zeta(t_0)$ shows an infinity of zeros as in Eq. (4), and thus, for each of these ensemble realizations, the same multiple crossings of the forward and backward manifolds. For this reason, we think that the averaged quantity $\langle\Delta_\zeta\rangle$ is a significant indicator in the presence of noise.

From Eq. (3), by standard methods we may obtain the Fokker Planck equation for the probability density function $P(X,\dot X,t)$:

$$\frac{\partial P}{\partial t}=-\sum_{ij}A_{ij}\frac{\partial}{\partial x_i}x_jP+\tfrac{1}{2}\sum_{ij}B_{ij}(t)\frac{\partial^2 P}{\partial x_i\partial x_j} , \qquad (5)$$

where

$$x\equiv\begin{bmatrix}X\\ V\end{bmatrix},\quad A(t)\equiv\begin{bmatrix}0 & 1\\ \omega^2(t) & 0\end{bmatrix},\quad B(t)\equiv\begin{bmatrix}0 & 0\\ 0 & \sigma^2 f^2(x_s(t))\end{bmatrix} .$$

A formal time-dependent solution may be written down for Eq. (5) (Ref. 13, page 228 and following). It is a time-dependent Gaussian, $P(X,\dot X,t)$. We write the second term in Eq. (2a) as $(Q\dot x)_\zeta\cos\phi$ where we have set $\phi=\Omega t_0$ and $(Q\dot x)_\zeta\equiv Q\int_{-\infty}^{\infty}\dot x(t)\sin(\Omega t)dt$, the $\dot x(t)$ $[=x_s(t)+\dot X(t)]$ being Gaussian random variables. The central limit theorem[13-15] suggests that $(Q\dot x)_\zeta$ is Gaussian distributed. Now consider the first integral in (2) which may be cast in the form $-(4/\tau)\int_0^\infty \dot x^2(t)dt$. Assuming weak damping, we may write this, approximately, as $-4\bar K=-4\langle K\rangle_\zeta$, where we define

$$\bar K\equiv\tau^{-1}\int_0^\tau \dot x^2(t)dt$$

in the $\tau\to\infty$ limit, and we have assumed ergodicity for the time-averaged kinetic energy. Thus,

$$\Delta_\zeta=-4\langle K\rangle_\zeta+(Q\dot x)_\zeta\cos\phi .$$

In this weak-noise limit, and for very low damping, Δ_ζ is Gaussian distributed in amplitude, $(Q\dot x)_\zeta$, and shifted by a constant ensemble-averaged kinetic energy. We emphasize that, in this approximation, the phase ϕ, the parameter of homoclinic crossings is *not* a random variable.

Let us now consider the ensemble-averaged correction to the Melnikov function. We require the quantities $\langle X\rangle$, $\langle\dot X\rangle$, and $\langle\dot X^2\rangle$. From Eq. (3) we observe that,

$$\langle\ddot X\rangle-\omega^2(t)\langle X\rangle=0 ,$$

as in Ref. 10. There, the WKB approximation was utilized to obtain

$$\langle X\rangle=\frac{h(t)}{h(0)}\langle X(0)\rangle ,\quad \langle\dot X\rangle=\frac{\dot h(t)}{\dot h(0)}\langle\dot X(0)\rangle , \qquad (6)$$

where we have defined $\langle x(0)\rangle=x_s(0)+\langle X(0)\rangle$ and we assume that $\langle\dot X(+\infty)\rangle=0$. Thus, $\langle\dot x(+\infty)\rangle=\dot x_s(+\infty)=0$ and we are lead to the condition

$$\frac{\langle X(0)\rangle}{h(0)}=\frac{\langle\dot X(0)\rangle}{\dot h(0)}=M ,$$

where M is a constant. Effectively, the noise shifts the point of saddle maximum and not its height, otherwise the stochastic orbits pass above the unstable saddle and $\langle\Delta_s\rangle$ diverges. In the above, we have set,[10]

$$h(t)=S_0^{1/6}[\omega^2(t)]^{-1/4}\mathrm{Ai}[(\tfrac{3}{2}S_0)^{2/3}] ,$$

$$S_0(t)=\int_{t_c}^{t}\omega(t')dt' ,$$

where t_c is defined by $\omega^2(t_c)=0$, Ai being the Airy function. The above WKB approximation has been verified numerically in Ref. 10.

From Eq. (5) we may write down[10] a solution for $\langle\dot X^2\rangle$. This differs from the results of Ref. 10 in having a contribution from a *time-dependent* diffusion coefficient, $\sigma^2 f^2(x_s(t))$. However, this term is even in t and we may readily show that it does not contribute to the integrals in $\langle\Delta_\zeta\rangle$. The remaining contribution is

$$-k\int_{-\infty}^{\infty}\langle\dot X^2(t)\rangle dt=-k\frac{\langle\dot X^2(0)\rangle}{\dot h^2(0)}\int_{-\infty}^{\infty}\dot h^2(t)dt$$
$$\equiv -kL^2\int_{-\infty}^{\infty}\dot h^2(t)dt .$$

Finally we have for the ensemble-averaged $\langle\Delta_\zeta\rangle$,

$$\langle\Delta_\zeta(t_0)\rangle=-k\int_{-\infty}^{\infty}\dot x_s^2(t)dt+Q\int_{-\infty}^{\infty}\dot x_s(t)g(t,t_0)dt$$
$$-2kM\int_{-\infty}^{\infty}\dot x_s(t)\dot h(t)dt+QM\int_{-\infty}^{\infty}\dot h(t)g(t,t_0)dt-kL^2\int_{-\infty}^{\infty}\dot h^2(t)dt . \qquad (7)$$

In the Duffing oscillator, for example, the third term is zero since $\dot x_s(t)$ is odd and $\dot h(t)$ is even.

Consider homoclinic tangency. Let $(Q/k)_0$ be the $\sigma^2=0$ value determined by

$$0=-\int_{-\infty}^{\infty}\dot{x}_s^2(t)dt+\left(\frac{Q}{K}\right)_0\int_{-\infty}^{\infty}\dot{x}_s(t)g(t,t_0)dt\,.$$

Let $(Q/k)-(Q/k)_0\equiv\delta$ be the ensemble-averaged noise shift of the homoclinic tangency. δ is determined by

$$0=-L^2\int_{-\infty}^{\infty}\dot{h}^2(t)dt+M\left(\frac{Q}{k}\right)_0\int_{-\infty}^{\infty}\dot{h}(t)g(t,t_0)dt+\delta\int_{-\infty}^{\infty}[\dot{x}_s(t)+M\dot{h}(t)]g(t,t_0)\,. \tag{8}$$

Now, M may be positive or negative. The quantity in square brackets in the third term above is positive since, by hypothesis, $\dot{x}_s(t)\gg M\dot{h}(t)$ for all time. $g(t,t_0)$ is taken as a simple sinusoidal periodic function and we set the phase factor to be positive in the second and third terms of (8). Hence we find,

$$M<0\rightarrow\delta\geq 0\,, \tag{9a}$$

$$M>0\rightarrow\delta\lesssim 0\,, \tag{9b}$$

$$L^2\int_{-\infty}^{\infty}\dot{h}^2(t)dt\lesssim|M|\left(\frac{Q}{k}\right)_0\int_{-\infty}^{\infty}\dot{h}(t)g(t,t_0)dt\,. \tag{9c}$$

In (9b) and (9c) we have a condition for ensemble-averaged *noise-induced crossing*. This condition may be written as

$$\frac{\langle\dot{X}^2(0)\rangle}{|\langle\dot{X}(0)\rangle|(Q/k)_0}<\frac{\dot{h}^{-1}(0)\int_{-\infty}^{\infty}\dot{h}(t)g(t,t_0)dt}{\dot{h}^{-2}(0)\int_{-\infty}^{\infty}\dot{h}^2(t)dt}\,. \tag{10}$$

We recall that our weak-noise assumption excludes higher-order terms in the second of Eq. (3). Hence, the transport equation for the mean value $\langle X(t)\rangle$ does not include higher moments of X and a Fokker Planck equation of the form (5) may be written down and solved for this problem. Since the time-dependent diffusion term does not contribute to the Melnikov function, we are lead to the condition (10) that depends solely on the initial values of the statistics of the velocity deviation, $\dot{X}(t)$ induced by the noise. For sinusoidal driving, the value of the right-hand side is [for the Duffing oscillator,[10] $(Q/k)_0=0.71$] approximately 0.85. Under these conditions, homoclinic crossing could, on average, occur for lower Q/k values than for the deterministic ($\sigma^2=0$) case.

This work was performed while W.C.S. was an A.S.E.E. visiting faculty member at Naval Ocean Systems Center.

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