RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 41, NUMBER 2 15 JANUARY 1990

Acausal filters for chaotic signals

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(Received 18 September 1989)

It is known that attempts to improve the signal-to-noise ratio of measured chaotic signals through low-pass filtering lead to wrong results from subsequent calculations of the attractor dimension. We show that the use of acausal filters avoids this problem. The benefit in recovering the chaotic signal from noise is demonstrated to be considerable.

Deterministic chaos is a ubiquitous phenomenon in all fields of science. For a characterization of chaotic dynamics, algorithms are widely used that give estimates of the process entropy and the attractor dimension from a single time series. Such procedures are by no means trivial because these algorithms are strictly proven only for conditions unattainable in real world experiments: One must work with a finite number of data points, each of which has finite resolution, and very often, experimental signals are marred by noise that makes it difficult to extract reliable information about the deterministic process. For example, an extraction of the attractor dimension is subject to systematic errors due to both additive and quantization noise.[1]

It is therefore desirable to reduce the noise as much as possible. A standard laboratory procedure to improve signal-to-noise ratio is the application of a low-pass filter. This relies on the fact that if the digitization is done reasonably, i.e., at such a rate that no high-frequency information is lost, it is automatically implied that the signal bandwidth is less than the Nyquist frequency. Noise, on the other hand, is often at least approximately white. In such a case a low-pass filter would attenuate the noise more than the signal, and thereby the ratio would improve.

Unfortunately, there is a severe penalty when this procedure is applied to chaotic signals. It has been suggested by Badii and Politi[2] that low-pass filtering of chaotic signals can lead to grossly false results in a subsequent evaluation of the filtered signals for the attractor dimension. The idea has been tested and confirmed with a computer-simulated filter in Ref. 3 and with real world filters in Ref. 4. In the latter paper it was also observed that the digitization noise leads to a different systematic error which was studied in more depth in Ref. 1. It was further shown in Ref. 4 that the entropy of the process is not affected by either of these systematic errors.

It might seem that one has the choice between relying on noisy signals that do not lead to well-defined dimension estimates, or "beautified" signals that do yield values but false ones. In this Rapid Communication we show that this dilemma can be avoided: there are filters that can achieve both, cleaning the signal without leading to false dimensions.

In their prediction Badii and Politi[2] considered a particular class of filters, namely those which can be described by differential equations. Adding such a filter to a dynamical system raises the number of differential equations describing the combined system. This implies an increase in the number of Lyapunov exponents. The new exponents can, depending on their values in comparison to the other exponents of the system, enter the Kaplan-Yorke equation and thus bring about a dimension increase.

Real world filters do not only attenuate different frequency components differently ("amplitude response"), they also cause frequency-dependent phase shifts ("phase response"). Both are inseparably connected through Kramers-Kronig dispersion relations which are a consequence of causality, i.e., the time ordering of cause and effect.

A simple *RC* low-pass filter, for example, contains a capacitor which in its electric field can store energy. It thus forms a memory element (with decay time $\tau = RC$). Therefore, at each instant in time the filter's output contains information about both the *present* and *past* filter's input, but *no* information about the *future* input. Such a filter is therefore equally temporally asymmetric as the flow of time, or arrow of time, itself.

It was speculated in Ref. 4 that the filter-induced dimension increase is deeply related to the phase shifts brought about by these "causal" filters. If that is true, filters that introduce no phase shifts might not affect the dimension. We show here that this is indeed the case. However, filters with a low-pass frequency response and a perfectly flat phase response are not consistent with the dispersion relations.

There is a class of filters called "acausal" because they are not subject to the time arrow; they therefore do not have to obey the dispersion relations.[5] Acausal filters can be used whenever a certain time span of a signal is first stored, then evaluated afterwards. Time reversal, e.g., amounts then to nothing more than going backwards through a file. The only case where acausal filters cannot be used is when filtering is required in *real time*.

In our experiments, we performed acausal filtering through manipulating measured signals in the frequency domain. The time domain signals were first Fourier transformed, then multiplied by a filter function. A low-pass filter requires a filter function that peaks at $\Omega = 0$, then rolls off towards higher frequencies. Finally, we transformed back to the time domain.

As a signal source we used an electronic chaos generator very similar to the one described in Ref. 6 and employed in Ref. 4. This source has the advantage that the Lyapunov exponents of the signal are known. Here we will use the same time units as in Ref. 4 for the Lyapunov exponents and filter rolloff frequencies. The chaotic signal $x(t)$ was measured with ten-bit resolution with a desktop computer, equipped with an analog-to-digital converter. Fourier transformation was performed for 2^{15} points with a fast Fourier transform routine on the same computer. The result are sine and cosine coefficients $S(\Omega)$ and $C(\Omega)$ which are related to the power spectrum $P(\Omega)$ through $P(\Omega)=S(\Omega)^2+C(\Omega)^2$. Both coefficients were multiplied by the same filter function $F(\Omega)=[1+(\Omega\tau)^2]^{-1/2}$. Note that using the same real function for both $S(\Omega)$ and $C(\Omega)$ creates no phase shifts whatsoever. Finally, the data were transformed back to the time domain.

The particular choice of $F(\Omega)$ was intended to facilitate a comparison with a causal first-order low-pass filter which has a square-root-of-Lorentzian amplitude response. For this comparison we filtered the test data also with a simulated RC filter. While this can be done in the frequency domain also (the filter function would then have to be complex), we found it more convenient to do it in the time domain, as in Refs. 2–4.

From the two differently filtered versions of the test data, correlation dimensions D_2 and entropies K_2 were then determined using the well-known algorithm due to Grassberger and Procaccia.[7] We used 10000 of the data points, again rounded to ten-bit precision for better comparison. The whole procedure was repeated several times for different rolloff frequencies $\eta=1/\tau$. We find that whatever η, the entropy is unaffected by either filter; this was to be expected from the discussion in Ref. 4. As for the dimension, our results are presented in Fig. 1.

The causal filter (open squares) leads to the increase as predicted in Ref. 2 (solid line). (As in Ref. 4, the data points fall below the solid line because the latter is for D_1, and $D_1 \geq D_2$.) In contrast, the acausal filter of the same η does not affect the dimension (filled squares). Let us remark, however, that for extremely small η ($\eta \leq 0.05$) where the causal filter still yields a dimension (if the wrong one), there is no useful plateau and thus no dimension estimate for the acausal filter. At this point it is not entirely clear whether this is really indigenous to the acausal filter, or caused by some other step in our procedure.

So far we have considered low-noise signals which are not very realistic for most experiments. In the next step we added about -30-dB white noise to the test data and repeated the procedure on this noisy signal. It now becomes apparent that the acausal filter does not only do no harm to the dimension estimate but that it is actually beneficial. Figure 2 shows the slope of the correlation integral of the Grassberger-Procaccia technique for embedding dimension $d=10$, for both types of filters. It is obvious that there is no useful plateau for the causally filtered data because noise and filter-induced dimension increase conspire to drive the slope up. In contrast, the data from the acausal filter of the same rolloff frequency give a

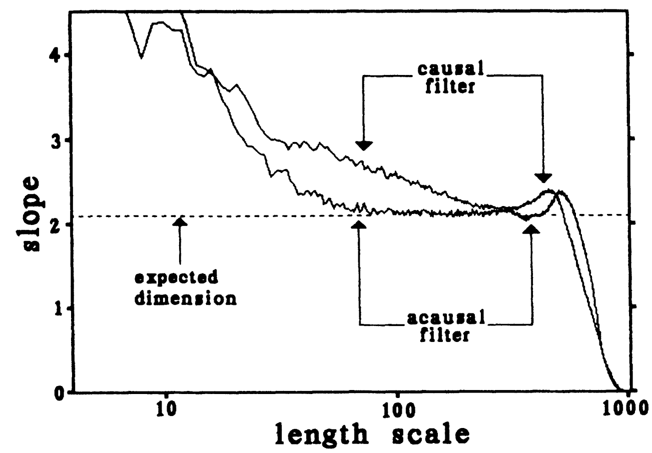

FIG. 2. Slope of correlation integral from Grassberger-Procaccia algorithm, obtained from noisy data. For clarity, only the slope for embedding dimension $d=10$ is shown. Filter rolloff frequency was $\eta=0.075$ for both traces. Only the acausal filter gives a plateau at the correct value of the dimension.

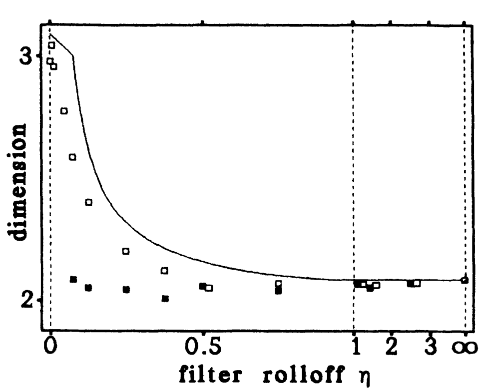

FIG. 1. Attractor dimension as a function of filter rolloff frequency η. Solid line: Lyapunov dimension D_1 as predicted in Ref. 2. Open squares: D_2 for the causal filter. Filled squares: D_2 for the acausal filter. D_1 is an upper bound for D_2. Note change of scale at $\eta=1$.

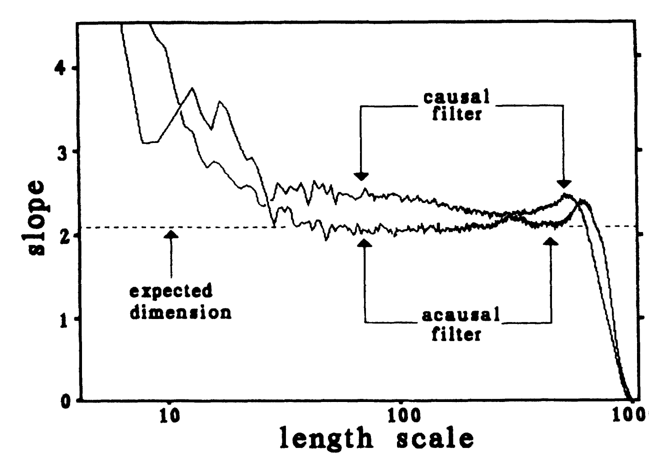

FIG. 3. Same as Fig. 2 except that coarsely digitized data (seven bits) were used and that $\eta=0.125$.

well-defined plateau which yields the same value as the low-noise data. The acausal filter can thus correctly recover the deterministic part of the signal. Its superiority over the causal filter is obvious.

Finally, we repeated the procedure after we rounded off the ten-bit precision low-noise test data to seven-bit resolution. This amounts to an increase in the quantization noise from about -60 to -42 dB. In general, quantization noise does not act the same way as uncorrelated additive noise,[1] but here the results are similar. We find that while the causal filter leads to the usual substantial dimension overestimate, the acausal filter again recovers the correct dimension (see Fig. 3). This is remarkable because in a very coarse-grained representation of phase space one would expect that most information about the intricate fractal structure of the attractor is lost beyond retrieval. However, the smoothing by the acausal filter in effect interpolates between the coarse steps and "guesses" at the true structure—successfully at least in the case tested here.

We conclude that processing chaotic experimental signals by an acausal filter prior to further evaluation is advantageous. Particularly in cases of bad signal-to-noise ratio, a substantial benefit can be expected.

More research is required, however, to determine exactly why the dimension increase is avoided. We think that an important point is the loss of asymmetry between past and future brought about by acausal filters, particularly by those with real coefficients as used here. While an attractor dimension, as a geometric concept, is certainly invariant under time reversal, the Kaplan-Yorke equation is not, because arranging the exponents in decreasing order presupposes the forward direction of the time arrow (note that time reversal inverts the Lyapunov spectrum). This suggests that the Kaplan-Yorke equation does not hold for the case considered here. Consequently, the above explanation for the usual dimension increase would not be applicable.

The valuable assistance of M. Dämmig with the computations is gratefully acknowledged.

[1]M. Möller, W. Lange, F. Mitschke, N. B. Abraham, and U. Hübner, Phys. Lett. A **138**, 176 (1989).

[2]R. Badii and A. Politi, in *Dimensions and Entropies in Chaotic Systems*, edited by G. Mayer-Kress (Springer-Verlag, Berlin, 1986), p. 67.

[3]R. Badii, G. Broggi, B. Derighetti, M. Ravani, S. Ciliberto, A. Politi, and M. A. Rubio, Phys. Rev. Lett. **60**, 979 (1988).

[4]F. Mitschke, M. Möller, and W. Lange, Phys. Rev. A **37**, 4518 (1988).

[5]See, e.g., W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes: The Art of Scientific Computing* (Cambridge Univ. Press, Cambridge, 1986), p. 437.

[6]F. Mitschke and N. Flüggen, Appl. Phys. B **35**, 59 (1984).

[7]P. Grassberger and I. Procaccia, Phys. Rev. Lett. **50**, 346 (1983).