Quantum theory of solitons in optical fibers. I. Time-dependent Hartree approximation

Y. Lai and H. A. Haus
Department of Electrical Engineering and Computer Science and Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 2 December 1988)

This paper is the first part of a two-part study on the quantum nonlinear Schrödinger equation [the second paper follows: Lai and Haus, Phys. Rev. A **39**, 854 (1989)]. The quantum nonlinear Schrödinger equation is solved analytically and is shown to have bound-state solutions. These bound-state solutions are closely related to the soliton phenomenon. This fact has not been pursued in the literature. In this paper we use the time-dependent Hartree approximation to construct approximate bound states and then superimpose these bound states to construct soliton states. This construction enables us to study the quantum effects of soliton propagation and soliton collisions.

I. INTRODUCTION

Recently a number of theoretical papers on the quantum effects of optical field propagation in nonlinear media have been published.[1–4] The motivation is to study the squeezing effects in optical fibers and to evaluate the possibility of using optical solitons in communication systems and laser gyros. The starting point of most analyses is the quantum nonlinear Schrödinger equation (QNSE), which is the quantum version of the classical nonlinear Schrödinger equation (CNSE). The CNSE is widely used in the study of pulse propagation in nonlinear optical fibers.[5] This equation has been solved analytically by the inverse scattering method and has been shown to have soliton solutions.[6] From the correspondence principle, it is natural to expect that the QNSE can also serve as a quantum model for pulse propagation in nonlinear optical fibers and that it can be solved analytically. However, previous work on this problem was carried out by linearizing the nonlinear equation. Linearization makes the problem more tractable but unfortunately also limits the validity of the results. In fact, the QNSE is also well known among quantum-statistical physicists and quantum-field theorists and, surprisingingly, has been solved analytically. In statistical physics, the QNSE is the evolution equation of a one-dimensional system of bosons with δ-function interactions in the second quantized form.[7] It was first solved by Bethe's ansatz method[8–13] in the 1960s. Since the original work of Bethe on the isotropic Heisenberg spin chain[14] in the 1930s, this method has been successfully applied to a number of models in statistical physics and quantum-field theory.[12,13] Recently the inverse scattering approach has been applied successfully to the solution of QNSE.[12,15–18] Both methods can be used to construct the eigenstates of the Hamiltonian. The quantum inverse scattering method constructs the creation operators of these eigenstates and derives their commutation relations. Bethe's ansatz method achieves this by solving the wave-function equation. When the coefficient of the nonlinear term in the QNSE is negative, there are bound-state solutions that are the eigenstates of the Hamiltonian with bound wave functions. Surprisingly, many treatments of this problem ended at this stage, leaving unsolved the important problem as to how these bound states are related to the soliton phenomenon. Nohl[19] was the first one to try to answer this question. Unsatisfied with Nohl's results, Wadati and Sakagami[20] presented an improved theory. Wadati and Sakagami introduced a wave packet which is a time-dependent superposition of the fundamental bound states and showed that the matrix element of the field operator for this wave packet approaches the classical fundamental soliton with zero velocity when the photon number is large. They then generalized their results to the moving solitons by a Galilean transformation. Although their results provide a good basis for the present work, their approach leaves some questions open.

(1) A soliton state should be a time-independent superposition of the bound states so that it is a solution of the governing equation.

(2) It is the expectation value of the field operator that corresponds to the classical soliton field, not the matrix element of the field operator.

(3) The construction should be generalized to higher-order soliton states to provide information about soliton collisions.

We have constructed soliton states that meet with the three criteria listed above. This construction enables us to study the quantum effects of soliton propagation and soliton collisions.

In this paper (paper I), we present an approximate solution by the time-dependent Hartree approximation. This approach was introduced by Yoon and Negele[21] to the study of one-dimensional bosons with δ-function interactions. Again they did not construct the soliton states. By following this approach, we construct approximate fundamental and higher-order soliton states. It is found that a soliton experiences phase-spreading effects when it propagates. The soliton collision effects are also studied. In the following paper (paper II), the soliton states will be constructed by superimposing the exact eigenstates of the Hamiltonian. There we will study the quantum effects of soliton propagation again and show that a soliton also experiences dispersion effects when it propagates.

II. QUANTIZATION OF THE NONLINEAR SCHRÖDINGER EQUATION

Under the paraxial and slowly varying envelope approximation, the evolution equation of a one-dimensional pulse propagating through a nonlinear, dispersive medium is given by[5]

$$\left[\frac{\partial}{\partial z}+\frac{1}{v_g}\frac{\partial}{\partial t}\right]A(z,t)=i\frac{1}{2}k''\frac{\partial^2}{\partial t^2}A(z,t)-i\kappa A^*(z,t)A(z,t)A(z,t) \,. \tag{2.1}$$

Here $A(z,t)$ is the envelope of the pulse, $v_g=1/k'$ is the group velocity, k' and k'' are the first and second derivatives of the propagation constant with respect to frequency, and κ expresses the magnitude of the Kerr nonlinearity. By a change of the variables, this equation can be reduced to the CNSE,

$$x\equiv v_g t-z \,, \tag{2.2a}$$

$$s\equiv\frac{1}{2}\frac{k''}{|k'|^2}z \,, \tag{2.2b}$$

$$\phi(s,x)\equiv I^{-1}A(z,t) \,, \tag{2.2c}$$

$$c\equiv\frac{\kappa|k'|^2I^2}{k''} \,, \tag{2.2d}$$

$$i\frac{\partial}{\partial s}\phi(s,x)=-\frac{\partial^2}{\partial x^2}\phi(s,x)+2c\phi^*(s,x)\phi(s,x)\phi(s,x) \,, \tag{2.3}$$

where x is the deviation from the pulse center moving with the velocity v_g, s is the normalized propagation distance, $\phi(s,x)$ is the normalized field envelope, and I is an intensity of normalization. To be consistent with the notation in the literature we use t instead of s as one of the independent variables:

$$i\frac{\partial}{\partial t}\phi(t,x)=-\frac{\partial^2}{\partial x^2}\phi(t,x)+2c\phi^*(t,x)\phi(t,x)\phi(t,x) \,. \tag{2.4}$$

The quantization is best perceived in Fourier transform space. If one defines by the Fourier transform

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(t,x)e^{i\beta x}dx\equiv a(t,\beta) \,, \tag{2.5a}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}a(t,\beta)e^{-i\beta x}d\beta\equiv\phi(t,x) \,, \tag{2.5b}$$

one obtains the equation of motion of the amplitude in Fourier transform space:

$$i\frac{\partial}{\partial t}a(t,\beta)=\beta^2a(t,\beta)+2c\int d\beta_1 d\beta_2 a^*(t,\beta_1)\times a(t,\beta_2)a(t,\beta+\beta_1-\beta_2). \tag{2.6}$$

By suitably choosing the value of I in (2.2c), one can normalize the field envelope so that it represents the photon flux at the "time" t. This enables us to identify $a(t,\beta)$ with the photon annihilation operator at the time t, $\hat{a}(t,\beta)$, and $a^*(t,\beta)$ with the creation operator $\hat{a}^\dagger(t,\beta)$. On the right-hand side of (2.6), the first term represents the dispersion effect and the second term represents the third-order nonlinearity. Note that the second term is in the form of a convolution. This is because for a broadband field one has to integrate over the Fourier-transform space. The quantization is accomplished by assignment of the commutation relations

$$[\hat{a}(t,\beta'),\hat{a}^\dagger(t,\beta)]=\delta(\beta-\beta') \,, \tag{2.7a}$$

$$[\hat{a}(t,\beta'),\hat{a}(t,\beta)]=[\hat{a}^\dagger(t,\beta'),\hat{a}^\dagger(t,\beta)]=0 \,. \tag{2.7b}$$

The quantized equation is

$$i\frac{\partial}{\partial t}\hat{a}(t,\beta)=\beta^2\hat{a}(t,\beta)+2c\int d\beta_1 d\beta_2\hat{a}^\dagger(t,\beta_1)\times\hat{a}(t,\beta_2)\hat{a}(t,\beta+\beta_1-\beta_2) \,. \tag{2.8}$$

Equation (2.8) can be derived from a well-defined Hamiltonian. That is, one can write (2.8) as

$$i\hbar\frac{d}{dt}\hat{a}(t,\beta)=[\hat{a}(t,\beta),\hat{H}] \,, \tag{2.9a}$$

with

$$\hat{H}=\hbar\left[\int\beta^2\hat{a}^\dagger(t,\beta)\hat{a}(t,\beta)d\beta+c\int\hat{a}^\dagger(t,\beta)\hat{a}^\dagger(t,\beta_1)\hat{a}(t,\beta_2)\times\hat{a}(t,\beta+\beta_1-\beta_2)d\beta d\beta_1 d\beta_2\right] \,. \tag{2.9b}$$

By defining new field operators as the inverse Fourier transforms of the annihilation and creation operators and applying the inverse Fourier transform to (2.8), one obtains the quantum nonlinear Schrödinger equation

$$i\frac{\partial}{\partial t}\hat{\phi}(t,x)=-\frac{\partial^2}{\partial x^2}\hat{\phi}(t,x)+2c\hat{\phi}^\dagger(t,x)\hat{\phi}(t,x)\hat{\phi}(t,x) \,. \tag{2.10}$$

The operators $\hat{\phi}(t,x)$ and $\hat{\phi}^\dagger(t,x)$ are annihilation and creation operators of photons at a "point" x and "time" t.

From the definition of the Fourier transform (2.5) and the commutation relations (2.7), it is easy to prove that the field operators satisfy the following commutation relations:

$$[\hat{\phi}(t,x'),\hat{\phi}^\dagger(t,x)]=\delta(x-x') \,, \tag{2.11a}$$

$$[\hat{\phi}(t,x'),\hat{\phi}(t,x)]=[\hat{\phi}^\dagger(t,x'),\hat{\phi}^\dagger(t,x)]=0 \,. \tag{2.11b}$$

With the help of (2.11), (2.10) can be written as

$$i\hbar\frac{d}{dt}\hat{\phi}(t,x)=[\hat{\phi}(t,x),\hat{H}] \,, \tag{2.12a}$$

with

$$\hat{H}=\hbar\left[\int \hat{\phi}_x^\dagger(t,x)\hat{\phi}_x(t,x)dx + c\int \hat{\phi}^\dagger(t,x)\hat{\phi}^\dagger(t,x)\hat{\phi}(t,x)\hat{\phi}(t,x)dx\right] . \qquad (2.12b)$$

The QNSE (2.10) is the operator evolution equation of a quantum system with the Hamiltonian (2.12b). Since the QNSE can be derived from a Hamiltonian, it is a well-defined operator equation.

III. EXPANSION IN FOCK SPACE AND THE UNCERTAINTY RELATIONS

A quantum problem can be solved in the Schrödinger picture or in the Heisenberg picture. At the end of Sec. II, we formulated our problem in the Heisenberg picture [(2.10), (2.12)]. In the Schrödinger picture, the problem is stated in terms of the time evolution of the state of the system $|\psi\rangle$:

$$i\hbar\frac{d}{dt}|\psi\rangle=\hat{H}_s|\psi\rangle , \qquad (3.1)$$

$$\hat{H}_s=\hbar\left[\int \hat{\phi}_x^\dagger(x)\hat{\phi}_x(x)dx + c\int \hat{\phi}^\dagger(x)\hat{\phi}^\dagger(x)\hat{\phi}(x)\hat{\phi}(x)dx\right] , \qquad (3.2)$$

where $\hat{\phi}(x)$ and $\hat{\phi}^\dagger(x)$ are the field operators in the Schrödinger picture and satisfy the following commutation relations:

$$[\hat{\phi}(x'),\hat{\phi}^\dagger(x)]=\delta(x-x') , \qquad (3.3a)$$

$$[\hat{\phi}(x'),\hat{\phi}(x)]=[\hat{\phi}^\dagger(x'),\hat{\phi}^\dagger(x)]=0 . \qquad (3.3b)$$

It is interesting to note that (3.1) is a linear equation whereas (2.10) is a nonlinear one. It is not obvious at the outset which equation can be solved more easily. In the literature, Bethe's ansatz method solved the problem in the Schrödinger picture[8–13] whereas the quantum inverse scattering method[12,15–18] solved the problem in the Heisenberg picture. In the Schrödinger picture, one may expand the quantum state in Fock space and substitute it into (3.1). The result is a wave-function equation that has many degrees of freedom (like the equations in many-particle physics). For the QNSE, this wave-function equation is in a simple form and can be solved analytically. Therefore we follow this approach in the present work.

Any quantum state of this system can be expanded in Fock space as follows:

$$|\psi\rangle=\sum_n a_n\int\frac{1}{\sqrt{n!}}f_n(x_1,\ldots,x_n,t) \times\hat{\phi}^\dagger(x_1)\cdots\hat{\phi}^\dagger(x_n)\,dx_1\cdots dx_n|0\rangle . \qquad (3.4)$$

The state $|\psi\rangle$ is a superposition of states produced from the vacuum state by creating photons at the points $x_1,x_2,\ldots,x_n$ with the weighting functions f_n. Since photons are bosons, f_n should be a symmetric function of x_j. We require a_n and f_n to satisfy the following normalization conditions:

$$\sum_n |a_n|^2=1 , \qquad (3.5)$$

$$\int |f_n(x_1,\ldots,x_n,t)|^2dx_1\cdots dx_n=1 . \qquad (3.6)$$

Substituting (3.4) and (3.2) into (3.1) and using (3.3), we obtain an equation for $f_n(x_1,\ldots,x_n,t)$:

$$i\frac{d}{dt}f_n(x_1,\ldots,x_n,t) =\left[-\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}+2c\sum_{1\le i<j\le n}\delta(x_j-x_i)\right] \times f_n(x_1,\ldots,x_n,t) . \qquad (3.7)$$

This is just the Schrödinger equation for a one-dimensional system of bosons with δ-function interactions.[7] The t dependence in (3.7) can be factored out by assuming a solution of the form

$$f_n(x_1,\ldots,x_n,t)=f_n(x_1,\ldots,x_n)e^{-iE_nt} . \qquad (3.8)$$

The equation for $f_n(x_1,\ldots,x_n)$ is

$$\left[-\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}+2c\sum_{1\le i<j\le n}\delta(x_j-x_i)\right]f_n(x_1,\ldots,x_n) =E_nf_n(x_1,\ldots,x_n) . \qquad (3.9)$$

The solution of (3.7) constructs the eigenstates of the Hamiltonian. The question may be asked as to the relation between these eigenstates and the soliton phenomenon. It is well known in quantum electrodynamics that the photon number $\hat{N}$ and the phase $\hat{\theta}$ of an optical field have to satisfy an (approximate) uncertainty relation,

$$\langle\Delta\hat{N}^2\rangle\langle\Delta\hat{\theta}^2\rangle\ge\tfrac{1}{4} . \qquad (3.10)$$

In the Appendix we prove that the total momentum $\hat{P}$ and the mean position $\hat{\bar{x}}$ also have to satisfy an uncertainty relation,

$$\langle\Delta\hat{P}^2\rangle\langle\Delta\hat{\bar{x}}^2\rangle\ge\frac{\hbar^2}{4} , \qquad (3.11)$$

where the mean position operator $\hat{\bar{x}}$ is defined by

$$\hat{\bar{x}}=\left[\int x\hat{\phi}^\dagger(x)\hat{\phi}(x)dx\right]\hat{N}^{-1} , \qquad (3.12)$$

with the photon-number operator $\hat{N}$ defined by

$$\hat{N}=\int\hat{\phi}^\dagger(x)\hat{\phi}(x)dx . \qquad (3.13)$$

The total momentum operator $\hat{P}$ is

$$\hat{P}=-i\frac{\hbar}{2}\int[\hat{\phi}^\dagger(x)\hat{\phi}_x(x)-\hat{\phi}_x^\dagger(x)\hat{\phi}(x)]dx . \qquad (3.14)$$

It is easy to verify that $\hat{H}$, $\hat{N}$, and $\hat{P}$ commute and therefore have common eigenstates. The uncertainty relations imply that the eigenstates of the Hamiltonian, which are also the eigenstates of $\hat{N}$ and $\hat{P}$, have a random phase and a random mean position. These uncertainty relations also suggest that one must superimpose the eigenstates of the Hamiltonian in order to construct states with a mean phase and a mean position.

The bosons diffuse as expressed in (3.7) by the diffusion operator for the jth particle, $\partial^2/\partial x_j^2$. Depending on the sign of c, the interaction among the bosons enhances or opposes the free diffusion and thus makes the problem more complicated. For this reason, it is worthwhile to look at the equation in the absence of the interaction and to obtain a full analytical solution of the problem. The solution corresponds to the phenomenon of an optical pulse in a linear, dispersive fiber.

IV. DIFFUSION OF NONINTERACTING BOSONS

When $c=0$, (3.7) becomes

$$i\frac{d}{dt}f_n(x_1,\ldots,x_n,t)=-\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}f_n(x_1,\ldots,x_n,t)\ . \tag{4.1}$$

The eigensolutions are of the exponential form

$$e^{-iE_nt}\exp\left[i\sum_{j=1}^{n}k_jx_j\right]\ , \tag{4.2a}$$

with

$$E_n=\sum_{j=1}^{n}k_j^2\ . \tag{4.2b}$$

This energy can be seen to be the sum of the kinetic energies of particles (with mass $=\frac{1}{2}$) as represented by the momentum operator squared.

To satisfy the symmetry condition, all the permutation terms should be included. Therefore the general form of the solution is

$$f_{nk_1,\ldots,k_n}(x_1,\ldots,x_n,t)=e^{-iE_nt}\sum_{\{Q\}}\exp\left[i\sum_{j=1}^{n}k_{Q(j)}x_j\right]\ , \tag{4.3}$$

where the summation over $\{Q\}$ is the summation over all possible permutations of $[1,2,\ldots,n]$ and $Q(j)$ is the j the component of Q.

Using (4.3), we can construct the eigenstates of the Hamiltonian,

$$\begin{aligned}|k_1,\ldots,k_n,t\rangle&=\mathcal{N}_n\int\frac{1}{\sqrt{n!}}f_{nk_1,\ldots,k_n}(x_1,\ldots,x_n,t)\\&\quad\times\hat{\phi}^\dagger(x_1)\cdots\hat{\phi}^\dagger(x_n)dx_1\cdots dx_n|0\rangle\\&\equiv e^{-iE_nt}|k_1,\ldots,k_n\rangle\ .\end{aligned} \tag{4.4}$$

The normalization constant $\mathcal{N}_n$ can be determined from the following normalization condition:

$$\langle k_1',k_2',\ldots,k_n',t|k_1,\ldots,k_n,t\rangle=\frac{1}{n!}\sum_{\{Q\}}\delta(k_1-k'_{Q(1)})\cdots\delta(k_n-k'_{Q(n)})\ . \tag{4.5a}$$

It is

$$\mathcal{N}_n=\frac{1}{(2\pi)^{n/2}}\ . \tag{4.5b}$$

To construct a pulse, one has to superimpose these eigenstates

$$|\psi\rangle=\sum_n a_n\int g(k_1,\ldots,k_n)|k_1,\ldots,k_n,t\rangle dk_1\cdots dk_n\ . \tag{4.6}$$

Here $g(k_1,\ldots,k_n)$ is a symmetric function of $k_1,\ldots,k_n$. To satisfy the normalization condition

$$\langle\psi|\psi\rangle=1\ , \tag{4.7}$$

we require

$$\sum_n|a_n|^2=1\ , \tag{4.8}$$

$$\int|g_n(k_1,\ldots,k_n)|^2dk_1\cdots dk_n=1\ . \tag{4.9}$$

A natural choice for a_n is a Poisson distribution, and for g_n, a product of distributions:

$$a_n=\frac{\alpha_0^n}{\sqrt{n!}}e^{-|\alpha_0|^2/2}\ , \tag{4.10}$$

$$g_n(k_1,\ldots,k_n)=\prod_{j=1}^{n}g(k_j)\ . \tag{4.11}$$

Using (4.4)–(4.6), (4.10)–(4.11), and the fact that the multiple integrals can be written as products of integrals, (4.6) can be expressed in closed form:

$$|\psi\rangle=e^{-|\alpha_0|^2/2}\exp\left[\int\left[\frac{1}{\sqrt{2\pi}}\int\alpha_0g(k)e^{-ik^2t}e^{ikx}dk\right]\hat{\phi}^\dagger(x)dx\right]|0\rangle\ . \tag{4.12}$$

This is a coherent state with the mean value

$$\langle\psi|\hat{\phi}(x)|\psi\rangle=\frac{1}{\sqrt{2\pi}}\int\alpha_0g(k)e^{-k^2t}e^{ikx}dk\ . \tag{4.13}$$

The expectation value is the classical solution of a pulse on a linear, dispersive fiber; $g(k)$ is its Fourier transform at $t=0$. The pulse shape is the Fourier transform of the distribution in momentum space. Because of the nonlinear phase term in (4.13), the pulse will disperse. It is illuminating to see the changes when the nonlinear Kerr interaction is turned on.

V. CONSTRUCTION OF FUNDAMENTAL SOLITON STATES AND PHASE SPREADING

Equation (3.7) can be solved approximately by using the time-dependent Hartree approximation.[21] This approximation is valid when the number of particles is large. The basis of the Hartree approximation is the assumption that every particle "sees" the same potential caused by the interaction with other particles. Therefore we can use a single-particle wave function to describe a system of particles. To be explicit, we define a Hartree wave function by the following ansatz:

$$f_n^{(H)}(x_1,\ldots,x_n,t)=\prod_{j=1}^{n}\Phi_n(x_j,t)\ . \tag{5.1}$$

The functions $\Phi_n^{(H)}$ are to be determined by minimizing the following functional:

$$\begin{aligned}I&=\int f_n^{*(H)}(x_1,\ldots,x_n,t)\left[i\frac{\partial}{\partial t}+\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}-2c\sum_{1\le i<j\le n}\delta(x_j-x_i)\right]f_n^{(H)}(x_1,\ldots,x_n,t)dx_1\cdots dx_n\\&=n\int\Phi_n^*\left[i\frac{\partial}{\partial t}\Phi_n+\frac{\partial^2}{\partial x^2}\Phi_n-(n-1)c\Phi_n^*\Phi_n\Phi_n\right]dx\ .\end{aligned}\tag{5.2}$$

It turns out that the above functional reaches its minimum value if Φ_n obeys the classical nonlinear Schrödinger equation with the nonlinearity scaled by $n-1$:[21]

$$i\frac{\partial}{\partial t}\Phi_n=-\frac{\partial^2}{\partial x^2}\Phi_n+2(n-1)c\Phi_n^*\Phi_n\Phi_n\ . \tag{5.3}$$

This fact is one of the connections between quantum theory and classical theory.

Equation (5.3) has the following fundamental soliton solution:[6]

$$\begin{aligned}\Phi_n(x,t)=2|c(n-1)|^{-1/2}\eta\exp[&-4i(\xi^2-\eta^2)t\\&-2i\xi(x-x_0)]\\&\times\operatorname{sech}[2\eta(x-x_0+4\xi t)]\ .\end{aligned}\tag{5.4}$$

Contrary to the classical case, η cannot be arbitrary because Φ_n has to satisfy the normalization condition (5.5)

$$\int|\Phi_n(x,t)|^2dx=1\ . \tag{5.5}$$

This leads to the following quantization condition:

$$\eta=\frac{n-1}{4}|c|\approx\frac{n}{4}|c|\ . \tag{5.6a}$$

Substituting (5.6a) into (5.4) and setting

$$\xi=-p/2\ , \tag{5.6b}$$

where p plays the role of momentum, one has

$$\begin{aligned}\Phi_{np}=\frac{\sqrt{n-1}}{2}|c|^{1/2}\exp&\left[i\frac{(n-1)^2}{4}|c|^2t-ip^2t\right.\\&\left.+ip(x-x_0)\right]\\&\times\operatorname{sech}\left[\frac{(n-1)}{2}|c|(x-x_0-2pt)\right]\ .\end{aligned}\tag{5.7}$$

With (5.7), we can construct the Hartree product eigenstates according to (5.1).

$$|n,p,t\rangle_H=\frac{1}{\sqrt{n!}}\left[\int\Phi_{np}(x,t)\hat{\phi}^\dagger(x)dx\right]^n|0\rangle\ . \tag{5.8}$$

A superposition of these states using a Poissonian distribution of n gives the fundamental soliton state

$$\begin{aligned}|\psi_s\rangle_H&=\sum_n\frac{\alpha_0^n}{\sqrt{n!}}e^{-|\alpha_0|^2/2}|n,p,t\rangle_H\\&=\sum_n\frac{\alpha_0^n}{n!}e^{-|\alpha_0|^2/2}\left[\int\Phi_{np}(x,t)\hat{\phi}^\dagger(x)dx\right]^n|0\rangle\ .\end{aligned}\tag{5.9}$$

If the photon number is large,

$$n_0=|\alpha_0|^2\gg1\ , \tag{5.10a}$$

the nonlinearity not excessive,

$$|c|\ll1\ , \tag{5.10b}$$

and the time of observation limited,

$$n_0\sqrt{n_0}|c|^2t\ll1\ , \tag{5.10c}$$

then the summation in (5.9) can be equated to an exponential and $|\psi_s\rangle_H$ can be recognized to be a coherent state:

$$\begin{aligned}|\psi_s\rangle_H&\approx\sum_n\frac{\alpha_0^n}{n!}e^{-|\alpha_0|^2/2}\left[\int\Phi_{n_0p}(x,t)\hat{\phi}^\dagger(x)dx\right]^n|0\rangle\\&=e^{-|\alpha_0|^2/2}\exp\left[\int\alpha_0\Phi_{n_0p}(x,t)\hat{\phi}^\dagger(x)dx\right]|0\rangle\ .\end{aligned}\tag{5.11}$$

Here we have ignored the n dependence of Φ_{np} by replacing the variable n by its average n_0. The mean field is

$${}_H\langle\psi_s|\hat{\phi}(x)|\psi_s\rangle_H\approx\alpha_0\Phi_{n_0p}(x,t)\approx\frac{n_0-1}{2}|c|^{1/2}\exp\left[i\frac{(n_0-1)^2}{4}|c|^2t-ip^2t+ip(x-x_0)\right]\operatorname{sech}\left[\frac{n_0-1}{2}|c|(x-x_0-2pt)\right]\ . \tag{5.12a}$$

This is just the classical solution. If the time of observation is long enough, then the n dependence of the phase cannot be ignored. The mean field becomes

$$ {}_H\langle \psi_s|\hat{\phi}(x)|\psi_s\rangle_H \approx \sum_n e^{-|\alpha_0|^2}\frac{|\alpha_0|^{2n}}{n!}\frac{\alpha_0\sqrt{n}}{2}|c|^{1/2}\exp\left[i\frac{n^2}{4}|c|^2t-ip^2t+ip(x-x_0)\right]\mathrm{sech}\left[\frac{n}{2}|c|(x-x_0-2pt)\right] . \tag{5.12b} $$

Equation (5.12b) makes a very important statement. The expectation value of the field is the average of a set of classical solitons. This is a surprising result, because the field propagates in a nonlinear medium, and hence a simple superposition of solutions as the expectation value of the field was not anticipated. Since in (5.12b) components of different n's have different phase velocities, a soliton experiences phase spreading when it propagates.

Note that we have used a single value of the "momentum" p, not a superposition. However, $|n,p,t\rangle_H$ is not an eigenstate of the momentum operator $\hat{P}$ and thus a distribution of momenta is in fact associated with the state. In the following paper, we shall find that a distribution of momenta is necessary to construct a soliton state. Classically the self-phase modulation and the dispersion balance exactly to form a soliton. Quantum mechanically only the mean values of the two effects are in balance. There still are higher-order phase-spreading effects and higher-order dispersion effects. It has been shown in Ref. 22 that when a monochromatic coherent wave passes through a Kerr medium, the self-phase-modulation spreads its phase distribution and "squeezes" it. This kind of "squeezing" effect is different from that which occurs in a degenerate parametric amplifier or in a four-wave mixing process. In Ref. 22 the quasi-probability-density[23] (QPD) was used to visualize this effect. Here we also define a QPD for the field amplitude at the point x and time t:

$$ Q(\alpha,x,t)\equiv|\langle\alpha,x|\psi_s\rangle|^2 , \tag{5.13} $$

where

$$ |\alpha,x\rangle\equiv e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}[\hat{\phi}^\dagger(x)]^n|0\rangle \tag{5.14} $$

is a local coherent state at the point x.

Substituting (5.9) into (5.13), we have

$$ \begin{aligned} Q(\alpha,x,t)&=e^{-|\alpha|^2-|\alpha_0|^2}\left|\sum_{n=0}^{\infty}\frac{(\alpha^*\alpha_0)^n}{n!}[\Phi_{np}(x,t)]^n\right|^2 \\ &\approx e^{-|\alpha|^2-|\alpha_0|^2}\left|\sum_{n=0}^{\infty}\frac{(\alpha^*\alpha_0)^n}{n!}\exp\left[i\frac{n(n^2-1)}{4}|c|^2t-inp^2t+inp(x-x_0)\right]\right. \\ &\qquad\left.\times\left[\frac{n_0-1}{2}|c|^{1/2}\mathrm{sech}\left[\frac{n_0-1}{2}|c|(x-x_0-2pt)\right]\right]^n\right|^2 . \end{aligned} \tag{5.15} $$

Here we have ignored the n dependence of the amplitude but retained the n dependence of the phase. The evolution of this QPD at the peak of the soliton is calculated and plotted in Fig. 1. The effect of the self-phase-modulation can be clearly seen from this plot. The magnitude of this effect in the side lobes of a soliton would be less than at the peak because the field amplitude is less.

One may also define a QPD for the field amplitude as a function of "wave vector" β and time t:

$$ Q(\alpha,\beta,t)=|\langle\alpha,\beta|\psi_s\rangle|^2 , \tag{5.16} $$

where

$$ |\alpha,\beta\rangle=e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}[\hat{a}^\dagger(\beta)]^n|0\rangle \tag{5.17} $$

is a coherent state of wave vector β. For the soliton state $|\psi_s\rangle_H$ in (5.11),

$$ Q(\alpha,\beta,t)=e^{-|\alpha|^2-|\alpha_0|^2}\left|\sum_{n=1}^{\infty}\frac{(\alpha^*\alpha_0)^n}{n!}[\Psi_{np}(\beta,t)]^n\right|^2 , \tag{5.18} $$

where $\Psi_{np}(\beta,t)$ is the Fourier transform of $\Phi_{np}(x,t)$:

$$ \Psi_{np}(\beta,t)\equiv\mathcal{F}[\Phi_{np}(x,t)] . \tag{5.19} $$

The evolution of $Q(\alpha,\beta,t)$ is basically the same as $Q(\alpha,x,t)$. The magnitude of the self-phase-modulation

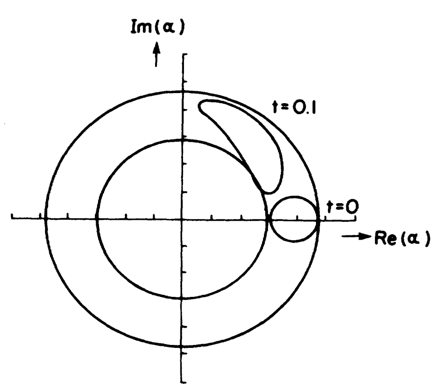

FIG. 1. Half-power contours of the quasi-probability-density at the peak of the soliton. $n_0=16$, $|c|=0.25$, $p=0$, $t=0$, and 0.1.

effect is greatest at the peak of the spectrum. Recently, Drummond and Carter[3] studied soliton propagation effects by solving numerically the linearized stochastic differential equation which is equivalent to the linearized quantum operator equation. They also concluded that the soliton will be squeezed and the effect is greatest at the carrier frequency. However, because they had to linearize the equation, their result can be correct only when the effect is not too large. As has been pointed out in Ref. 22, the squeezing effect due to self-phase-modulation looks like the ordinary squeezing effect when its magnitude is small.

The magnitude of the phase spreading can be estimated from (5.12b). After propagating a period of time t_{PS}, the phase distribution is doubled when

$$\frac{n_0\Delta n}{2}|c|^2 t_{PS}\approx\Delta\theta , \tag{5.20}$$

where Δn is the bandwidth of the photon number distribution and $\Delta\theta$ is the original bandwidth of the phase distribution. For a Poisson distribution, $\Delta n=\sqrt{n_0}$ and $\Delta\theta\geq 1/2\Delta n$.

To develop an estimate for the magnitude of the effect, we compare t_{PS} with the soliton period. From (5.12a), the soliton period is

$$t_s\approx\frac{8\pi}{n_0^2|c|^2} . \tag{5.21}$$

Therefore,

$$\frac{t_{PS}}{t_s}\approx\frac{n_0\Delta\theta}{4\pi\Delta n}\leq\frac{1}{8\pi} \tag{5.22}$$

for a Poisson distribution. From (5.22), one can see that the broadening of the phase distribution due to self-phase-modulation is a significant effect which has a characteristic length less than a soliton period.

The Hartree approximation predicts phase spreading due to the self-phase-modulation effect. We know that the self-phase-modulation effect is caused by the uncertainty of photon number. One may expect that the uncertainty of momentum should cause a dispersion effect of its own. This dispersion effect is lost under the Hartree approximation. We shall study this effect in the following paper.

VI. CONSTRUCTION OF HIGHER-ORDER SOLITON STATES AND SOLITON COLLISION

In this section we use the Hartree approximation to construct two-soliton states and study soliton collision effects. The construction is not as straightforward as that of the fundamental soliton states in Sec. V because the two-soliton states in collision and two-soliton states not in collision have to be treated differently. When a two-soliton state is in collision, all the photons occupy the same space and interact. Every photon behaves in the same way and therefore has the same wave function. However, when a two-soliton state is not in collision, it consists of two independent groups of photons. Photons in different groups behave differently and therefore have different wave functions, although photons in the same group still interact and can be assumed to have the same wave function. Based on the above argument, we construct a two-soliton state that has $n=n_1+n_2$ photons with n_1 and n_2 photons bound together, respectively. We can assume that the total wave function is

$$f^{(c)}_{n_1n_2}(x_1,\ldots,x_{n_1+n_2},t)=\prod_{j=1}^{n_1+n_2}\Phi_{n_1n_2}(x_j,t) \tag{6.1}$$

in collision and

$$f^{(0)}_{n_1n_2}(x_1,\ldots,x_{n_1+n_2},t) =\sum_{\{Q\}}\prod_{j=1}^{n_1}\Phi^{(1)}_{n_1}(x_{Q(j)},t)\prod_{j=n_1+1}^{n_1+n_2}\Phi^{(2)}_{n_2}(x_{Q(j)},t) \tag{6.2}$$

not in collision. In the latter expansion the summation is over Q, over all possible permutations of $[1,2,\ldots,n_1+n_2]$ with the grouping of photons into $[1,2,\ldots,n_1]$ and $[n_1+1,n_1+2,\ldots,n_1+n_2]$ unchanged. The summation appears because $f^{(0)}_{n_1n_2}$ has to be symmetric with respect to the x_j's. All the wave functions Φ_{n_1,n_2}, $\Phi^{(1)}_{n_1}$,$\Phi^{(2)}_{n_2}$ satisfy the normalization condition (5.5). The connection between $\Phi_{n_1n_2}$ and $\Phi^{(1)}_{n_1}$,$\Phi^{(2)}_{n_2}$ can be established by noting that in a sense $\Phi_{n_1n_2}$ is the "mean" wave function of a photon. When the two-soliton state is not in collision, since there are n_1 photons with wave function $\Phi^{(1)}_{n_1}$ and n_2 photons with wave function $\Phi^{(2)}_{n_2}$, we can conclude that the asymptotic approximation of $\Phi_{n_1n_2}$ should be

$$\Phi_{n_1n_2}\to\left[\frac{n_1}{n_1+n_2}\right]^{1/2}\Phi^{(1)}_{n_1}+\left[\frac{n_2}{n_1+n_2}\right]^{1/2}\Phi^{(2)}_{n_2} . \tag{6.3}$$

We shall use (6.3) to establish the connection between the wave functions before and after collision. This approach is somewhat analogous to the WKB method in quantum mechanics. By substituting (6.1) into (5.2) and minimizing the functional, one gets

$$i\frac{\partial}{\partial t}\Phi_{n_1n_2}=-\frac{\partial^2}{\partial x^2}\Phi_{n_1n_2}+2(n_1+n_2-1)c|\Phi_{n_1n_2}|^2\Phi_{n_1n_2} . \tag{6.4}$$

The derivation is the same as that of (5.3). Substituting (6.2) into (5.2) and minimizing the functional, we have

$$i\frac{\partial}{\partial t}\Phi^{(1)}_{n_1}=-\frac{\partial^2}{\partial x^2}\Phi^{(1)}_{n_1}+2(n_1-1)c|\Phi^{(1)}_{n_1}|^2\Phi^{(1)}_{n_1} , \tag{6.5}$$

$$i\frac{\partial}{\partial t}\Phi^{(2)}_{n_2}=-\frac{\partial^2}{\partial x^2}\Phi^{(2)}_{n_2}+2(n_1-1)c|\Phi^{(2)}_{n_2}|^2\Phi^{(2)}_{n_2} . \tag{6.6}$$

In the derivation of (6.5) and (6.6), we have used the fact that $\Phi^{(1)}_{n_1}$ and $\Phi^{(2)}_{n_2}$ are two well-separated functions. This approximation is used frequently in the derivation of this section.

Note that if one substitutes (6.3) into (6.4) and separates $\Phi_{n_1}^{(1)}$ and $\Phi_{n_2}^{(2)}$, one obtains (6.5) and (6.6) again. This proves that (6.3) is consistent with the criteria of the Hartree approximation. Moreover, (6.5) and (6.6) are the same equations as (5.3). This justifies our expectation that a two-soliton state not in collision is the product state of two fundamental soliton states.

The solutions of (6.5) and (6.6) have been obtained in Sec. V. They are

$$\Phi_{n_j}^{(j)}=\left[\frac{n_j-1}{2}\right]^{1/2}|c|^{1/2}\exp\left[i\frac{(n_j-1)^2}{4}|c|^2t-ip_j^2t+ip_j(x-x_{j0})+i\theta_j\right]\operatorname{sech}\left[\frac{n_j-1}{2}|c|(x-x_{j0}-2p_jt)\right], \tag{6.7}$$

with $j=1,2$. However, the phases and mean positions can be different before and after collision. The difference can be determined by noting that before and after collision,

$$\left[\frac{n_1}{n_1+n_2}\right]^{1/2}\Phi_{n_1}^{(1)}+\left[\frac{n_2}{n_2+n_2}\right]^{1/2}\Phi_{n_2}^{(2)}$$

is the asymptotic approximation of the same Φ_{n_1,n_2}, i.e., the asymptotic solution of the CNSE (6.4). It has been shown that the CNSE has two-soliton solutions. Before collision, a two-soliton solution is like two fundamental soliton solutions. After collision, it is still like two fundamental soliton solutions except for a phase shift and a position shift. According to Zakharov and Shabat[6] the magnitude of these shifts for the first soliton are

$$\begin{aligned}\delta\theta_1(n_1,p_1,n_2,p_2)&=-2\arg\left[\frac{\zeta_1-\zeta_2}{\zeta_1-\zeta_2^*}\right]\\&\approx-2\left[\tan^{-1}\left[\frac{\frac{1}{2}|c|(n_1+n_2)}{p_2-p_1}\right]-\tan^{-1}\left[\frac{\frac{1}{2}|c|(n_2-n_1)}{p_2-p_1}\right]\right],\end{aligned} \tag{6.8}$$

$$\begin{aligned}&\delta x_1(n_1,p_1,n_2,p_2)\\&\quad=\frac{1}{\eta_1}\ln\left[\frac{|\zeta_1-\zeta_2|}{|\zeta_1-\zeta_2^*|}\right]\\&\quad\approx\frac{2}{n_1|c|}\left[\ln\left[(p_2-p_1)^2+\frac{|c|^2}{4}(n_2-n_1)^2\right]-\ln\left[(p_2-p_1)^2+\frac{|c|^2}{4}(n_2+n_1)^2\right]\right].\end{aligned} \tag{6.9}$$

Here $\zeta_1=\xi_1+i\eta_1$, $\zeta_2=\xi_2+i\eta_2$, and we have used the definition (5.6a), and (5.6b) for η and ξ. The shifts for the second soliton are analogous.

With these solutions, one can construct the following Hartree states:

$$\begin{aligned}|n_1,p_1,n_2,p_2,t\rangle&=\mathcal{N}_{n_1n_2}\int\left[\sum_{\{Q\}}\prod_{j=1}^{n_1}\Phi_{n_1}^{(1)}(x_j,t)\prod_{j=n_1+1}^{n_1+n_2}\Phi_{n_2}^{(2)}(x_j,t)\right]\hat\phi^\dagger(x_1)\cdots\hat\phi^\dagger(x_{n_1+n_2})dx_1\cdots dx_{n_1+n_2}|0\rangle\\&=\frac{(n_1+n_2)!}{n_1!n_2!}\mathcal{N}_{n_1n_2}\left[\int\Phi_{n_1}^{(1)}(x,t)\hat\phi^\dagger(x)dx\right]^{n_1}\left[\int\Phi_{n_2}^{(2)}(x,t)\hat\phi^\dagger(x)dx\right]^{n_2}|0\rangle.\end{aligned} \tag{6.10}$$

$\mathcal{N}_{n_1n_2}$ is to be determined from

$$\langle n_1,p_1,n_2,p_2,t|n_1,p_1,n_2,p_2,t\rangle=1. \tag{6.11}$$

The result is

$$\mathcal{N}_{n_1n_2}=\frac{\sqrt{n_1!n_2!}}{(n_1+n_2)!}. \tag{6.12}$$

Therefore,

$$|n_1,p_1,n_2,p_2,t\rangle=\frac{1}{\sqrt{n_1!n_2!}}\left[\int \Phi^{(1)}_{n_1}(x,t)\hat{\phi}^\dagger(x)dx\right]^{n_1}\times\left[\int \Phi^{(2)}_{n_2}(x,t)\hat{\phi}^\dagger(x)dx\right]^{n_2} . \quad (6.13)$$

The matrix elements of the field operator for these states are

$$\langle n_1,p_1,n_2,p_2,t|\hat{\phi}(x)|n_1+1,p_1,n_2,p_2,t\rangle \approx\sqrt{n_1+1}\Phi^{(1)}_{n_1+1}(x,t) , \quad (6.14)$$

$$\langle n_1,p_1,n_2,p_2,t|\hat{\phi}(x)|n_1,p_1,n_2+1,p_2,t\rangle \approx\sqrt{n_2+1}\Phi^{(2)}_{n_2+1}(x,t) . \quad (6.15)$$

The other elements are zero. Here $\Phi^{(j)}_{n_j}(x,t)=\Phi^{(0j)}_{n_j}(x,t)$ before collision and $\Phi^{(j)}_{n_j}(x,t)=e^{i\delta\theta_j}\Phi^{(0j)}_{n_j}(x-\delta x_j,t)$ after collision. $\Phi^{(0j)}_{n_j}(x,t)$ $(j=1,2)$, is the fundamental soliton solution of (6.5) and (6.6), respectively.

The two-soliton states can be constructed by superimposing over all n_1 and n_2,

$$|\Psi_s\rangle=\sum_{n_1,n_2} a_1(n_1)a_2(n_2)|n_1,p_1,n_2,p_2,t\rangle . \quad (6.16)$$

The natural choices for $a_1(n_1),a_2(n_2)$ are Poisson distributions,

$$a_1(n_1)=\frac{(\alpha_{10})^{n_1}}{\sqrt{n_1!}}e^{-|\alpha_{10}|^2/2} , \quad (6.17)$$

$$a_2(n_2)=\frac{(\alpha_{20})^{n_2}}{\sqrt{n_2!}}e^{-|\alpha_{20}|^2/2} . \quad (6.18)$$

The mean field can be calculated to be

$$\langle\Psi_s|\hat{\phi}(x)|\Psi_s\rangle\approx\sum_{n_1,n_2}|a_1(n_1)|^2|a_2(n_2)|^2[\alpha_{10}\Phi^{(1)}_{n_1+1}(x,t)+\alpha_{20}\Phi^{(2)}_{n_2+1}(x,t)]$$
$$\approx\left[\sum_{n_1}|a_1(n_1)|^2\alpha_{10}\Phi^{(01)}_{n_1+1}(x,t)\right]+\left[\sum_{n_2}|a_2(n_2)|^2\alpha_{20}\Phi^{(02)}_{n_2+1}(x,t)\right] \quad (6.19a)$$

before collision and

$$\langle\Psi_s|\hat{\phi}(x)|\Psi_s\rangle\approx\left[\sum_{n_1,n_2}|a_1(n_1)|^2|a_2(n_2)|^2\alpha_{10}e^{i\delta\theta_1}\Phi^{(01)}_{n_1+1}(x-\delta x_1,t)\right]$$
$$+\left[\sum_{n_1,n_2}|a_1(n_1)|^2|a_2(n_2)|^2\alpha_{20}e^{i\delta\theta_2}\Phi^{(02)}_{n_2+1}(x-\delta x_2,t)\right] \quad (6.19b)$$

after collision.

This result also contains the quantum fluctuations produced in the collision. The $\delta\theta_i$'s and δx_i's $(i=1,2)$ are functions of n_j $(j=1,2)$ and thus are determined probabilistically.

VII. CONCLUSIONS

We have set up the solution of the nonlinear Schrödinger equation in the Schrödinger formulation, looking for eigenfunction solutions of given energy. In this way, the problem is reduced to a linear problem. This is analogous to, but not congruent with, the inverse scattering approach to the solution of the classical nonlinear Schrödinger equation, which also reduces the problem to the solution of linear equations.

In solving the problem, we made the Hartree approximation according to which the bosons move in the common potential produced by them collectively. When constructing a solitonlike solution with a phase that has nonzero expectation value, a superposition of eigenstates of different photon number was necessary. The distribution of photon number and the self-phase-modulation led to the phase spreading of a soliton. This effect cannot be ignored because it has a characteristic length less than a soliton period.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation Grant No. EET-8700474 and by the Charles Stark Draper Laboratories, Grant No. DL-H-2854018.

APPENDIX: UNCERTAINTY RELATION OF SOLITON POSITION AND MOMENTUM

In this appendix, we prove the uncertainty relation (3.11). The photon-number operator $\hat{N}$, the total momentum operator $\hat{P}$, and the mean position operator $\hat{x}$ have been defined in (3.13), (3.14), and (3.12). With the help of the commutation relations (3.3), the commutator of $\hat{x}$ and $\hat{P}$ can be calculated as follows:

$$[\hat{x},\hat{P}]=-i\frac{\hbar}{2}\int dx'\int dx\{[x'\hat{\phi}^\dagger(x')\hat{\phi}(x'),\hat{\phi}^\dagger(x)\hat{\phi}_x(x)]-[x'\hat{\phi}^\dagger(x')\hat{\phi}(x'),\hat{\phi}^\dagger_x(x)\hat{\phi}(x)]\}\hat{N}^{-1} . \quad (A1)$$

Omitting terms that integrate to zero,

$$[x'\hat{\phi}^\dagger(x')\hat{\phi}(x'),\hat{\phi}^\dagger(x)\hat{\phi}_x(x)]=x'\hat{\phi}^\dagger(x')\hat{\phi}(x')\hat{\phi}^\dagger(x)\hat{\phi}_x(x)-x'\hat{\phi}^\dagger(x)\hat{\phi}_x(x)\hat{\phi}^\dagger(x')\hat{\phi}(x')$$
$$=x'\hat{\phi}^\dagger(x')\hat{\phi}_x(x)\delta(x-x')-x'\hat{\phi}^\dagger(x)\hat{\phi}(x')\frac{\partial}{\partial x}\delta(x-x')$$
$$=x'\hat{\phi}^\dagger(x')\hat{\phi}_x(x)\delta(x-x')+x'\hat{\phi}_x^\dagger(x)\hat{\phi}(x')\delta(x-x')$$
$$=[x\hat{\phi}^\dagger(x)\hat{\phi}_x(x)+x\hat{\phi}_x^\dagger(x)\hat{\phi}(x)]\delta(x-x')$$
$$=\left[\frac{\partial}{\partial x}[x\hat{\phi}^\dagger(x)\hat{\phi}(x)]-\hat{\phi}^\dagger(x)\hat{\phi}(x)\right]\delta(x-x')\ , \qquad \text{(A2a)}$$

$$[x'\hat{\phi}^\dagger(x')\hat{\phi}(x'),\hat{\phi}_x^\dagger(x)\hat{\phi}(x)]=-[x'\hat{\phi}^\dagger(x')\hat{\phi}(x'),\hat{\phi}^\dagger(x)\hat{\phi}_x(x)]\ . \qquad \text{(A2b)}$$

Therefore,

$$[\hat{\bar{x}},\hat{P}]=i\hbar\left[\int\hat{\phi}^\dagger(x)\phi(x)dx\right]\hat{N}^{-1}=i\hbar\ . \qquad \text{(A3)}$$

In the derivation of (A3), we have used

$$[\hat{N},\hat{P}]=0\ , \qquad \text{(A4)}$$

$$\left[\int x'\hat{\phi}^\dagger(x')\hat{\phi}(x')dx',\hat{N}\right]=0\ . \qquad \text{(A5)}$$

From (A3),

$$\langle\Delta\hat{\bar{x}}^2\rangle\langle\Delta\hat{P}^2\rangle\geq\frac{\hbar^2}{4}\ . \qquad \text{(A6)}$$

[1] M. J. Potasek and B. Yurke, Phys. Rev. A **35**, 3974 (1987).
[2] M. J. Potasek and B. Yurke, Phys. Rev. A **38**, 1335 (1988).
[3] P. D. Drummond and S. J. Carter, J. Opt. Soc. Am. **4**, 1565 (1987).
[4] S. J. Carter, P. D. Drummond, M. D. Reid, and R. M. Shelby, Phys. Rev. Lett. **58**, 1841 (1987).
[5] For example, see H. A. Haus, *Waves and Fields in Optoelectronics* (Prentice-Hall, Englewood Cliffs, NJ, 1984), Chap. 10.
[6] V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **34**, 61 (1971) [Sov. Phys.—JETP **34**, 62 (1972)].
[7] For example, see K. Huang, *Statistical Mechanics* (Wiley, New York, 1963), Appendix.
[8] E. H. Lieb and W. Linger, Phys. Rev. **130**, 1605 (1963).
[9] I. B. McGuire, J. Math. Phys. **5**, 622 (1964).
[10] C. N. Yang, Phys. Rev. Lett. **19**, 1312 (1967).
[11] C. N. Yang, Phys. Rev. **168**, 1920 (1967).
[12] For a review, see H. P. Thacker, Rev. Mod. Phys. **53**, 253 (1981).
[13] P. Garbaczewski, *Classical and Quantum Field Theory of Exactly Soluble Nonlinear Systems* (World Scientific, Singapore, 1985).
[14] H. A. Bethe, Z. Phys. **71**, 205 (1931).
[15] H. B. Thacker and D. Wilkinson, Phys. Rev. D **19**, 3660 (1979).
[16] D. B. Creamer, H. B. Thacker, and D. Wilkinson, Phys. Rev. D **21**, 1523 (1980).
[17] M. Göckeler, Z. Phys. C **7**, 263 (1981).
[18] M. Wadati, in *Dynamical Problems in Soliton Systems,* Vol. 30 of *Springer Series in Synergetics,* edited by S. Takeno (Springer, Berlin, 1985).
[19] C. R. Nohl, Ann. Phys. (N.Y.) **96**, 234 (1976).
[20] M. Wadati and M. Sakagami, J. Phys. Soc. Jpn. **53**, 1933 (1984).
[21] B. Yoon and J. W. Negel, Phys. Rev. A **16**, 1451 (1977).
[22] M. Kitagawa and Y. Yamamoto, Phys. Rev. A **34**, 3974 (1986).
[23] R. J. Glauber, in *Quantum Optics and Electronics,* edited by C. M. DeWitt (Gordon and Breach, New York, 1965), p. 138.