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Bifurcation in the phase probability distribution of a highly squeezed state

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We calculate the phase distribution of a highly squeezed state using both a definition of a phase eigenstate and the area-of-overlap principle. This probability curve undergoes a transition from a single- to a double-peaked distribution when we decrease the product of squeeze and displacement parameters.

A squeezed state,[1,2] $|\psi_{sq}\rangle$, of a single mode of the electromagnetic field, or in its most elementary version of a harmonic oscillator described by the two conjugate dimensionless variables, coordinate x, and momentum p, exhibits an asymmetrical redistribution of the corresponding Heisenberg uncertainty fluctuations. The possible enhancement in sensitivity of devices, such as a gravitational wave detector,[3,4] a ring laser gyroscope,[5] or of quantum noise limited communication,[6] has boosted the experimental and theoretical study of such nonclassical states. The remarkable property of an oscillatory photon number probability, $W_m[|\psi_{sq}\rangle]$, of a highly squeezed state resulting from interference in phase space[7–10] represents only one side of the coin of complementarity in quantum mechanics.[11,12] Distribution in phase, $W_\varphi[|\psi_{sq}\rangle]$, that is, in the variable conjugate to m, constitutes the other. But what is the striking feature of $W_\varphi[|\psi_{sq}\rangle]$? A bifurcation, that is, a transition from a single-peaked phase probability curve to a double-peaked curve induced by either a decrease of the displacement parameter α or an increase of the squeeze s, that is the central result reported in this Rapid Communication. Insight into this bifurcation phenomenon, which we have not been able to find in the pertinent literature,[13,14] springs[15] from (i) a mathematical definition of a phase eigenstate[16]—the foundation of a recently proposed[17,18] phase operator in quantum mechanics—and (ii) the area-of-overlap principle.[7–10,19,20]

The question of a phase variable, that is, of a phase operator in quantum mechanics, is a long-standing problem since the decisive year 1926—the dawn of the Bohr-Sommerfeld *Atommechanik*[21] and the rise of the Schrödinger-Born-Heisenberg-Jordan quantum mechanics.[22,23] In contrast to *Atommechanik,* which makes heavy use of action-angle variables m and φ, this new *Undulationsmechanik* or *Matrizenmechanik* is formulated in terms of the conjugate variables x and p. However, when Dirac[24] in the same year quantizes the radiation field relying on m and φ, London[25,26] already recognizes the impossibility of constructing a Hermitian matrix representing the phase operator in the number state bases. In the late 1950's and 1960's, masers and lasers producing electromagnetic waves with relatively well-defined phases stimulate[27] the search for such operators. Again Louisell[28] points out the pathological character of the Dirac phase operator: its matrix elements in the number state bases are indefinite. Since then many new phase operators have been suggested as summarized in the review of Ref. 27.

A recent approach[14,17,18] starts from the definition of states[16]

$$|\varphi\rangle = \lim_{r\to\infty} |\varphi^{(r)}\rangle , \tag{1a}$$

where

$$|\varphi^{(r)}\rangle \equiv [2\pi(r+1)]^{-1/2} \sum_{m=0}^{r} \exp(im\varphi)\,|m\rangle . \tag{1b}$$

Here $|m\rangle$ denotes the mth number state.

The so-defined states display[16] properties expected from states of well-defined phase. However, the limit $r\to\infty$ in Eq. (1a) has to be taken with great care, that is, only after all relevant calculations have been performed with large but finite r. The failure of all earlier attempts to find a phase operator result[17,18] from taking this limit at a premature stage.

With the help of the phase eigenstates, Eq. (1b), we now calculate the phase probability amplitude,

$$w_\varphi^{(r)}[|\psi_{sq}\rangle] \equiv \langle\varphi^{(r)}|\psi_{sq}\rangle = [2\pi(r+1)]^{-1/2} \sum_{m=0}^{r} \langle m|\psi_{sq}\rangle \exp(-im\varphi) , \tag{2}$$

of a squeezed state[8]

$$\psi_{sq}(x) \equiv \langle x|\psi_{sq}\rangle = (s/\pi)^{1/4} \exp[-(s/2)(x-\sqrt{2}\alpha)^2] . \tag{3}$$

Here $s>0$ and α denote the squeeze and displacement parameter, respectively. According to Eq. (2) the phase probability *amplitude*, $w_\varphi^{(r)}[|\psi_{sq}\rangle]$, of a squeezed state is the one-sided, finite, discrete Fourier transform of the photon number probability *amplitude*[6–8]

$$w_m[|\psi_{sq}\rangle]\equiv\langle m|\psi_{sq}\rangle=\left[\frac{2\sqrt{s}}{s+1}\left(\frac{s-1}{s+1}\right)^m(2^m m!)^{-1}\exp\left(-\frac{2s}{s+1}\alpha^2\right)\right]^{1/2}\mathrm{H}_m\left[\frac{s}{(s^2-1)^{1/2}}\sqrt{2}\alpha\right]. \tag{4}$$

Here H_m denotes the mth Hermite polynomial.[29]

In the limit of strong squeezing in the x variable,[8,19] that is, for $s=2/\epsilon\gg 1$, where $0<\epsilon\ll 1$, the quantity $w_m[|\psi_{sq}\rangle]$, Eq. (4) allows the asymptotic representation[8,19,20]

$$w_m[|\psi_{sq}\rangle]\cong\begin{cases}0 \text{ for } m<\alpha^2-\frac{1}{2},\\ \mathcal{A}_m^{1/2}\exp(i\phi_m)+\mathcal{A}_m^{1/2}\exp(-i\phi_m) \text{ for } m>\alpha^2-\frac{1}{2},\end{cases} \tag{5a}$$

where

$$\mathcal{A}_m\equiv[\epsilon/(4\pi)]^{1/2}\frac{\exp[-\epsilon(m+\frac{1}{2}-\alpha^2)]}{(m+\frac{1}{2}-\alpha^2)^{1/2}}, \tag{5b}$$

and

$$\phi_m\equiv(m+\tfrac{1}{2})\arctan[(m+\tfrac{1}{2}-\alpha^2)^{1/2}/\alpha]-\alpha(m+\tfrac{1}{2}-\alpha^2)^{1/2}-\pi/4. \tag{5c}$$

Hence, for $m>\alpha^2$ the expansion coefficients w_m [Eq. (4)] decay and the sum [Eq. (2)] converges even without the factor $(r+1)^{-1/2}$, that is,

$$w_\varphi[|\psi_{sq}\rangle]\equiv\lim_{r\to\infty}[(r+1)^{1/2}w_\varphi^{(r)}]=(2\pi)^{-1/2}\sum_{m=0}^{\infty}w_m[|\psi_{sq}\rangle]\exp(-im\varphi). \tag{6}$$

Thus the phase distribution $W_\varphi[|\psi_{sq}\rangle]$ of the squeezed state Eq. (3) reads

$$W_\varphi[|\psi_{sq}\rangle]\equiv|w_\varphi[|\psi_{sq}\rangle]|^2, \tag{7}$$

where w_φ follows from Eqs. (4) and (6). Before we evaluate this distribution we note that the expression Eq. (6) is a special case of the *phase functional*[15] $w_\varphi[|\psi\rangle]$, an expression for the *phase probability amplitude* of an arbitrary state $|\psi\rangle$ whose *photon number probability amplitudes* $w_m[|\psi\rangle]$ decay such that the discrete m-Fourier transform exists.

The term $(m!)^{-1/2}$ in $w_m[|\psi_{sq}\rangle]$, Eq. (4), and hence in the sum Eq. (6) rules out an exact analytical treatment. We therefore evaluate the sum numerically. In Fig. 1 we display the so-calculated phase distribution $W_\varphi[|\psi_{sq}\rangle]$ in its dependence on the displacement parameter α for a fixed squeeze, $s=21$. In the limit of appropriately large α values the phase probability $W_\varphi[|\psi_{sq}\rangle]$ exhibits a single maximum at $\varphi^{(0)}=0$, whereas in the neighborhood of $\alpha_c=3$ a bifurcation occurs: Values of α below α_c introduce two maxima located at nonzero phase values. In the limit of a highly squeezed vacuum $\alpha=0$ two narrow peaks located at phases $\varphi_{\max}^{(\pm)}(\alpha=0)=\pm\pi/2$ make their appearance.

More insight into this bifurcation phenomenon is offered by an approximate analytical treatment of the sum Eq. (6),[30] which capitalizes on the approximate photon number probability amplitude $w_m[|\psi_{sq}\rangle]$, Eq. (5), together with the stationary phase approximation.[31] The resulting periodic approximate phase distribution[15]

$$W_\varphi[|\psi_{sq}\rangle]\cong\begin{cases}\pi^{-1/2}(\epsilon\alpha^2)^{1/2}\dfrac{\exp(-\epsilon\alpha^2\tan^2\varphi)}{\cos^2\varphi} \text{ for } -\pi/2<\varphi<\pi/2,\\ 0 \text{ for } \pi/2<\varphi<3\pi/2,\end{cases} \tag{8}$$

is thus a Gaussian distribution in the variable $\tan\varphi$. The uncertainty $\Delta\varphi$ in phase, that is, the width of $W_\varphi[|\psi_{sq}\rangle]$ defined by the exponential falloff of the Gaussian reads as

$$\Delta\varphi=\arctan[(\epsilon\alpha^2)^{-1/2}]. \tag{9}$$

Moreover, for $\epsilon\alpha^2>1$ the distribution, Eq. (8), exhibits a single maximum located at $\varphi^{(0)}=0$, whereas for $\epsilon\alpha^2<1$ this maximum turns into a minimum and two maxima at $\varphi_{\max}^{(\pm)}=\pm\arccos[(\epsilon\alpha^2)^{1/2}]$ emerge, in full accord with the numerical evaluation of Eq. (6) leading to Fig. 1. Moreover, this approximate treatment suggests the appearance of the same bifurcation scenario when we keep the displacement parameter α constant, but increase the squeeze, that is, decrease ϵ.

We conclude by approaching the question of the phase distribution, $W_\varphi[|\psi_{sq}\rangle]$, of a highly squeezed state from yet another angle. The guiding principle in the present search for $W_\varphi[|\psi_{sq}\rangle]$ is the concept of area-of-overlap-in-phase space.[7–10,19,20] The quantum-mechanical scalar product $w_\varphi[|\psi_{sq}\rangle]=\langle\varphi|\psi_{sq}\rangle$ between two quantum states such as the phase eigenstate $|\varphi\rangle$ and a squeezed state $|\psi_{sq}\rangle$ is identical to the area of overlap between the two states represented in *phase space*. But can one represent in phase space a squeezed state $|\psi_{sq}\rangle$ and a phase state?

The Gaussian Wigner-cigar[8,10]

$$P_{sq}^{(W)}(x,p)=\pi^{-1}\exp[-s(x-\sqrt{2}\alpha)^2-s^{-1}p^2] \tag{10}$$

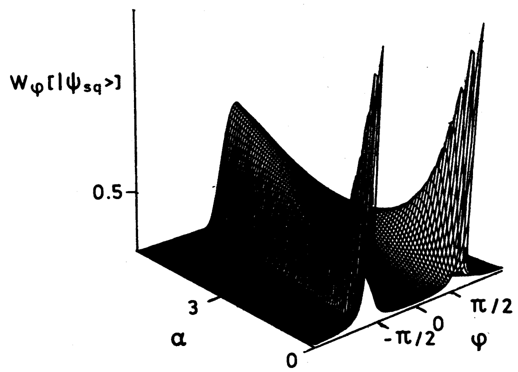

FIG. 1. The probability, $W_\varphi[|\psi_{sq}\rangle]$, to find the phase of an oscillator to lie between the value φ and $\varphi+d\varphi$ when the oscillator is in a state $|\psi_{sq}\rangle$, Eq. (3), highly squeezed in the x variable ($s=21$) undergoes a bifurcation for decreasing displacement α of the state: For values of α appropriately larger than 3 the probability curve exhibits a *single* maximum located at the phase angle $\varphi^{(0)}=0$ whereas below this critical value *two* symmetrically located maxima at nonvanishing phases $\varphi_{max}^{(\pm)}(\alpha)$ occur. In the limit of a highly squeezed vacuum $\alpha=0$ the distribution $W_\varphi[|\psi_{sq}\rangle]$ is strongly peaked at the phase values $\varphi_{max}^{(+)}(\alpha=0)=\pi/2$ and $\varphi_{max}^{(-)}(\alpha=0)=-\pi/2$. A similar bifurcation scenario makes its appearance when instead of reducing α and keeping s constant we increase the squeezing while leaving the displacement unaltered. The curves presented here result from a numerical evaluation of the phase probability expression, Eqs. (4), (6), and (7), based on the mathematical definition of a phase eigenstate, Eq. (1).

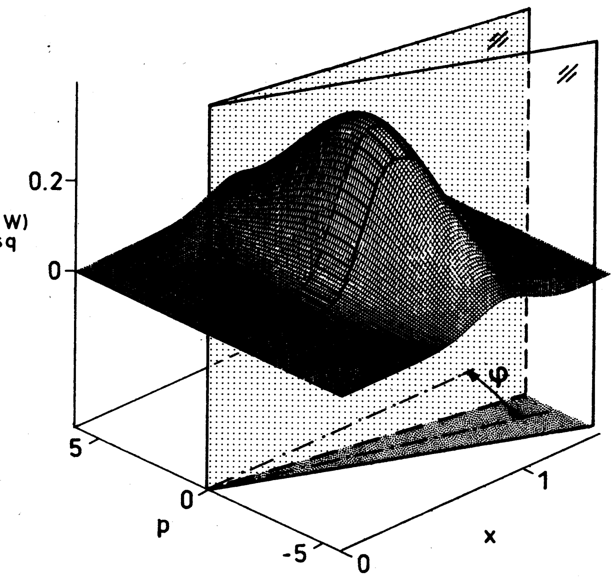

FIG. 2. In its most elementary version we associate with a state $|\varphi\rangle$ of well-defined phase φ a ray propagating in x-p oscillator phase space, emerging from its origin and directed under an angle φ relative to the x axis shown by the dashed-dotted lines. A more appropriate visualization of $|\varphi\rangle$ starts from a divergent beam of solid angle $d\varphi$ propagating along this center line as illustrated by the dark wedge-shaped phase-space slice. The Gaussian Wigner-cigar, $P_{sq}^{(W)}$, of Eq. (10)—depicted here by the hump shape—serves as a representation of a highly squeezed state $|\psi_{sq}\rangle$, Eq. (3), of squeeze $s=21$, and displacement parameter $\alpha=0.7$. The area-of-overlap-in-phase-space principle associates with the phase probability, $W_\varphi[|\psi_{sq}\rangle]=|\langle\varphi|\psi_{sq}\rangle|^2$, of this highly squeezed state, the weighted area, that is, the volume of the shaded phase space slice cut out of this hump by the wedge forming planes of glass—the knife edges of the phase state. The volume of this slice is a measure of phase probability—no simpler pictorial representation offers itself.

shown in Fig. 2 is the obvious answer to the first question. Less obvious, however, is the response to the second inquiry. In a first attempt we associate with a state $|\varphi\rangle$ of well-defined phase, a ray[32] in x-p oscillator phase space emerging from the origin and directed under an angle φ with the x axis as depicted in Fig. 2 by the dashed-dotted lines. More appropriate, however, is the picture[15] of $|\varphi\rangle$ as a diverging beam shown in Fig. 2 by the dark phase space wedge. Then the phase probability $W_\varphi[|\psi_{sq}\rangle]$ is identical to the phase space slice cut out of the Gaussian cigar, Eq. (10) by the phase state wedge represented in Fig. 2 by two panes and given in polar coordinates by

$$W_\varphi[|\psi_{sq}\rangle]\cong\int_0^\infty d\rho\int_{-\pi}^{\pi}d\varphi'\rho\delta(\varphi-\varphi')\times P_{sp}^{(W)}(x=\rho\cos\varphi';p=\rho\sin\varphi')\,. \tag{11}$$

When we substitute Eq. (10) into Eq. (11) and perform the integration in the limit of large squeezing $s=2/\epsilon$ where $0<\epsilon\ll 1$, we arrive[15] at Eq. (8).

Area of overlap between the phase-space slice and the Gaussian cigar—no simpler visualization of the squeezed state phase probability offers itself. Moreover, this formalism provides deeper insight into Eq. (8). It readily identifies the phase uncertainty $\Delta\varphi$, Eq. (9), as the angle in phase space determined by the height $(2/\epsilon)^{1/2}$ of the Gaussian cigar Eq. (10) defined by its contour line of exponential falloff and its displacement $\sqrt{2}\alpha$, from the origin. Even the bifurcation phenomenon depicted in Fig. 1 and hidden in the numerical evaluation of the sum Eq. (6) becomes more transparent in this approach. Consider the Gaussian cigar of Fig. 2 corresponding to a state highly squeezed in the x variable and elongated in p. When located far away from the origin of phase space, that is, when the displacement $\sqrt{2}\alpha$ of the cigar is much larger than its height, $(2/\epsilon)^{1/2}$, this angle $\Delta\varphi(\alpha)$ is small and hence the phase is well localized around the phase zero. In the extreme case of the squeezed vacuum $\alpha=0$ the elongation of the Gaussian cigar along the p axis introduces an asymmetry in phase space, confines the phase to the values $+\pi/2$ or $-\pi/2$, and gives rise to the two maxima of the phase distribution. Since the probability curve has to be a continuous function of the displacement α, a bifurcation must occur between these two extremes, that is, when the height, $(2/\epsilon)^{1/2}$, is identical to the displacement $\sqrt{2}\alpha$ of the cigar, or when $\epsilon\alpha^2=1$.

This striking bifurcation phenomenon arising in the phase distribution of a highly squeezed state remains to be detected in the realm of quantum optics.

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