Quantum theory of solitons in optical fibers. II. Exact solution

Y. Lai and H. A. Haus
Department of Electrical Engineering and Computer Science and Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 2 December 1988)

In the preceding paper [paper I of a two-part study; Lai and Haus, Phys. Rev. A **40**, 844 (1989)] we have used the time-dependent Hartree approximation to solve the quantum nonlinear Schrödinger equation. In the present paper, the eigenstates of the Hamiltonian are constructed exactly by Bethe's ansatz method and are superimposed to construct exact soliton states. Both fundamental and higher-order soliton states are constructed and their mean fields are calculated. The quantum effects of soliton propagation and soliton collisions are studied in the framework of this construction. It is shown that a soliton experiences dispersion as well as phase spreading. The magnitude of this dispersion is estimated and is shown to be very small when the average photon number of the soliton is much larger than unity. The phase and position shifts due to a collision and the uncertainty of these shifts are also calculated.

I. INTRODUCTION

In the preceding paper (paper I),[1] it was shown that the quantum nonlinear Schrödinger equation (QNSE) is equivalent to the evolution equation of one-dimensional bosons with δ-function interactions. By applying the time-dependent Hartree approximation to this equation, we constructed approximate eigenstates of the Hamiltonian. These eigenstates are the exact eigenstates of the photon-number operator but not the exact eigenstates of the momentum and the Hamiltonian operator. From the uncertainty relations, they can be interpreted as soliton states with a mean position and a random phase. The uncertainty relations also suggest that one has to superimpose these states to construct a soliton state with a mean phase. It was shown that due to the uncertainty of the photon number, a soliton experiences phase spreading when it propagates. However, the Hartree approximation suppresses the effect of the momentum uncertainty. A distribution of momentum must be associated with a soliton with a mean position. Just as the uncertainty of photon number causes the phase-spreading effect, this momentum uncertainty causes a dispersion effect of its own. We study this effect in the present paper by using the exact eigenstates of the Hamiltonian.

It is surprising that the QNSE can be solved exactly. It was first solved by Bethe's ansatz method[2-6] and then by the quantum inverse scattering method.[7-10] In the present paper we follow Bethe's ansatz method to construct the eigenstates of the Hamiltonian. We then superimpose these eigenstates to construct soliton states. Both fundamental and higher-order soliton states are constructed and their mean fields are calculated to justify the construction. All the classical results can be recovered in the limit of large photon number. It is found that due to the uncertainty of momentum, a soliton experiences dispersion when it propagates and the magnitude of this effect is significant only after many soliton periods. The phase and position shifts due to a collision and the uncertainty of these shifts are also calculated.

II. EXACT SOLUTION FOR EIGENSTATES WITH KERR INTERACTIONS

In paper I, we have shown that the problem can be stated in the Schrödinger picture as follows:

$$i\hbar\frac{d}{dt}|\psi\rangle = H_s|\psi\rangle \ , \tag{2.1}$$

with

$$H_s = \hbar\left[\int \hat{\phi}_x^\dagger(x)\hat{\phi}_x(x)dx + c\int \hat{\phi}^\dagger(x)\hat{\phi}^\dagger(x)\hat{\phi}(x)\hat{\phi}(x)dx\right] . \tag{2.2}$$

Here $\hat{\phi}(x)$ and $\hat{\phi}^\dagger(x)$ are the field operators in the Schrödinger picture and satisfy the following commutation relations:

$$[\hat{\phi}(x'),\hat{\phi}^\dagger(x)] = \delta(x-x') \ , \tag{2.3a}$$

$$[\hat{\phi}(x'),\hat{\phi}(x)] = [\hat{\phi}^\dagger(x'),\hat{\phi}^\dagger(x)] = 0 \ . \tag{2.3b}$$

Expanding $|\psi\rangle$ in Fock space and substituting it into (2.1), one has

$$|\psi\rangle = \sum_n a_n \int \frac{1}{\sqrt{n!}} f_n(x_1,\ldots,x_n,t) \times \hat{\phi}^\dagger(x_1)\cdots\hat{\phi}^\dagger(x_n)dx_1\cdots dx_n|0\rangle \ , \tag{2.4}$$

$$\sum_n |a_n|^2 = 1 \ , \tag{2.5}$$

$$\int |f_n(x_1,\ldots,x_n,t)|^2 dx_1\cdots dx_n = 1 \ , \tag{2.6}$$

$$i\frac{d}{dt}f_n(x_1,\ldots,x_n,t)$$
$$=\left[-\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}+2c\sum_{1\le i<j\le n}\delta(x_j-x_i)\right]\times f_n(x_1,\ldots,x_n,t)\ . \quad (2.7)$$

Here $f_n(x_1,\ldots,x_n,t)$ is a symmetric function of x_j and (2.5) and (2.6) are the normalization conditions for a_n and f_n, respectively. Equation (2.7) is just the Schrödinger equation for a one-dimensional system of bosons with δ-function interactions. The t dependence in (2.7) can be factored out by assuming a solution of the form

$$f_n(x_1,\ldots,x_n,t)=f_n(x_1,\ldots,x_n)e^{-iE_nt}\ . \quad (2.8)$$

The equation for $f_n(x_1,\ldots,x_n)$ is

$$\left[-\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}+2c\sum_{1\le i<j\le n}\delta(x_j-x_i)\right]f_n(x_1,\ldots,x_n)=E_nf_n(x_1,\ldots,x_n)\ . \quad (2.9)$$

It turns out that Eq. (2.9) can be solved exactly.

Since f_n is a symmetric and continuous function, it is enough to specify its value in the region $x_1\le x_2\le\cdots\le x_n$. In the regions $x_j\ne x_i$, all the δ functions in (2.9) vanish and the solutions of (2.9) are of the exponential form

$$\exp\left[i\sum_{j=1}^{n}k_jx_j\right]\ . \quad (2.10)$$

To satisfy the symmetry condition, all the permutation terms should be included. Therefore, the general form of the solutions is

$$f_n(x_1,\ldots,x_n)=\sum_{\{Q\}}A_Q\exp\left[i\sum_{j=1}^{n}k_{Q(j)}x_j\right]\ , \quad (2.11)$$

where the summation over $\{Q\}$ is the summation over all possible permutations of $[1,2,\ldots,n]$ and $Q(j)$ is the jth component of Q. The δ functions in Eq. (2.9) impose boundary conditions at the boundaries $x_j=x_i$. At these boundaries, there is a discontinuity in the slope of the function f_n. We show in Appendix A that these boundary conditions impose the relation among the A_Q's:

$$A_{Q'}=\frac{k_{Q(j+1)}-k_{Q(j)}+ic}{k_{Q(j+1)}-k_{Q(j)}-ic}A_Q\ . \quad (2.12)$$

Here Q' is the permutation derived from Q by interchanging the jth and $(j+1)$th components.

$$Q,Q'\in\{Q\}\ ,$$
$$Q(l)=Q'(l)\ \text{ for } l\ne j,j+1\ ,$$
$$Q(j)=Q'(j+1)\ ,$$
$$Q(j+1)=Q'(j)\ .$$

Reintroducing the t dependence, one has

$$f_n(x_1,\ldots,x_n,t)=e^{-iE_nt}\sum_{\{Q\}}A_Q\exp\left[i\sum_{j=1}^{n}k_{Q(j)}x_j\right] \quad (2.13)$$

for $x_1\le x_2\le\cdots\le x_n$ with the energy expressed by

$$E_n=\sum_{j=1}^{n}k_j^2\ . \quad (2.14)$$

In general, k_j must be real because the wave functions cannot be infinite. However, for negative c, a rising exponential for $x_i<x_j$ can be matched to a falling exponential for $x_i>x_j$. Thus negative values of c make "bound" states possible, states that cluster around the planes $x_i=x_j$ in multidimensional space. No such solutions exist for positive c. To be explicit, in the case of $c<0$, bound state solutions exist if k_j satisfies the following condition:

$$k_j=p+i\frac{c}{2}(n-2j+1),\ \ j=1,2,\ldots,n\ . \quad (2.15)$$

The reason why we need condition (2.15) can be seen by substituting it into (2.12). We find that all the A_Q vanish except $A_{[1,2,\ldots,n]}$. Therefore

$$f_{np}(x_1,\ldots,x_n)=\mathcal{N}_n\exp\left[ip\sum_{j=1}^{n}x_j+\frac{c}{2}\sum_{1\le i<j\le n}|x_j-x_i|\right]\ , \quad (2.16)$$

$$\mathcal{N}_n=A_{[1,2,\ldots,n]}\ . \quad (2.17)$$

If any other A_Q is nonzero, the wave function is not bound. This fact thus leads to the quantization condition (2.15). f_{np} of (2.16) is symmetric in the x_i's and applies to all regions.

If any pair of x_j values is widely separated, the wave function (2.16) is very small. This is why these solutions are called bound-state wave functions. With (2.16), one can construct the bound states that are the eigenstates of the Hamiltonian (2.2).

$$|n,p\rangle=\frac{1}{\sqrt{n!}}\int f_{n,p}(x_1,\ldots,x_n)\hat{\phi}^\dagger(x_1)\cdots\hat{\phi}^\dagger(x_n)\times dx_1\cdots dx_n|0\rangle\equiv\hat{R}^\dagger(n,p)|0\rangle\ , \quad (2.18)$$

with the eigenvalue

$$E(n,p)=np^2-\frac{|c|^2}{12}n(n^2-1)\ . \quad (2.19)$$

The energy is the sum of the net kinetic energy of the bosons with momentum p each and (negative) potential energy due to the binding force of the Kerr nonlinearity $(|c|^2/12)(n^2-1)$. The dependence on n follows from the functional dependence of the nonlinearity which is quadratic in $\hat{\phi}^\dagger(x)\hat{\phi}(x)$. Reintroducing the t dependence, we have

$$|n,p,t\rangle=e^{-iE(n,p)t}|n,p\rangle\ . \quad (2.20)$$

In Appendix B, we show that these states can be normalized,

$$\langle n',p'|n,p\rangle=\delta_{nn'}\delta(p-p')\ , \qquad (2.21)$$

and the normalization constant $\mathcal{N}_n$ in (2.16) is

$$\mathcal{N}_n=\frac{\sqrt{(n-1)!}}{\sqrt{2\pi}}|c|^{(n-1)/2}\ . \qquad (2.22)$$

It is easy to prove that $|n,p,t\rangle$ is also the eigenstate of the photon-number operator $\hat{N}$ and the momentum operator $\hat{P}$:

$$\hat{N}|n,p,t\rangle=n|n,p,t\rangle\ , \qquad (2.23)$$

$$\hat{P}|n,p,t\rangle=\hbar np|n,p,t\rangle\ . \qquad (2.24)$$

The matrix elements of the field operator for these eigenstates are[10,11]

$$\langle n',p'|\hat{\phi}(x)|n+1,p\rangle$$

$$=\delta_{nn'}\frac{\sqrt{n(n+1)}}{2\pi}|c|^{(4n-1)/2}n!(n-1)!e^{i[(n+1)p-np']x}\times\prod_{j=1}^{n}\frac{1}{\left[(p-p')^2+|c|^2\frac{(2n-2j+1)^2}{4}\right]} \qquad (2.25)$$

$$\approx\delta_{nn'}\frac{\sqrt{n(n+1)}}{2}|c|^{-1/2}e^{i[(n+1)p-np']x}\times\operatorname{sech}\left[\frac{\pi}{|c|}(p-p')\right]\ , \qquad (2.26)$$

$$\langle n',p'|\hat{\phi}^\dagger(x)\hat{\phi}(x)|n,p\rangle$$

$$=\delta_{nn'}\frac{n}{2\pi}|c|^{2(n-1)}n!(n-1)!e^{i(np-np')x}\times\prod_{j=1}^{n-1}\frac{1}{[(p-p')^2+j^2|c|^2]} \qquad (2.27)$$

$$\approx\delta_{nn'}\frac{n^2}{2|c|}e^{in(p-p')x}\frac{p-p'}{\sinh\left[\frac{\pi}{|c|}(p-p')\right]}\ . \qquad (2.28)$$

The details of the calculation can be found in Appendix C. Computer calculations show that (2.26) and (2.28) are good approximations even when n is not large.

Note that the Fourier transform of a sech is also a sech, and the Fourier transform of a sech^2 is proportional to $k/\sinh k$. Thus the approximate expressions (2.26) and (2.28) suggest the construction of a sech-shape pulse by superposition of different momentum states.

III. CONSTRUCTION OF FUNDAMENTAL SOLITON STATES AND QUANTUM DISPERSION

The criteria for the construction of localized soliton states are the following: (1) a soliton state should be a solution of the governing Eq. (2.1), and (2) the mean value of the field operator should look like a soliton. If we tried to construct a soliton state out of an eigenstate of the Hamiltonian, we would obtain from (2.25)–(2.28)

$$\langle n,p,t|\hat{\phi}(x)|n,p,t\rangle=0\ , \qquad (3.1)$$

$$\langle n,p,t|\hat{\phi}^\dagger(x)\hat{\phi}(x)|n,p,t\rangle \text{ is independent of } x\ . \qquad (3.2)$$

It is clear that $|n,p,t\rangle$ is not a localized soliton state because it does not satisfy the second condition. However, on the basis of the uncertainty relations as discussed in paper I of this study, we can interpret it as a soliton state with a random phase and a random mean position. Indeed, $|n,p,t\rangle$ is an eigenstate of $\hat{N}$ and $\hat{P}$, and the uncertainty relations require that its phase and mean position should be random; thus the mean field is zero and the mean intensity is constant. To construct localized soliton states with a mean phase and a mean position, one has to superimpose the eigenstates of the Hamiltonian

$$|\psi\rangle=\sum_n a_n\int g_n(p)|n,p,t\rangle dp\ . \qquad (3.3)$$

Here we require

$$\sum_n|a_n|^2=1\ , \qquad (3.4)$$

$$\int|g_n(p)|^2dp=1\ . \qquad (3.5)$$

The natural choices for a_n and $g_n(p)$ are a Poisson distribution and a Gaussian distribution, respectively,

$$a_n=\frac{\alpha_0^n}{\sqrt{n!}}e^{-|\alpha_0|^2/2}\ , \qquad (3.6)$$

$$g_n(p)=\frac{1}{(\Delta p)^{1/2}(\pi)^{1/4}}\exp\left[-\frac{1}{2}\frac{|p-p_0|^2}{(\Delta p)^2}\right]e^{-inpx_0}\equiv g(p)e^{-inpx_0}\ . \qquad (3.7)$$

To justify our construction we calculate the mean value of the field operator. The result is given below and the details are presented in Appendix D:

$$\langle\psi_s|\hat{\phi}(x)|\psi_s\rangle\approx\sum_n\frac{|\alpha_0|^{2n}}{n!}\exp(-|\alpha_0|^2)\int\frac{1}{(\Delta p)\sqrt{\pi}}\exp\left[-\frac{(p-p_0)^2}{(\Delta p)^2}\right]\times\frac{\alpha_0\sqrt{n}}{2}|c|^{1/2}\left[\exp\left[i\frac{|c|^2n(n+1)}{4}t-ip^2t+ip(x-x_0)\right]\times\operatorname{sech}(\tfrac{1}{2}|c|(n+\tfrac{1}{2})(x-x_0-2pt)\right]dp\ . \qquad (3.8)$$

In the derivation of (3.8), we made the approximations (2.26) and (2.28) and

$$n_0 = |\alpha_0|^2 \gg 1 \ , \tag{3.9a}$$

$$|c| \ll 1 \ , \tag{3.9b}$$

$$\Delta p \gg |c| \ , \tag{3.10a}$$

$$|c|^2 t \ll 1 \ . \tag{3.10b}$$

Condition (3.10a) ensures that the mean field has a sech shape and condition (3.10b) ignores the higher-order dispersion effect (see Appendix D).

Equation (3.8) makes a very important statement. The expectation value of the field is the average of a set of classical soliton solutions with different group and phase velocities. The phase velocities depend on the photon number, the group velocities depend on the momentum. This is a surprising result, because the field propagates in a nonlinear medium, and hence a simple superposition of solutions as the expectation value of the field was not anticipated. The result has valuable predictive value. Since the superposition is of many different pulse shapes with different phases, a spreading of the phase and amplitude is to be expected. In paper I of this study we have studied the phase spreading effect and have shown that the magnitude of phase spreading is significant with a characteristic length less than a soliton period. The magnitude of dispersion can be estimated as follows: From (3.8), one can expect that the width of a soliton is doubled when

$$(\Delta p)t \approx \frac{1}{|c| n_0/2} \ . \tag{3.11}$$

Therefore, the characteristic "time" of dispersion is

$$t_{\text{dis}} \approx \frac{2}{|c| n_0 (\Delta p)} \ . \tag{3.12}$$

Comparing (3.12) with the soliton period

$$t_s \approx \frac{8\pi}{n_0^2 |c|^2} \ , \tag{3.13}$$

one has

$$\frac{t_{\text{dis}}}{t_s} \approx \frac{n_0 |c|}{4\pi(\Delta p)} \ . \tag{3.14}$$

Note that in order to localize the soliton, the required bandwidth is condition (3.10a). However, from (3.14) one can see that if

$$|c| \ll \Delta p \ll n_0 |c| \ , \tag{3.15}$$

then the soliton is localized and the dispersion effect is significant only after many soliton periods. Since c is usually very small and $n_0|c|$ is large, condition (3.15) is usually satisfied. To clarify this result, let us compare a soliton with a pulse in a linear, dispersive medium. A pulse in a linear, dispersive medium can be considered to be a superposition of plane waves $\exp(-ik^2 t + ikx)$,[1] whereas from (3.8), a soliton can be considered to be a superposition of sech pulses with a width much smaller than the inverse of the momentum bandwidth. The dispersion effect is proportional to the bandwidth Δp of the momentum. In a linear medium, a bandwidth of the order of $1/\Delta x$ is necessary to construct a pulse with a width Δx because the distribution of momentum is the Fourier transform of the pulse waveform. However, to construct a soliton with a width Δx, the bandwidth can be much less than $1/\Delta x$ as indicated by (3.15) (note that $n_0|c|$ is of the order of the inverse of the soliton width). Therefore, the dispersion effect of a soliton can be much less than that of a pulse in a linear, dispersive medium.

IV. CONSTRUCTION OF HIGHER-ORDER SOLITON STATES AND SOLITON COLLISION

We have already used the Hartree approximation to construct approximate higher-order soliton states and studied soliton collision effects. However, more insight about the quantum nature of solitons can be gained by using the exact solutions of (2.7). In this section we construct two-soliton states. Other higher order-soliton states can be constructed in the same way.

We start from the general solution (2.16) with $n = n_1 + n_2$. If one chooses

$$k_j = p_1 + \frac{ic}{2}(n_1 - 2j + 1) \quad j = 1, \dots, n_1 \ , \tag{4.1}$$

$$k_{n_1+j} = p + \frac{ic}{2}(n_2 - 2j + 1) \quad j = 1, \dots, n_2 \ , \tag{4.2}$$

then

$$f_{n_1 p_1 n_2 p_2}(x_1, \dots, x_{n_1+n_2}) = \sum_{\{Q\}} A_Q F_Q(x_1, \dots, x_{n_1}, \dots, x_{n_1+n_2}) \ . \tag{4.3}$$

Here F_Q is a symmetric function of x_j.

$$F_Q(x_1, \dots, x_{n_1+n_2}) = \exp\left[ip_1 \sum_{j=1}^{n_1} x_{Q^{-1}(j)} + ip_2 \sum_{j=n_1+1}^{n_1+n_2} x_{Q^{-1}(j)} \right] \exp\left[\frac{c}{2} \sum_{1 \le i < j \le n_1} (x_{Q^{-1}(j)} - x_{Q^{-1}(i)}) \right]$$
$$\times \exp\left[\frac{c}{2} \sum_{n_1+1 \le i < j \le n_1+n_2} (x_{Q^{-1}(j)} - x_{Q^{-1}(i)}) \right] \tag{4.4}$$

for $x_1 \le x_2 \le \cdots \le x_{n_1+n_2}$.

In (2.7) the summation over $\{Q\}$ is the summation over all possible permutations of $[1,2,\ldots,n_1+n_2]$. However, because of the special values of k_j in (4.1)–(4.2), A_Q is zero if the order of $[1,2,\ldots,n_1]$ or $[n_1+1,\ldots,n_1+n_2]$ is permuted (see Sec. III). Therefore, in (4.3) the summation over $\{Q\}$ is the summation over all possible permutations of $[1,2,\ldots,n_1+n_2]$ with the order of $[1,\ldots,n_1]$ and $[n_1+1,\ldots,n_1+n_2]$ unchanged. In (4.4), Q^{-1}, the inverse of Q, appears because we have converted the permutation over k into the permutation over x.

The coefficients A_Q in (4.3) also have to satisfy (2.12). It can be seen from (2.12) that they differ from one another only by a certain phase. As an example and also for later use, we calculate the relation between

$$A_{\text{in}} = A_{[1,2,\ldots,n_1,n_1+1,\ldots,n_1+n_2]}$$

and

$$A_{\text{out}} = A_{[n_1+1,\ldots,n_1+n_2,1,\ldots,n_1]}$$

(Refs. 5 and 12). The details are given in Appendix E:

$$A_{\text{out}} = e^{i\theta(n_1,p_1,n_2,p_2)} A_{\text{in}} , \tag{4.5}$$

with

$$\theta(n_1,p_1,n_2,p_2) = -\left[4\sum_{j=1}^{n_1-1} \tan^{-1}\left[\frac{|c|(n_2-n_1+2j)/2}{p_2-p_1}\right] + 2\tan^{-1}\left[\frac{|c|(n_2-n_1)/2}{p_2-p_1}\right] + 2\tan^{-1}\left[\frac{(n_2+n_1)/2}{p_2-p_1}\right]\right] . \tag{4.6}$$

With (4.3), one can construct the bound state,

$$\begin{aligned} |n_1,p_1,n_2,p_2\rangle &= \sum_{\{Q\}} A_Q \int_{-\infty}^{\infty} F_Q(x_1,\ldots,x_{n_1+n_2}) \prod_{j=1}^{n_1+n_2} \hat{\phi}^\dagger(x_j)dx_j|0\rangle \\ &= (n_1+n_2)!\sum_{\{Q\}} A_Q \int_{x_1\le x_2 \cdots \le x_{n_1+n_2}} F_Q(x_1,\ldots,x_{n_1+n_2}) \prod_{j=1}^{n_1+n_2} \hat{\phi}^\dagger(x_j)dx_j|0\rangle . \end{aligned} \tag{4.7}$$

Reintroducing the t dependence, one has

$$|n_1,p_1,n_2,p_2,t\rangle = e^{-iE(n_1,p_1,n_2,p_2)t}|n_1,p_1,n_2,p_2\rangle , \tag{4.8}$$

with

$$E(n_1,p_1,n_2,p_2) = n_1p_1^2+n_2p_2^2-\frac{|c|^2}{12}n_1(n_1^2-1)-\frac{|c|^2}{12}n_2(n_2^2-1) . \tag{4.9}$$

The localized two-soliton states can be constructed by superimposing the bound states,

$$|\psi_s\rangle = \sum_{n_1,n_2} a_1(n_1)a_2(n_2) \times \int\int g_{n_1}(p_1)g_{n_2}(p_2) \times |n_1,p_1,n_2,p_2,t\rangle dp_1dp_2 , \tag{4.10}$$

with

$$a_1(n_1) \equiv \frac{(\alpha_{10})^{n_1}}{\sqrt{n_1!}}e^{-|\alpha_{10}|^2/2} , \tag{4.11a}$$

$$a_2(n_2) \equiv \frac{(\alpha_{20})^{n_2}}{\sqrt{n_2!}}e^{-|\alpha_{20}|^2/2} , \tag{4.11b}$$

$$g_{n_1}(p_1) \equiv \frac{1}{(\Delta p_1)^{1/2}(\pi)^{1/4}}\exp\left[-\frac{1}{2}\frac{(p_1-p_{10})^2}{(\Delta p_1)^2}\right]e^{-in_1p_1x_{10}} \equiv g_1(p_1)e^{-in_1p_1x_{10}} , \tag{4.11c}$$

$$g_{n_2}(p_2) \equiv \frac{1}{(\Delta p_1)^{1/2}(\pi)^{1/4}}\exp\left[-\frac{1}{2}\frac{(p_2-p_{20})^2}{(\Delta p_2)^2}\right]e^{-in_2p_2x_{20}} \equiv g_2(p_2)e^{-in_2p_2x_{20}} . \tag{4.11d}$$

Without loss of generality, we assume $p_{10} > p_{20}$.

The above construction is justified by studying the two-soliton state before collision and after collision. In the two limits, the two-soliton state is composed of two well-separated fundamental solitons. To be explicit, we show in Appendix F that before collision, the two-soliton state is approximately equal to

$$|\psi_s\rangle \approx \left[\sum_{n_1} a_1(n_1)\int g_{n_1}(p_1)e^{-iE(n_1,p_1)t}\hat{R}^\dagger(n_1,p_1)dp_1\right]\left[\sum_{n_2} a_2(n_2)\int g_{n_2}(p_2)e^{-iE(n_2,p_2)t}\hat{R}^\dagger(n_2,p_2)dp_2\right]|0\rangle , \tag{4.12}$$

where the two large parentheses are identified as the creation operators for fundamental solitons [see (3.3)] and $\hat{R}^\dagger(n,p)$ has been defined in (2.18).

After collision,

$$|\psi_s\rangle \approx \sum_{n_1,n_2} a_1(n_1)a_2(n_2)\int\int e^{i\theta(n_1,p_1,n_2,p_2)} g_{n_1}(p_1)g_{n_2}(p_2)\exp[-iE(n_1,p_1)t-iE(n_2,p_2)t] \times \hat{R}^\dagger(n_1,p_2)\hat{R}^\dagger(n_2,p_2)dp_1dp_2|0\rangle \;, \quad (4.13)$$

with $\theta(n_1,p_1,n_2,p_2)$ defined in (4.6) and $E(n,p)$ defined in (2.19). Note that the only difference between (4.13) and (4.12) is the phase factor $\theta(n_1,p_1,n_2,p_2)$. To see the effect of this factor, we write (4.13) as

$$\begin{aligned}|\psi_s\rangle \approx \sum_{n_1,n_2} a_1(n_1)a_2(n_2)\exp&\left[i\theta(n_{10},p_{10},n_{20},p_{20})+i\frac{\partial\theta}{\partial n_1}(n_1-n_{10})+i\frac{\partial\theta}{\partial n_2}(n_2-n_{20})\right]\\ &\times\left[\int g_{n_1}(p_1)\exp\left[i\frac{\partial\theta}{\partial p_1}(p_1-p_{10})-iE(n_1,p_1)t\right]\hat{R}^\dagger(n_1,p_1)dp_1\right]\\ &\times\left[\int g_{n_2}(p_2)\exp\left[i\frac{\partial\theta}{\partial p_2}(p_2-p_{20})-iE(n_2,p_2)t\right]\hat{R}^\dagger(n_2,p_2)dp_2\right]|0\rangle \;. \end{aligned} \quad (4.14)$$

Here we have used the expansion

$$\begin{aligned}\theta(n_1,p_1,n_2,p_2)\approx&\,\theta(n_{10},p_{10},n_{20},p_{20})\\ &+\frac{\partial\theta}{\partial n_1}(n_1-n_{10})+\frac{\partial\theta}{\partial n_2}(n_2-n_{20})\\ &+\frac{\partial\theta}{\partial p_2}(p_1-p_{10})+\frac{\partial\theta}{\partial p_2}(p_2-p_{20})\;.\end{aligned} \quad (4.15)$$

All the derivatives of θ are evaluated at $(n_{10},n_{20},p_{10},p_{20})$.

It is now clear that the two-soliton state after collision is still composed of two well-separated fundamental solitons except for a phase shift and a "position" shift. The mean phase shift for the first soliton is

$$\begin{aligned}\delta\theta_1 &\approx \frac{\partial\theta}{\partial n_1}(n_{10},p_{10},n_{20},p_{20})\\ &\approx \theta(n_{10}+1,p_{10},n_{20},p_{20})-\theta(n_{10},p_{10},n_{20},p_{20})\;,\end{aligned} \quad (4.16)$$

and the position shift is[12]

$$\delta x_1 \approx \frac{1}{n_{10}}\frac{\partial\theta(n_{10},p_{10},n_{20},p_{20})}{\partial p_1}\;. \quad (4.17)$$

For the second soliton the phase shift is

$$\begin{aligned}\delta\theta_2 &\approx \frac{\partial\theta}{\partial n_2}(n_{10},p_{10},n_{20},p_{20})\\ &\approx \theta(n_{10},p_{10},n_{20}+1,p_{20})-\theta(n_{10},p_{10},n_{20},p_{20})\;,\end{aligned} \quad (4.18)$$

and the position shift is

$$\delta x_2 \approx \frac{1}{n_{20}}\frac{\partial\theta(n_{10},p_{10},n_{20},p_{20})}{\partial p_2}\;. \quad (4.19)$$

In Appendix G we show that when n_{10} and n_{20} are large and $|c|$ is small, the magnitude of $\delta\theta_1$ and δx_1 in (4.16) and (4.17) approach the classical results.

The increase of the uncertainties due to a collision can be estimated by expanding $\theta(n_1,p_1,n_2,p_2)$ to second order. The phase uncertainty for the first soliton is

$$\begin{aligned}\Delta\theta_1 \approx& \left|\frac{\partial^2\theta}{\partial n_1^2}\right|\Delta n_1+\left|\frac{\partial^2\theta}{\partial n_1\partial p_1}\right|\Delta p_1\\ &+\left|\frac{\partial^2\theta}{\partial n_1\partial n_2}\right|\Delta n_2+\left|\frac{\partial^2\theta}{\partial n_1\partial p_2}\right|\Delta p_2\;,\end{aligned} \quad (4.20)$$

and the position uncertainty is

$$\begin{aligned}\Delta x_1 \approx& \left|\frac{\partial^2\theta}{\partial n_1\partial p_1}\right|\Delta n_1+\left|\frac{\partial^2\theta}{\partial p_1^2}\right|\Delta p_1\\ &+\left|\frac{\partial^2\theta}{\partial p_1\partial n_2}\right|\Delta n_2+\left|\frac{\partial^2\theta}{\partial p_1\partial p_2}\right|\Delta p_2\;.\end{aligned} \quad (4.21)$$

V. CONCLUSIONS

With the exact eigenstates of the Hamiltonian describing the nonlinear Schrödinger equation, we constructed solitonlike solutions. These have finite expectation values of amplitude and phase. Because of the uncertainty principle connecting the phase to the photon number and the position to the momentum, superpositions in terms of $|n,p,t\rangle$ eigenstates were required. The mean field of a soliton was shown to be a superposition of the classical soliton solutions with different group velocities and phase velocities. Contrary to the classical soliton theory, a soliton experiences phase spreading and dispersion when it propagates. The phase-spreading effect is quite significant because its characteristic length is less than a soliton period. On the other hand, the dispersion effect is quite small with a characteristic length about n_0 soliton periods. Here n_0 is the expectation value of the photon number in the soliton.

The same formalism was extended to study solitons in collision, by setting up the quantum states of a higher-order soliton. The phase shift and position shift of the colliding solitons were determined, along with their quantum fluctuations.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation Grant No. EET-8700474 and by the Charles Stark Draper Laboratories, Grant No. DL-H-2854018.

APPENDIX A: RECURSION FORMULA FOR PERMUTATION COEFFICIENTS

In this appendix, we prove (2.12). The region $x_1 \le x_2 \le \cdots \le x_n$ has $n-1$ boundaries. Let us consider the first boundary $x_1 = x_2$. On one side of this boundary is region 1: $x_1 < x_2 < \cdots < x_n$. On the other side is region 2: $x_2 < x_1 < \cdots < x_n$. The form of the solution in region 1 is given by (2.11) and in region 2 the solution is the same except that A_Q is changed to $A_{Q'}$. Here Q' is the permutation derived from Q by interchanging $Q(1)$ and $Q(2)$. By defining new variables,

$$x \equiv x_1 - x_2 \ , \tag{A1a}$$

$$y \equiv x_1 + x_2 \ , \tag{A1b}$$

one can rewrite the solution as

$$\begin{aligned} f(x,y,x_3,\ldots,x_n) &= \sum_{\{Q\}} A_Q \exp\left[i\frac{k_{Q(1)}-k_{Q(2)}}{2}x + i\frac{k_{Q(1)}+k_{Q(2)}}{2}y \right] \exp\left[i\sum_{j=3}^{n} k_{Q(j)}x_j \right] \quad \text{in region 1} \ , \\ &= \sum_{\{Q\}} A_{Q'} \exp\left[i\frac{k_{Q(1)}-k_{Q(2)}}{2}x + i\frac{k_{Q(1)}+k_{Q(2)}}{2}y \right] \exp\left[i\sum_{j=3}^{n} k_{Q(j)}x_j \right] \quad \text{in region 2} \ . \end{aligned} \tag{A2}$$

Equation (3.9) now becomes

$$\left\{ 2\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] - 2c\delta(x) + \sum_{j=3}^{n} \frac{\partial^2}{\partial x_j^2} - 2c \sum_{3 \le i < j \le n} \delta(x_j - x_i) - 2c \sum_{j=3}^{n} \left[\delta\left[x_j - \frac{x+y}{2} \right] + \delta\left[x_j - \frac{y-x}{2} \right] \right] \right\} f_n(x,y,\ldots,x_n) = E_n f_n(x_1,\ldots,x_n) \ . \tag{A3}$$

Integrating (A3) over x from 0^- to 0^+ gives us the boundary condition to be satisfied:

$$\left[\frac{\partial}{\partial x}\bigg|_{0^+} - \frac{\partial}{\partial x}\bigg|_{0^-} \right] f_n(x,y,\ldots,x_n) = c f_n(0,y,\ldots,x_n) \ . \tag{A4}$$

Substituting (A2) into (A4), we obtain

$$\sum_{\{Q\}} A_{Q'} \left[i\frac{k_{Q(1)}-k_{Q(2)}}{2} \right] \exp\left[i\frac{k_{Q(1)}+k_{Q(2)}}{2}y \right] - \sum_{\{Q\}} A_Q \left[i\frac{k_{Q(1)}-k_{Q(2)}}{2} \right] \exp\left[i\frac{k_{Q(1)}+k_{Q(2)}}{2}y \right] = \frac{c}{2}\left[\sum_{\{Q\}} A_{Q'} \exp\left[i\frac{k_{Q(1)}+k_{Q(2)}}{2}y \right] + \sum_{\{Q\}} A_Q \exp\left[i\frac{k_{Q(1)}+k_{Q(2)}}{2}y \right] \right] \ . \tag{A5}$$

Equation (A5) is satisfied when

$$A_{Q'}\left[i\frac{k_{Q(1)}-k_{Q(2)}}{2} \right] - A_Q\left[i\frac{k_{Q(1)}-k_{Q(2)}}{2} \right] = \frac{c}{2}(A_{Q'} + A_Q) \ . \tag{A6}$$

Therefore, $A_{Q'}$ and A_Q are related by

$$A_{Q'} = \frac{k_{Q(2)} - k_{Q(1)} + ic}{k_{Q(2)} - k_{Q(1)} - ic} A_Q \ . \tag{A7}$$

By considering the jth boundary $x_j = x_{j+1}$ (2.12) can be proved in the same way.

APPENDIX B: NORMALIZATION OF EIGENSTATES

In this appendix, we prove (2.21). From (2.18), it follows that, if $n' \ne n$, then

$$\langle n',p' | n,p \rangle = 0 \ , \tag{B1}$$

because states of different n involve an unequal number of operators. For equal n's but different p's, one has from (2.16) and (2.18),

$$\langle n,p' | n,p \rangle = \mathcal{N}_n^2 \int \exp\left[c \sum_{1 \le i < j \le n} |x_j - x_i| + i(p-p') \times \sum_{j=1}^{n} x_j \right] dx_1 \cdots dx_n \ . \tag{B2}$$

Noting that the integrand is a symmetric function of the x_j's, we reduce the integration region to $x_1 \le x_2, \le \cdots \le x_n$:

$$\langle n,p'|n,p\rangle=\mathcal{N}_n^2 n!\int_{-\infty}^{\infty}dx_n\int_{-\infty}^{x_n}dx_{n-1}\cdots\int_{-\infty}^{x_2}dx_1\exp\left[-c\sum_{j=1}^{n}(n-2j+1)x_j+i(p-p')\sum_{j=1}^{n}x_j\right]$$
$$=\mathcal{N}_n^2 n!\prod_{l=1}^{n-1}\frac{1}{l[-c(n-l)+i(p-p')]}\int_{-\infty}^{\infty}e^{in(p-p')x_n}dx_n$$
$$=\mathcal{N}_n^2 n!\prod_{l=1}^{n-1}\frac{1}{l[-c(n-l)]}\frac{2\pi}{n}\delta(p-p')$$
$$=\mathcal{N}_n^2 n!\frac{2\pi}{n}\prod_{l=1}^{n-1}\left[\frac{1}{l^2|c|}\right]\delta(p-p')$$
$$=2\pi\frac{\mathcal{N}_n^2}{(n-1)!}|c|^{-(n-1)}\delta(p-p')\ . \tag{B3}$$

By substituting (2.22) into (B3), (2.21) is proved.

APPENDIX C: MATRIX ELEMENTS OF FIELD OPERATORS

In this appendix we prove (2.25)–(2.28). The calculation is basically the same as in Ref. 11. From (2.16) and (2.18), we have

$$\langle n,p'|\hat{\phi}(x)|n+1,p\rangle=\frac{1}{\sqrt{n!}}\frac{1}{\sqrt{(n+1)!}}\int f^*_{np'}(x'_1,\ldots,x'_n)f_{n+1,p}(x_1,\ldots,x_{n+1})$$
$$\times\langle 0|\hat{\phi}(x'_1)\cdots\hat{\phi}(x'_n)\hat{\phi}(x)\hat{\phi}^\dagger(x_1)\cdots\hat{\phi}^\dagger(x_{n+1})|0\rangle dx'_1\cdots dx'_n dx_1\cdots dx_{n+1}$$
$$=\frac{1}{\sqrt{n!}}\frac{1}{\sqrt{(n+1)!}}(n+1)!\int f^*_{n,p'}(x_1,\ldots,x_n)f_{N+1,p}(x_1\cdots x_n,x)dx_1\cdots dx_n$$
$$=\sqrt{n+1}\mathcal{N}_n\mathcal{N}_{n+1}\int\exp\left[i(p-p')\sum_{j=1}^{n}x_j+c\sum_{1\le i<j\le n}|x_j-x_i|\right]$$
$$\times\exp\left[ipx+\frac{c}{2}\sum_{j=1}^{n}|x-x_j|\right]dx_1\cdots dx_n\ .$$

Since the integrand is a symmetric function of the x_j's, we reduce the integration region to $\cup_m\{-\infty\le x_1\le x_2\le\cdots\le x_m\le x\le x_{m+1}\le\cdots\le x_n\le\infty\}$ so that the absolute values can be removed.

$$\langle n,p'|\hat{\phi}(x)|n=1,p\rangle=\sqrt{n+1}\mathcal{N}_n\mathcal{N}_{n+1}n!\sum_{m=0}^{n}\int_{-\infty}^{x}dx_m\int_{-\infty}^{x_m}dx_{m-1}\cdots\int_{-\infty}^{x_2}dx_1$$
$$\times\int_x^{\infty}dx_{m+1}\int_{x_{m+1}}^{\infty}dx_{m+1}\cdots\int_{x_{n-1}}^{\infty}dx_n$$
$$\times\exp\left[i(p-p')\sum_{j=1}^{n}x_j-c\sum_{j=1}^{n}(n-2j+1)x_j\right]\exp\left[ipx+\frac{c}{2}(2m-n)x-\frac{c}{2}\sum_{j=1}^{m}x_j+\frac{c}{2}\sum_{j=m+1}^{n}x_j\right]$$
$$=\sqrt{n+1}\mathcal{N}_n\mathcal{N}_{n+1}n!(2/|c|)^n\exp\{i[(n+1)p-np']x\}$$
$$\times\sum_{m=0}^{n}\frac{1}{m!(n-m)!}\prod_{r=1}^{m}1\Big/\left[2n-2r+1+i\frac{2}{|c|}(p-p')\right]\prod_{r=1}^{n-m}1\Big/\left[2n+2r+1-i\frac{2}{|c|}(p-p')\right]\ . \tag{C1}$$

With the following identity,

$$\sum_{m=0}^{n}\frac{1}{m!(n-m)!}\prod_{j=1}^{n-m}(2j-1+ia)\prod_{j=1}^{m}(2j-1-ia)=2^n\ , \tag{C2}$$

we have

$$\sum_{m=0}^{n}\frac{1}{m!(n-m)!}\prod_{r=1}^{m}1\Big/\left[2n-2r+1+i\frac{2}{|c|}(p-p')\right]\prod_{r=1}^{n-m}1\Big/\left[2n-2r+1-i\frac{2}{|c|}(p-p')\right]$$
$$=\left[\prod_{r=1}^{n}1\Big/\left[(2n-2r+1)^2+\frac{4}{|c|^2}(p-p')^2\right]\right]\sum_{m=0}^{n}\frac{1}{m!(n-m)!}\prod_{r=m+1}^{n}\left[2n-2r+1+i\frac{2}{|c|}(p-p')\right]$$
$$\times\prod_{r=n-m+1}^{n}\left[2n-2r+1-i\frac{2}{|c|}(p-p')\right]$$
$$=2^n\prod_{r=1}^{n}1\Big/\left[(2n-2r+1)^2+\frac{4}{|c|^2}(p-p')^2\right]. \tag{C3}$$

Using (2.22) and (C3), Eq. (C1) is reduced to (2.25). Using the identities

$$\operatorname{sech}\tfrac{1}{2}\pi y=\prod_{j=1}^{\infty}\frac{(2j-1)^2}{y^2+(2j-1)^2}, \tag{C4}$$

$$\frac{(2n)!!}{(2n-1)!!}\equiv\frac{\prod_{j=1}^{n}(2j)}{\prod_{j=1}^{n}(2j-1)}=\sqrt{n\pi}\quad n\to\infty\quad\text{(Wallis formula)}, \tag{C5}$$

$$\prod_{j=n+1}^{\infty}\left[1+\frac{1}{(2j-1)^2}a^2\right]=1\quad n\to\infty. \tag{C6}$$

Equation (2.25) is reduced to (2.26).

The mean intensity can be calculated in the same way:

$$\langle n,p'|\hat\phi^\dagger(x)\hat\phi(x)|n,p\rangle=\mathcal{N}_n\mathcal{N}_n n\int\exp\left[i(p-p')\sum_{j=1}^{n-1}x_j+c\sum_{1\le i<j\le n-1}|x_j-x_i|\right]\exp\left[c\sum_{j=1}^{n-1}|x-x_j|\right]dx_1\cdots dx_{n-1}$$
$$=\mathcal{N}_n\mathcal{N}_n n(n-1)!\sum_{m=1}^{n-1}\int_{-\infty\le x_1\le\cdots x_m\le x\le x_{m+1}\le\cdots x_{n-1}\le\infty}dx_1\cdots dx_n$$
$$\times\exp\left[i(p-p')\sum_{j=1}^{n-1}x_j-c\sum_{j=1}^{n-1}(n-2j)x_j+c(2m-n+1)x-c\sum_{j=1}^{m}x_j+c\sum_{j=m+1}^{n-1}x_j\right]$$
$$=\mathcal{N}_n\mathcal{N}_n n(n-1)!\sum_{m=0}^{n-1}\frac{1}{m!(n-1-m)!}\prod_{r=1}^{m}\frac{1}{[|c|(n-r)+i(p-p')]}\prod_{r=1}^{n-1-m}\frac{1}{[|c|(n-r)-i(p-p')]}$$
$$=\mathcal{N}_n\mathcal{N}_n n(n-1)!|c|^{-(n-1)}\sum_{m=0}^{n-1}\frac{1}{m!(n-1-m)!}$$
$$\times\prod_{r=1}^{m}1\Big/\left[(n-r)+i\frac{1}{|c|}(p-p')\right]\prod_{r=1}^{n-1-m}1\Big/\left[(n-r)-i\frac{1}{|c|}(p-p')\right]. \tag{C7}$$

With the identity

$$\sum_{m=0}^{n-1}\frac{1}{m!(n-1-m)!}\prod_{l=1}^{n-1-m}(l+ia)\prod_{l=1}^{m}(l-ia)=n, \tag{C8}$$

we have

$$\sum_{m=0}^{n-1}\frac{1}{m!(n-1-m)!}\prod_{r=1}^{m}1\Big/\left[(n-r)+i\frac{1}{|c|}(p-p')\right]\prod_{r=1}^{n-1-m}1\Big/\left[(n-r)-i\frac{1}{|c|}(p-p')\right]$$
$$=n\prod_{r=1}^{n}1\Big/\left[(n-r)^2+\frac{(p-p')^2}{|c|^2}\right]. \tag{C9}$$

With (C9), (C7) can be reduced to (2.27). Using

$$\sinh\pi x=\pi x\prod_{l=1}^{\infty}\left[1+\frac{x^2}{l^2}\right], \tag{C10}$$

$$\prod_{l=n}^{\infty}\frac{1}{(1+a^2/l^2)}=1 \quad n\to\infty , \tag{C11}$$

Eq. (2.27) can be reduced to (2.28).

Another way to calculate these matrix elements is to use the quantum inverse scattering method. Note that the normalization condition used in Refs. 10 and 11 is

$$\langle n',p'|n,p\rangle=\frac{2\pi}{n}\delta_{nn'}\delta(p-p') , \tag{C12a}$$

whereas we use

$$\langle n',p'|n,p\rangle=\delta_{nn'}\delta(p-p') . \tag{C12b}$$

APPENDIX D: THE MEAN FIELD OF THE $N=1$ SOLITON

In this appendix, we prove (3.8). Using the matrix elements expressions (2.25), (2.26), and the time dependence (2.20) with (2.19), we have

$$\begin{aligned}\langle\psi_s|\hat{\phi}(x)|\psi_s\rangle&=\sum_n a_n^* a_{n+1}\int\int g_n^*(p')g_{n+1}(p)\langle n,p',t|\hat{\phi}(x)|n+1,p,t\rangle dp\,dp'\\&=\sum_n a_n^* a_{n+1}\exp\left[i\frac{|c|^2 n(n+1)}{4}t\right]\\&\quad\times\int\int g(p')g(p)\frac{\sqrt{n(n+1)}}{2}|c|^{-1/2}\exp\{-i[(n+1)p-np']x_0\}\\&\quad\times\exp\{-i[(n+1)p^2-np'^2]t+i[(n+1)p-np']x\}\operatorname{sech}\left[\frac{\pi}{|c|}(p-p')\right]dp\,dp' ,\end{aligned} \tag{D1}$$

where a_n and $g(p)$ are given in (3.6) and (3.7). We now define new variables,

$$p_1=\frac{p-p'}{2} , \tag{D2a}$$

$$p_2=\frac{p+p'}{2} , \tag{D2b}$$

and express (D1) in terms of them:

$$\begin{aligned}\langle\psi_s|\hat{\phi}(x)|\psi_s\rangle&\approx\sum_n a_n^* a_{n+1}\exp\left[i\frac{|c|^2 n(n+1)}{4}t\right]\\&\quad\times\int\int\frac{1}{2(\Delta p)\sqrt{\pi}}\exp\left[-\left[\frac{p_1^2}{(\Delta p)^2}+\frac{p_2^2}{(\Delta p)^2}\right]\right]\frac{\sqrt{n(n+1)}}{2}|c|^{-1/2}\\&\quad\times\exp\{i[2(n+\tfrac{1}{2})p_1(x-x_0)+p_2(x-x_0)-p_1^2t-p_2^2t-4(n+\tfrac{1}{2})p_1p_2t]\}\operatorname{sech}\left[\frac{2\pi}{|c|}p_1\right]dp_1dp_2 .\end{aligned} \tag{D3}$$

By neglecting the term ip_1^2t in the phase and assuming $\Delta p\gg|c|$, we can carry out the integration over p_1. Dropping the subscript on p_2 we have

$$\begin{aligned}\langle\psi_s|\hat{\phi}(x)|\psi_s\rangle&\approx\sum_n a_n^* a_{n+1}\exp\left[i\frac{|c|^2 n(n+1)}{4}t\right]\\&\quad\times\int\frac{1}{(\Delta p)\sqrt{\pi}}\exp\left[-\frac{(p-p_0)^2}{(\Delta p)^2}\right]\left[\frac{\sqrt{n(n+1)}}{2}|c|^{1/2}\exp\{i[p(x-x_0)-p^2t]\}\right.\\&\quad\left.\times\operatorname{sech}[\tfrac{1}{2}(n+\tfrac{1}{2})|c|(x-x_0-2p)]\right]dp .\end{aligned} \tag{D4}$$

Substituting a_n in (3.6) into (D4), (3.8) is proved. Since we neglect the term ip_1^2t, (D4) and (3.8) are correct only when $(\Delta p)^2t\ll1$. The effect of this term is to make the dispersion effect more serious than that predicted by (3.8) when t is large.

APPENDIX E: CALCULATION OF THE PHASE FACTOR

In this appendix, we prove (4.5) and (4.6). To go from the permutation $[1,2,\ldots,n_1,\ldots,n_1+n_2]$ to $[n_1+1,\ldots,n_1+n_2,1,\ldots,n_1]$ we have to permute $n_1 \times n_2$ times. The effect of every permutation is a multiplication by the factor expressed in (2.12). The total effect is the product of all these factors. Therefore,

$$A_{\text{out}}=S(n_1,p_1,n_2,p_2)A_{\text{in}} , \tag{E1}$$

with

$$\begin{aligned}
S(n_1,p_1,n_2,p_2)&=\prod_{j=1}^{n_1}\prod_{l=1}^{n_2}\frac{(k_{n_1+l}-k_j)+ic}{(k_{n_1+l}-k_j)-ic}\\
&=\prod_{j=1}^{n_1}\prod_{l=1}^{n_2}\frac{\left[(p_2-p_1)+i\frac{c}{2}[n_2-n_1+2(j-l+1)]\right]}{\left[(p_2-p_1)+i\frac{c}{2}[n_2-n_1+2(j-l-1)]\right]}\\
&=\prod_{j=1}^{n_1}\frac{\left[(p_2-p_1)+i\frac{c}{2}(n_2-n_1+2j)\right]\left[(p_2-p_1)+i\frac{c}{2}(n_2-n_1+2j-2)\right]}{\left[(p_2-p_1)-i\frac{c}{2}(n_2+n_1-2j)\right]\left[(p_2-p_1)-i\frac{c}{2}(n_2+n_1-2j+2)\right]} .
\end{aligned} \tag{E2a}$$

Changing index j in the denominator to $n+1-j$, we have

$$\begin{aligned}
S(n_1,p_1,n_2,p_2)&=\prod_{j=1}^{n_1}\frac{\left[(p_2-p_1)+i\frac{c}{2}(n_2-n_1+2j)\right]\left[(p_2-p_1)+i\frac{c}{2}(n_2-n_1+2j-2)\right]}{\left[(p_2-p_1)-i\frac{c}{2}(n_2-n_1+2j-2)\right]\left[(p_2-p_1)-i\frac{c}{2}(n_2-n_1+2j)\right]}\\
&=\left[\prod_{j=1}^{n_1-1}\left[\frac{(p_2-p_1)+\frac{ic}{2}(n_2-n_1+2j)}{(p_2-p_1)-\frac{ic}{2}(n_2-n_1+2j)}\right]^2\right]\left[\frac{p_2-p_1+\frac{ic}{2}(n_2-n_1)p_2-p_1+\frac{ic}{2}(n_2+n_1)}{p_2-p_1-\frac{ic}{2}(n_2-n_1)p_2-p_1-\frac{ic}{2}(n_2+n_1)}\right]\\
&\equiv e^{i\theta(n_1,p_1,n_2,p_2)} .
\end{aligned} \tag{E2b}$$

Equation (4.6) follows directly from (E2b). In the literature, the factor S is usually called the S matrix.[5,12]

APPENDIX F: TWO-SOLITON STATES IN TWO LIMITS

In this appendix, we prove (4.12) and (4.13). Let us consider the terms in (4.10).

$$\begin{aligned}
&\int\int g_{n_1}(p_1)g_{n_2}(p_2)|n_1,p_1,n_2,p_2t\rangle dp_1dp_2\\
&=(n_1+n_2)!\sum_{\{Q\}}A_Q\int_{x_1\le x_2\le\cdots\le x_{n_1+n_2}}\left[\int g_{n_1}(p_1)\exp\left[-iE(n_1,p_1)t+ip_1\sum_{j=1}^{n_1}x_{Q^{-1}(j)}\right]dp_1\right.\\
&\qquad\left.\times\left[\exp\left[\frac{c}{2}\sum_{1\le i<j\le n_1}|x_{Q^{-1}(j)}-x_{Q^{-1}(i)}|\right]\right]\right]\\
&\qquad\times\left[\int g_{n_2}(p_2)\exp\left[-iE(n_2,p_2)t+ip_2\sum_{j=n_1+1}^{n_1+n_2}x_{Q^{-1}(j)}\right]dp_2\right.\\
&\qquad\left.\times\left[\exp\left[\frac{c}{2}\sum_{n_1+1\le i<j\le n_1+n_2}|x_{Q^{-1}(j)}-x_{Q^{-1}(i)}|\right]\right]\right]\prod_{j=1}^{n_1+n_2}\hat{\phi}^\dagger(x_j)dx_j|0\rangle
\end{aligned} \tag{F1a}$$

$$=(n_1+n_2)!\sum_{\{Q\}} A_Q \exp\left[i\left[-n_1p_{10}^2t-n_2p_{20}^2t+\frac{|c|^2}{12}n_1(n_1^2-1)t+\frac{|c|^2}{12}n_2(n_2^2-1)t\right]\right]$$

$$\times\int_{x_1\le x_2\le\cdots\le x_{n_1+n_2}}\exp\left[ip_{10}\sum_{j=1}^{n_1}x_{Q^{-1}(j)}-in_1p_{10}x_{10}+ip_{20}\sum_{j=n_1+1}^{n_1+n_2}x_{Q^{-1}(j)}-in_2p_{20}x_{20}\right]$$

$$\times\left\{\int g_1(p_{10}+p_1')\exp\left[in_1p_1'\left[\frac{1}{n_1}\sum_{j=1}^{n_1}x_{Q^{-1}(j)}-x_{10}-2p_{10}t\right]\right]dp_1'\right\}$$

$$\times\left[\exp\left[\frac{c}{2}\sum_{1\le i<j\le n_1}|x_{Q^{-1}(j)}-x_{Q^{-1}(i)}|\right]\right]$$

$$\times\left\{\int g_2(p_{20}+p_2')\exp\left[in_2p_2'\left[\frac{1}{n_2}\sum_{j=n_1+1}^{n_1+n_2}x_{Q^{-1}(j)}-x_{20}-2p_{20}t\right]\right]dp_2'\right\}$$

$$\times\left[\exp\left[\frac{c}{2}\sum_{n_1+1\le i<j\le n_1+n_2}|x_{Q^{-1}(j)}-x_{Q^{-1}(j)}|\right]\right]\prod_{j=1}^{n_1+n_2}\hat{\phi}^\dagger(x_j)dx_j|0\rangle\ . \qquad \text{(F1b)}$$

Here we have set $p_1=p_{10}+p_1'$, $p_2=p_{20}+p_2'$, and linearized the nonlinear phase term in (F1a). This is equivalent to ignoring the quantum dispersion effect. The terms in (F1b) can be significant only when the following conditions are satisfied:

$$\frac{1}{n_1}\sum_{j=1}^{n_1}x_{Q^{-1}(j)}\approx x_{10}+2p_{10}t\ , \qquad \text{(F2a)}$$

$$\frac{1}{n_2}\sum_{j=n_1+1}^{n_1+n_2}x_{Q^{-1}(j)}\approx x_{20}+2p_{20}t\ , \qquad \text{(F2b)}$$

$\{x_{Q^{-1}(j)}:\ j=1,\ldots,n_1\}$ are grouped together , (F2c)

$\{x_{Q^{-1}(j)}:\ j=n_1+1,\ldots,n_1+n_2\}$

are grouped together , (F2d)

and the constraint is obeyed

$$x_1\le x_2\le\cdots\le x_{n_1+n_2}\ . \qquad \text{(F2e)}$$

Before and after collision, $x_{10}+2p_{10}t$ and $x_{20}+2p_{20}t$ are far apart. Therefore, from (F2a)–(F2e), we can conclude that (1) before collision, only the term corresponding to $Q=[1,2,\ldots,n_1+n_2]$ is important. After collision, only the term corresponding to $[n_1+1,\ldots,n_1+n_2,1,\ldots,n_1]$ is important. (2) the integration over $\{x_{Q^{-1}(j)}: j=1,\ldots,n_1\}$ and $\{x_{Q^{-1}(j)}: j=n_1+1,\ldots,n_1+n_2\}$ can be decoupled. With the two approximations and relation (4.5), Eqs. (4.12) and (4.13) follow directly from (4.10) and (F1a).

APPENDIX G: PHASE SHIFT AND POSITION SHIFT DUE TO SOLITON COLLISION

In this appendix, we demonstrate how (4.16) and (4.17) approach the classical results given by the Eqs. (6.8) and (6.9) in paper I. With (4.6), we have

$$\theta(n_1+1,p_1,n_2,p_2)-\theta(n_1,p_1,n_2,p_2)\approx\tfrac{1}{2}[\theta(n_1+2,p_1,n_2,p_2)-\theta(n_1,p_1,n_2,p_2)]$$

$$=-\frac{1}{2}\left[2\tan^{-1}\left[\frac{\frac{1}{2}|c|(n_1+n_2+2)}{p_2-p_1}\right]+2\tan^{-1}\left[\frac{\frac{1}{2}|c|(n_1+n_2)}{p_2-p_1}\right]+2\tan^{-1}\left[\frac{\frac{1}{2}|c|(n_2-n_1-2)}{p_2-p_1}\right]+2\tan^{-1}\left[\frac{\frac{1}{2}|c|(n_2-n_1)}{p_2-p_1}\right]\right]$$

$$\approx-\left[2\tan^{-1}\left[\frac{\frac{1}{2}|c|(n_1+n_2)}{p_2-p_1}\right]+2\tan^{-1}\left[\frac{\frac{1}{2}|c|(n_2-n_1)}{p_2-p_1}\right]\right]$$

and

$$\frac{1}{n_1}\frac{\partial\theta(n_1,p_1,n_2,p_2)}{\partial p_1}=-\frac{1}{n_1}\left\{4\sum_{j=1}^{n_1-1}\frac{\frac{1}{2}|c|(n_2-n_1+2j)}{(p_2-p_1)^2+[\frac{1}{2}|c|(n_2-n_1+2j)]^2}+2\frac{\frac{1}{2}|c|(n_2-n_1)}{(p_2-p_1)^2+[\frac{1}{2}|c|(n_2-n_1)]^2}+2\frac{\frac{1}{2}|c|(n_2+n_1)}{(p_2-p_1)^2+[\frac{1}{2}|c|(n_2+n_1)]^2}\right\}$$

$$\approx-\frac{4}{n_1}\int_{(n_2-n_1)/2}^{(n_2+n_1)/2}\frac{|c|x}{(p_2-p_1)^2+|c|^2x^2}dx$$

$$=-\frac{2}{n_1|c|}\left[\ln\left[(p_2-p_1)^2+\frac{|c|^2(n_2+n_1)^2}{4}\right]-\ln\left[(p_2-p_1)^2+\frac{|c|^2(n_2-n_1)^2}{4}\right]\right].$$

[1] Y. Lai and H. A. Haus, preceding paper, Phys. Rev. A **40**, 844 (1989).
[2] E. H. Lieb and W. Linger, Phys. Rev. **130**, 1605 (1963).
[3] I. B. McGuire, J. Math. Phys. **5**, 622 (1964).
[4] C. N. Yang, Phys. Rev. Lett. **19**, 1312 (1967).
[5] C. N. Yang, Phys. Rev. **168**, 1920 (1967).
[6] H. P. Thacker, Rev. Mod. Phys. **53**, 253 (1981).
[7] H. B. Thacker and D. Wilkinson, Phys. Rev. D **19**, 3660 (1979).
[8] D. B. Creamer, H. B. Thacker, and D. Wilkinson, Phys. Rev. D **21**, 1523 (1980).
[9] M. Göckeler, Z. Phys. C **7**, 263 (1981).
[10] M. Wadati, in *Dynamical Problems in Soliton Systems*, Vol. 30 of *Springer Series in Synergetics*, edited by S. Takeno (Springer, Berlin, 1985).
[11] M. Wadati and M. Sakagami, J. Phys. Soc. Jpn. **53**, 1933 (1984).
[12] B. Yoon and J. W. Negele, Phys. Rev. A **16**, 1451 (1977).