Path-integral formulation for stochastic processes driven by colored noise

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A detailed discussion of the path-integral formalism for stochastic processes described by a stochastic differential equation driven by a nonwhite noise is given. The path-integral representation in the configuration space of the transition probability for a process driven by Ornstein-Uhlenbeck noise is derived. We show how to treat in this approach any kind of initial conditions, including the question of the coupling with the noise at initial time. Known approximations are reobtained in this context. Markovian approximations based on the Lagrangian are also discussed. The stationary distribution of the process in the weak-noise limit is obtained from the Lagrangian without relying on the use of Fokker-Planck or Markovian approximations.

I. INTRODUCTION

Path-integral methods arose from the context of stochastic processes. They have become a standard technique to analyze continuous Markov stochastic processes described by Fokker-Planck equations.[1] These processes are equivalently described by stochastic differential equations of the Langevin type driven by a Gaussian white noise. The colored noise problem refers to the calculation of the statistical properties of a stochastic process described by a stochastic differential equation driven by a nonwhite noise. In this case the process is non-Markovian. The analysis of this type of non-Markovian processes in terms of a path integral formulation is rare[2,3] up to recent times.[4,5] It seems, however, to be a rather useful alternative point of view to study some of the issues of the colored noise problem.[5] In this paper we give a detailed discussion of the path-integral formalism for such stochastic processes, together with some relevant applications. We will particularly deal with processes driven by an Ornstein-Uhlenbeck noise, but some of our results hold for more general cases.

A review of results concerning stationary distributions, steady-state dynamics, and transient dynamics of colored noise-driven stochastic processes is given in Ref. 6. The recent literature in this problem is overwhelmingly extensive (see, as a representative sample Refs. 6–10) and rather controversial, mainly in connection with the calculation of activation rates and mean first passage times in bistable systems. The ubiquitous presence of statements about confusion reveals that a satisfactory theoretical understanding has not been achieved. The basic difficulty of the problem is the non-Markovian nature of the process, so that well-known Fokker-Planck techniques can not be, in principle, used. However, most of the theoretical approaches proposed so far force the description of the process, in one way or another, into a Fokker-Planck formulation. Such Fokker-Planck approximations arise, via functional methods, cumulant expansions, or other techniques, as a result of truncations or resummations of expansions in the correlation time τ of the noise. Although early analogical[11] and numerical[12] simulations were useful to check qualitative predictions of the theory, the validity of these approximations is often difficult to assess. The advantage of the path-integral formulation is that it takes a rather different approach to the problem, avoiding, in principle, the question of a Fokker-Planck approximation. Using the path-integral formulation it is possible to obtain some exact results and methods which do not use τ as an expansion parameter. An example of these results is the exact expression for the action integral.[3] A remarkable fact is that this action integral does not contain any terms involving powers of τ of order higher than τ^2. The problems associated with τ expansions can then be, in principle, bypassed in this formulation. We follow here, as in Ref. 3, a path-integral formulation in the configuration space, while other recent formulations[4(a)–4(c)] introduce a phase-space representation. The advantage of a configuration-space representation is that the stochastic process is discussed in the space of physical interest without the introduction of additional unphysical variables. The formulation of Ref. 4(d) is also in the configuration space, but it seems to use paths defined in the time interval $(-\infty, +\infty)$. In such formulation an arbitrary initial preparation of the non-Markovian stochastic process cannot be considered.

The summary of contents and main results of this paper are as follows. In Sec. II we analyze different possible alternative ways to derive the path-integral representation of the transition probability for a process driven by Ornstein-Uhlenbeck noise. Two particular routes are useful to understand two main points. One of them clarifies the fact that the action integral contains contributions in τ only up to order τ^2. This is related to the inversion of the correlation function of the noise which, however, contains terms to all orders in τ. A second route permits a detailed discussion of preparation effects and boundary terms of the action integral. We show how to treat in the path integral approach any kind of initial conditions including the question of the coupling with the noise at initial time. Preparation effects are crucial in non-Markovian processes due to the memory of the dynamics. Technical aspects are relegated to two appendixes. Appendix A contains the calculation of the inversion of the correlation function of the noise. Appendix B substantiates our results through a discretized version of the path integral.

In Sec. III we address the problem of deriving approximate evolution equations for the probability density starting from a path-integral formulation. Most of the results here are valid for arbitrary colored noise. The general presentation features a cumulant expansion of the path-integral expression for a generating function. Depending on the type of cumulants used one arrives to integro-differential or differential equations for the probability density. We discuss how different known approximations are reobtained in this context. As a separate question we examine the conditions to obtain a Markovian Fokker-Planck approximation from the path integral formulation of Sec. II. We conclude that the adiabatic approximation of Ref. 10 folds within these conditions, being consistent with the the Markovian Fokker-Planck approximation for the dynamics. Our analysis indicates how to implement such approximation in more general cases.

The calculation of the stationary distribution of the process in the weak-noise limit is considered in Sec. IV. The novelty of this calculation is that it neither makes use of Fokker-Planck or Markovian approximations, nor relies explicitly on τ expansions. It directly addresses the variational problem which appears in the path-integral formulation without invoking appropriate evolution equations for the probability density. The problem is discussed from two alternative points of view. One is the direct calculation of the minimizing path and the other a Hamilton-Jacobi type formulation. This second formulation permits a treatment which is rather independent of particular models. Our result for the nonequilibrium potential identifies the contents of an expression obtained in a variety of independent calculations. A technical discussion of the properties of the minimizing path and its relation with the minimizing path in phase space is given in Appendix C.

II. PATH-INTEGRAL REPRESENTATION FOR PROCESSES DRIVEN BY ORNSTEIN-UHLENBECK NOISE

We consider a stochastic process characterized by a stochastic differential equation (SDE) for a variable $q(t)$ of the form

$$\dot{q}(t)=f(q)+g(q)\xi(t) , \quad (2.1)$$

where $\xi(t)$ is an Ornstein-Uhlenbeck process. That is, Gaussian with zero mean and correlation

$$C(t,t')\equiv\langle\xi(t)\xi(t')\rangle=\frac{D}{\tau}\exp(-|t-t'|/\tau) , \quad (2.2)$$

where D is the noise intensity and τ its correlation time. Equation (2.1), with specified initial conditions, defines a non-Markovian process in the configuration space of the variable $q(t)$. In the limit $\tau\rightarrow 0$, $C(t,t')\rightarrow 2D\,\delta(t-t')$ and $q(t)$ becomes a Markovian diffusion process[13] whose probability density $P(q,t)$ obeys the following Fokker-Planck equation[14]

$$\partial_t P(q,t)=-\partial_q f(q)P(q,t)+D\partial_q g(q)\partial_q g(q)P(q,t) . \quad (2.3)$$

The process $q(t)$ might be described by Markovian methods at the price of enlarging the space of variables considered. Indeed, $\xi(t)$ can be considered as an additional variable driven by a Gaussian white noise $\eta(t)$,

$$\dot{\xi}(t)=-\tau^{-1}\xi(t)+\tau^{-1}\eta(t) , \quad (2.4)$$

where $\langle\eta(t)\eta(t')\rangle=2D\,\delta(t-t')$. The set of equations (2.1) and (2.4) define a Markovian diffusion process in (q,ξ) space characterized by the following Fokker-Planck equation:

$$\partial_t P(q,\xi,t)=-\partial_q[f(q)+g(q)\xi]P(q,\xi,t) +\partial_\xi\frac{\xi}{\tau}P(q,\xi,t)+\frac{D}{\tau^2}\partial_\xi^2 P(q,\xi,t) . \quad (2.5)$$

We are interested in a path-integral representation of the process $q(t)$ in the q-configuration space of interest. The transition probability from q_0 at t_0, to q at time t is shown below to have the general expression

$$P(q,t/q_0,t_0)=\int_{q_0}^{q}D_g[q(s)]\exp\left[-\int_{t_0}^{t}dsL-\tau[\dot{q}_0-f(q_0)]^2/[2Dg(q_0)^2]\right] , \quad (2.6)$$

where $D_g[q(s)]$ is a shorthand notation for $D[q(s)/\int g(q(s))dq(s)]$.

The integral is over all the paths going from (q_0,t_0) to (q,t). L is the Lagrangian-like function[15] which for (2.1) turns to be[3,5]

$$L=\{\tau[\ddot{q}-f'(q)\dot{q}-g'(q)\dot{q}(\dot{q}-f(q))/g(q)] +\dot{q}-f(q)\}^2/[4Dg(q)^2] . \quad (2.7)$$

In the limit $\tau=0$ it reduces to the well-known Lagrangian associated with (2.3) in the prepoint discretization.[16,17]

The Langrangian (2.7) depends on $\ddot{q}$, so that properly speaking, it is not a Lagrangian function. Such dependence, together with the appearance of the initial condition in (2.6), reflects the non-Markovian nature of the process associated. The Lagrangian (2.7) is an exact result containing all the required information on the process $q(t)$ which concerns statistical properties depending on a single time. A remarkable feature in (2.7) is that it only contains terms linear and quadratic in τ while most studies of the colored noise problem are based in expansions involving a power series in τ with infinite terms.

The derivation of (2.6) and (2.7) has two possible starting points which correspond to the original SDE representation (2.1) and to the Markovian description in an enlarged space given by (2.5). In both cases attention should be paid to the delicate problem of initial conditions and preparation effects. As we shall see later, the natural framework to discuss these problems is the enlarged Markovian description. The first starting point is the one followed in the original derivation of Ref. 3. Here attention is paid to the trajectory in q space and it follows the same line of thought that related work for white[18] or colored noise.[2] Let us recall such derivation for the particular case $g=1$.

Inserting the characteristic functional of ξ we get the probability density of a noise realization which is given by

$$\rho(\xi(s);t_0\le s\le t)$$
$$=N\exp\left[-\tfrac{1}{2}\int_{t_0}^{t}du\int_{t_0}^{t}du'\xi(u)R(u,u')\xi(u')\right], \qquad (2.8)$$

where $R(u,u')$ is the inverse of the correlation function $C(t,t')$ [see Eq. (A1)].

The transition probability of the non-Markovian process $q(t)$ is obtained from Eqs. (2.1) and (2.8) by integrating over all the paths $q(t)$ going from (q_0,t_0) to (q,t). The following expression follows in the prepoint discretization:

$$P(q,t/q_0,t_0)=\int_{q_0}^{q}D[q(s)]\exp\left[-\tfrac{1}{2}\int_{t_0}^{t}du\int_{t_0}^{t}du'[\dot q(u)-f(q(u))]R(u,u')[\dot q(u')-f(q(u'))]\right]. \qquad (2.9)$$

Now if we use the expression for R given by Eq. (A4) with $g=1$, it is easy to see that, after some integrations by parts, expression (2.6) is recovered in the additive case $g=1$. The multiplicative case can be treated in a similar way. This derivation makes clear the reason why L in (2.7) does not contain terms proportional to τ^n, $n>2$. Due to the exponential form of the correlation C, the expansion of its inverse R in powers of τ is cut at order τ^2. This can be understood from the fact that the inverse of the Fourier transform of $\exp(-|t|/\tau)/(2\tau)$ is $1+k^2\tau^2$.

In this derivation we have assumed that ξ is in the stationary state. To include any kind of initial conditions we must obtain the inverse of the nonstationary correlation function of the noise. The problem of initial conditions and preparation effects will be discussed later.

The second starting point to obtain (2.7) is the path-integral representation of the Markovian process described by (2.5). Such representation for an n-variable diffusion process is standard.[1,17] The only delicate point is that the diffusion matrix associated with (2.5) is singular since the q variable obeys a deterministic equation of motion in the (q,ξ) space. This prevents the use of the standard Lagrangian representation in configuration space. The path integral representation is then developed in an enlarged phase space[1,17] in which conjugate momenta $\hat q$ and $\hat\xi$ associated with the variable q and ξ are introduced. These conjugate momenta are the c-number variables associated with the operators introduced by Martin, Siggia, and Rose[19] to define response functions.[2,18,20,21] For the transition probability from (q_0,ξ_0) at time $t=t_0$ to (q,ξ) at time t we have, in the prepoint discretization,

$$P(q,\xi,t/q_0,\xi_0,t_0)=\int_{(q_0,\xi_0)}^{(q,\xi)}D[q(s)]D[\xi(s)]D[\hat q(s)]\times D[\hat\xi(s)]\exp(-S), \qquad (2.10)$$

where the action integral is

$$S=\int_{t_0}^{t}ds[i\dot q(s)\hat q(s)+i\dot\xi(s)\hat\xi-H(q(s),\xi(s),i\hat q(s),i\hat\xi(s))], \qquad (2.11)$$

and the Hamiltonian-like function H is given by

$$H=i\hat q(f(q)+g(q)\xi)-i\hat\xi\xi/\tau-D(i\hat\xi)^2/\tau^2. \qquad (2.12)$$

It is interesting to note that the Lagrangian function can be obtained through the Legendre transformation of the complex function (2.12) in which $i\hat q\to\dot q$, $i\hat\xi\to\dot\xi$. We have

$$L(q,\dot q,\xi,\dot\xi)=i\dot q\hat q+i\dot\xi\hat\xi-H(q,\xi,i\hat q,i\hat\xi)$$
$$=\frac{\tau^2}{4D}(\dot\xi+\xi/\tau)^2. \qquad (2.13)$$

In the present two-variable approach fluctuations are associated with the ξ variable, while q obeys a deterministic equation. This is the reason why a Lagrangian that weights different possible paths is in (2.13) independent of q. The weighting of paths can be transferred to the q space if one replaces ξ and $\dot\xi$ in (2.13) as obtained from (2.1). In this way one recovers the Lagrangian function (2.7). This short cut from (2.12) to the Lagrangian (2.7) is justified by making the integrals over ξ, $\hat\xi$, and $\hat q$ in (2.10).

This can be done in six different ways depending on the order of integration. Two of these ways (integration in the following orders: $\xi,\hat{\xi},\hat{q}$ and $\hat{\xi},\xi,\hat{q}$) require the inversion of the noise correlation function. As we have discussed when recalling the original derivation of (2.6), such procedures are particularly interesting because they make clear the reason why L in (2.7) does not contain terms proportional to τ^n, $n>2$. If we first integrate over the noise variable ξ and then over its conjugate momentum $\hat{\xi}$, we obtain

$$P(q,\xi,t/q_0,\xi_0,t_0)=[4\pi D\tau^{-1}\sinh(t-t_0)/\tau]^{-1/2}\int_{(q_0,\xi_0)}^{(q,\xi)} D[q(s)]D[\hat{q}(s)]e^{-S_1+i\int_{t_0}^{t}ds\,\hat{q}(s)(\dot{q}-f(q))}\,, \tag{2.14}$$

where

$$\begin{aligned}S_1=[4D\tau^{-1}\sinh(t-t_0)/\tau]^{-1}\Bigg[&\xi^2e^{t/\tau}+\xi_0^2e^{-t/\tau}-2\xi_0\xi-\frac{4D}{\tau}\xi\int_{t_0}^{t}ds\, i\hat{q}(s)g(q(s))\sinh(s-t_0)/\tau\\&-\frac{4D}{\tau}\xi_0\int_{t_0}^{t}ds\, i\hat{q}(s)g(q(s))\sinh(t-s)/\tau\\&+\frac{8D^2}{\tau^2}\int_{t_0}^{t}ds\int_{t_0}^{t}ds'\hat{q}(s)g(q(s))\sinh(t-s)/\tau\sinh(s'-t_0)/\tau\hat{q}(s')g(q(s'))\Bigg]\,. \end{aligned}\tag{2.15}$$

If we integrate over all final possible values of ξ, we get an equation similar to (2.14) and (2.15). Now, to take into account preparation effects we must keep the variable ξ_0 in (2.15) and afterwards integrate over ξ_0 and q_0 with a joint distribution. Then the integration over q in (2.14) and (2.15) requires the inversion of a time-dependent function. This function corresponds to the nonstationary noise correlation function we have mentioned when recalling the original derivation of (2.6). Preparation effects will be analyzed later in a more convenient way.

The transition probability in q space for the case of stationary noise is obtained from (2.14) integrating over all possible final values of ξ and over the initial values ξ_0 with the stationary distribution $P_{\rm st}(\xi_0)=(\tau/2\pi D)^{1/2}\exp(-\xi_0^2\tau/2D)$, resulting in

$$\begin{aligned}P(q,t/q_0,t_0)&=\int d\xi\, d\xi_0 P_{\rm st}(\xi_0)P(q,\xi,t/q_0,\xi_0,t_0)\\&=\int_{q_0}^{q}D[q(s)]D[\hat{q}(s)]\\&\qquad\times\exp[-A(q(s),\hat{q}(s))]\,,\end{aligned}\tag{2.16}$$

where

$$\begin{aligned}A(q(s),\hat{q}(s))=&-i\int ds\,\hat{q}(s)[\dot{q}(s)-f(q(s))]\\&+\tfrac{1}{2}\int ds\int ds'\hat{q}(s)\bar{C}(s,s')\hat{q}(s')\,,\end{aligned}\tag{2.17}$$

$$\bar{C}(s,s')=g(q(s))C(s,s')g(q(s'))\,. \tag{2.18}$$

The result (2.16)–(2.18) was also obtained by Phytian[2] much in the same spirit of our first derivation of (2.7) given here, that is, following the SDE representation (2.1) and without going to the enlarged Markovian representation. The result (2.16)–(2.18) is in fact exact for a general Gaussian noise. The final integration over $\hat{q}$ in (2.16) leads to an expression similar to (2.9), but with the inverse of $\bar{C}$, instead of the inverse of the noise correlation function. Taking into account the discussion after (2.9) we recover again (2.6) in the Ornstein-Uhlenbeck case.

To discuss preparation effects we include in (2.10) the initial and final conditions for q and ξ by the use of *ad hoc* δ functions

$$\begin{aligned}P(q,\xi,t/q_0,\xi_0,t_0)=\int &D[q(s)]D[\xi(s)]D[\hat{q}(s)]\\&\times D[\hat{\xi}(s)]\delta(q(t)-q)\delta(\xi(t)-\xi)\\&\times\delta(q(t_0)-q_0)\delta(\xi(t_0)-\xi_0)\\&\times\exp(-S)\,.\end{aligned}\tag{2.19}$$

In this way the integral is over all the paths. The shortest path to treat arbitrary initial conditions is the following. We first integrate over $\hat{q}$. This results in the appearance of the δ functional $\delta[\dot{q}(s)-f(q(s))-g(q(s))\xi(s)]$, which makes the integration over ξ immediate, yielding the expression

$$\begin{aligned}P(q,\xi,t/q_0,\xi_0,t_0)=\int &D_g[q(s)]D[\hat{\xi}(s)]\delta(q(t)-q)\delta(\dot{q}(t)-f(q(t))-g(q(t))\xi)\\&\times\delta(q(t_0)-q_0)\delta(\dot{q}(t_0)-f(q(t_0))-g(q(t_0))\xi_0)\exp\left[-\frac{D}{\tau^2}\int_{t_0}^{t}ds\,\hat{\xi}(s)^2\right]\\&\times\exp\left[i\int_{t_0}^{t}ds\,\hat{\xi}(s)\left[\frac{d}{ds}\{[\dot{q}(s)-f(q(s))]/g(q(s))\}+\frac{1}{\tau}[\dot{q}-f(q)]/g(q)\right]\right]\,.\end{aligned}\tag{2.20}$$

The Gaussian integral over $\hat{\xi}$ results in

$$P(q,\xi,t/q_0,\xi_0,t_0)=\int_{q_0}^{q} D_g[q(s)]\delta(\dot{q}(t)-f(q)-g(q)\xi)\delta(\dot{q}(t_0)-f(q_0)-g(q_0)\xi_0)\exp\left[-\int_{t_0}^{t} ds\, L\right] , \tag{2.21}$$

where the integral is over all the paths $q(t)$ going from (q_0,t_0) to (q,t) and L is given by (2.7). The derivation given here of (2.21) is rather formal. A derivation from (2.10) using the prepoint discretized version of the path integral is given in Appendix B.

If we integrate (2.21) over all possible final values of ξ and over the initial values ξ_0 with the stationary distribution of the Ornstein-Uhlenbeck noise, we recover (2.6). This transition probability in q space corresponds to initial decoupled conditions for q and ξ, the noise being in the stationary state.

If we consider arbitrary initial conditions given by a joint distribution $P_0(q_0,\xi_0)$, we get from (2.21)

$$P(q,t/q_0,t_0;P_0)=\int_{q_0}^{q} D_g[q(s)]P_0(q_0,[\dot{q}(t_0)-f(q(t_0))]/g(q(t_0)))\exp\left[-\int_{t_0}^{t} ds\, L\right] . \tag{2.22}$$

We have then found the path-integral representation for the non-Markovian transition probability density with arbitrary initial preparation.[22] When $P_0(q_0,\xi_0)$ is the stationary probability density, we obtain the stationary transition probability density. However, since the stationary probability density is not known, this is only a formal result.

As a final point we want to consider the extension of the above results to the case when a Gaussian white noise is added to Eq. (2.1).[10(b)] We consider the SDE

$$\dot{q}(t)=f(q)+g(q)\xi(t)+\eta(t) , \tag{2.23}$$

with $\langle \eta(t)\eta(t')\rangle=2\epsilon\delta(t-t')$. We can repeat the procedure employed to obtain the Lagrangian (2.7). If we consider the case $g=1$, the following expression is derived:

$$P(q,t/q_0,t_0)=\int_{q_0}^{q} D[q(s)]D[\hat{q}(s)]\exp\left[i\int_{t_0}^{t} ds\,\hat{q}(s)[\dot{q}(s)-f(q(s))]-\tfrac{1}{2}\int_{t_0}^{t} ds\int_{t_0}^{t} ds'\hat{q}(s)\tilde{C}(s,s')\hat{q}(s')\right] , \tag{2.24}$$

where $\tilde{C}$ is the correlation function of the Gaussian noise $\xi+\eta$,

$$\tilde{C}(s,s')=2\epsilon\delta(s-s')+\frac{D}{\tau}\exp(-|s-s'|/\tau) . \tag{2.25}$$

The integration over $\hat{q}$ leads to an expression similar to (2.9) but with the inverse of $\tilde{C}$ instead of the inverse of C. Using the expansion (A3) for the exponential term, one finds

$$\tilde{R}(s,s')=\frac{1}{2(D+\epsilon)}\delta(s-s') - \frac{D}{2\epsilon(D+\epsilon)}\sum_{j=1}^{\infty}\tau^{2j}\left[\frac{\epsilon}{\epsilon+D}\right]^{j}\delta^{2j}(s-s') , \tag{2.26}$$

where boundary terms have not been taken into account. The inverse of the correlation function of the Gaussian noise $\xi(t)+\eta(t)$ contains then all the powers in τ and $\delta^{(n)}(s-s')$. Therefore the Lagrangian contains arbitrary time derivations of q. The implications of this fact for the case of Markovian Fokker-Planck approximations will be discussed at the end of Sec. III.

III. EQUATIONS FOR THE PROBABILITY DENSITY

In this section we use the path-integral formalism to derive evolution equations for the probability density $P(q,t)$. Our aim is not to obtain new equations, but rather to discuss connections among several different approximations taking the path-integral expression for the probability density as a starting point. Functional methods have been successfully applied[6,12,23,24] to derive Fokker-Planck-like equations for $P(q,t)$. The discussion of these equations from a true path-integral representation of the probability is a quite natural procedure.

We consider stochastic differential equations with a general noise. The case of the Ornstein-Uhlenbeck noise is considered later as a particular case. Hence, we start with Eq. (2.1) but now $\xi(t)$ being any kind of noise. The probability density can be obtained by averaging $\delta(q-q(t,q_0,t_0,\xi(t)))$ over all possible realizations, i.e.,

$$P(q,t/q_0,t_0)=\langle \delta(q-q(t,q_0,t_0,\xi(t)))\rangle , \tag{3.1}$$

where $q(t,q_0,t_0,\xi(t))$ is the solution of (2.1) for a given realization $\xi(t)$ of the noise and with initial condition $q(t_0)=q_0$, independent of $\xi(t)$. The delta functional $\delta(q-q(t,q_0,t_0,\xi(t)))$ in path-integral representation reads

$$\delta(q-q(t,q_0,t_0,\xi(t)))=\int_{q_0} D[q(s)]D[\hat{q}(s)]\delta(q-q(t))\exp\left[-i\int_{t_0}^{t} ds\,\hat{q}(s)[\dot{q}(s)-f(q)-g(q)\xi(s)]\right] . \quad (3.2)$$

After averaging in (3.2) over the realizations of the noise we obtain the probability density as a path integral in phase space given by:

$$P(q,t/q_0,t_0)=\int_{q_0} D[q(s)]D[\hat{q}(s)]\delta(q-q(t))\exp\{-i\int_{t_0}^{t} ds\,\hat{q}(s)[\dot{q}(s)-f(q)]\}\Phi\{q(t),\hat{q}(t)\} , \quad (3.3)$$

including the characteristic functional of the stochastic term $g(q)\xi(t)$

$$\Phi\{q(t),\hat{q}(t)\}\equiv\left\langle \exp\left[i\int_{t_0}^{t} ds\,\hat{q}(s)g(q(s))\xi(s)\right]\right\rangle . \quad (3.4)$$

This functional allows us to establish a connection with other methods used to study colored-noise problems which are based on cumulant expansions.[25-27] These cumulants can be defined either by imposing a differential or an integro-differential equation for the evolution of $\Phi\{q(t),\hat{q}(t)\}$ (Ref. 25). Assuming an integro-differential form it is possible to obtain a closed equation for $P(q,t)$. Following the projection method of Ref. 27 we obtain an integro-differential equation for Φ,

$$\frac{d\Phi}{dt}=\sum_{n=1}^{\infty}\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\cdots\int_{t_0}^{t_{n-1}}dt_n\langle\xi(t)\cdots\xi(t_n)\rangle_T i\hat{q}(t)\cdots i\hat{q}(t_n)g(q(t_n))\Phi\{q(t_n),\hat{q}(t_n)\} , \quad (3.5)$$

where $\langle\xi(t)\cdots\xi(t_n)\rangle_T$ are Terwiel's cumulants defined as

$$\langle\xi(t)\cdots\xi(t_n)\rangle_T=[P\xi(t)(1-P)\cdots(1-P)\xi(t_n)] , \quad (3.6)$$

P being a projection operator that acts similar to averaging over ξ.

Taking the time derivative in (3.3) and substituting the expression of $d\Phi/dt$ given in (3.5) we obtain

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial q}f(q)P(q,t)+\sum_{n=1}^{\infty}\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\cdots\int_{t_0}^{t_{n-1}}dt_n\langle\xi(t)\cdots\xi(t_n)\rangle_T G_T(q,t,\cdots,t_n) , \quad (3.7)$$

where

$$G_T(q,t,\cdots,t_n)=\int D[q(s)]D[\hat{q}(s)]\delta(q-q(t))i\hat{q}(t)g(q(t))\cdots i\hat{q}(t_n)g(q(t_n))$$
$$\times\exp\left[-i\int_{t_0}^{t}ds\,\hat{q}(s)[\dot{q}-f(q)]\right]\Phi\{q(t_n),\hat{q}(t_n)\} . \quad (3.8)$$

Taking into account the formal expression of the deterministic evolution

$$\int_{q_{i+1}}^{q_i} D[q(s)]D[\hat{q}(s)]\exp\left[-i\int_{t_{i+1}}^{t_i} ds\,\hat{q}(s)[\dot{q}(s)-f(q)]\right]\equiv\exp\left[\int_{t_{i+1}}^{t_i}ds\frac{\partial}{\partial q_{i+1}}f(q_{i+1})\right]\delta(q_i-q_{i+1}) \quad (3.9)$$

and operating in (3.8), we obtain

$$G_T(q,t,\cdots,t_n)=\frac{\partial}{\partial q}g(q)\exp\left[\int_{t_1}^{t}ds\frac{\partial}{\partial q}f(q)\right]\cdots\frac{\partial}{\partial q}g(q)\exp\left[\int_{t_n}^{t_{n-1}}ds\frac{\partial}{\partial q}f(q)\right]P(q,t_n) . \quad (3.10)$$

Substituting this expression in (3.7) we obtain a formally exact integro-differential equation for $P(q,t)$ (Ref. 27). Retaining only the second-order cumulant in this expansion we obtain the Bourret approximation.[28] Such approximation becomes exact in the case in which $\xi(t)$ is a dichotomic Markov process. In this case Terwiel's cumulants of order higher than 2 vanish. The Bourret approximation can be advantageously used in time-dependent problems where a convolution term is easily treated. However, this approximation introduces spurious boundary conditions which are usually difficult to handle. This fact makes desirable to find a differential equation for $P(q,t)$. In the limit of small-noise intensity a differential equation can be formally derived from the Bourret equation replacing $P(q,t')$ by

$$\exp\left[-\int_{t'}^{t}ds\frac{\partial}{\partial q}f(q)\right]P(q,t)$$

in the integral part of the equation. The resulting equation is the lowest-order approximation of Van Kampen's expansion[26] (see also Refs. 6, 12, and 29). The complete cumulant expansion of Van Kampen's is difficult to obtain directly from a functional formalism but can be recovered from (3.10) (Ref. 27).

A second alternative to derive evolution equations for the probability density is assuming a differential evolution equation for $\Phi\{q(t),\hat{q}(t)\}$:[25]

$$\frac{d\Phi}{dt}=\sum_{n=1}^{\infty}\int_{t_0}^{t}dt_1\cdots\int_{t_0}^{t_{n-1}}dt_n\langle\xi(t)\cdots\xi(t_n)\rangle_c i\hat{q}(t)g(q(t))\cdots i\hat{q}(t_n)g(q(t_n))\Phi\{q(t),\hat{q}(t)\}\ . \tag{3.11}$$

In this case $\langle\xi(t)\cdots\xi(t_n)\rangle_c$ are ordinary cumulants. Taking as before the time derivative in (3.3) and substituting $d\Phi/dt$ given by (3.11), we obtain an equation similar to (3.7) now with ordinary cumulants instead of Terwiel's cumulants, and G_T substituted by $G_c(q,t,t_1,\ldots,t_n)$, where

$$G_c(q,t,\ldots,t_n)=\int D[q(s)]D[\hat{q}(s)]\delta(q-q(t)i\hat{q}(t)g(q(t))\cdots i\hat{q}(t_n)g(q(t_n))$$
$$\times\exp\left[-i\int_{t_0}^{t}ds\,\hat{q}(s)[\dot{q}-f(q)]\right]\Phi\{q(t),\hat{q}(t)\}\ . \tag{3.12}$$

This last expression in terms of functional derivatives reads

$$G_c(q,t,\ldots,t_n)=\frac{\partial}{\partial q}g(q)\left\langle\frac{\delta}{\delta\xi(t_1)}\cdots\frac{\delta}{\delta\xi(t_n)}\delta(q-q(t,q_0,t_0,\xi(t)))\right\rangle\ . \tag{3.13}$$

The replacement of (3.13) in (3.7) yields an equation obtained several years ago by applying the generalized Novikov formula[24] to the averaged term $\langle\xi(t)\delta(q-q(t,q_0,t_0,\xi))\rangle$ of a stochastic Liouville equation.

A main difference between the treatment based on the differential equation (3.11) and the one based on (3.5) is that in the differential equation treatment the characteristic functional is not decoupled in (3.12). As a consequence a further approximation seems to be necessary in order to obtain a closed equation for $P(q,t)$. This has been extensively studied in the Ornstein-Uhlenbeck noise case where ordinary cumulants of order higher than 2 vanish, so that only the leading term remains,

$$G_c(q,t,t_1)=\frac{\partial}{\partial q}g(q)\left\langle\frac{\delta}{\delta\xi(t_1)}\delta(q-q(t))\right\rangle$$
$$=\frac{\partial}{\partial q}g(q)\frac{\partial}{\partial q}\left\langle\frac{\delta q(t)}{\delta\xi(t_1)}\delta(q-q(t))\right\rangle\ . \tag{3.14}$$

Equations (3.7) with (3.14) were obtained by operator methods in Ref. 30 and by functional methods in Ref. 12. They are the starting point of the small-τ and small-D approximations reviewed in Ref. 6. More recent attempts to obtain a closed equation for $P(q,t)$ taking (3.14) as a starting point are due to Fox[23] and Hänggi.[31] Fox uses an approximation of the response function $\delta q(t)/\delta\xi(t_1)$ for small $(t-t_1)$ which restricts the validity of the equation for small correlation time, much in the same spirit of the small-τ approximation.[6] Hänggi considers the decoupling ansatz

$$\left\langle\frac{\delta q}{\delta\xi(t_1)}\delta(q-q(t))\right\rangle\sim\left\langle\frac{\delta q}{\delta\xi(t_1)}\right\rangle P(q,t)$$

valid for small-noise intensity. This approximation is not restricted by the smallness of the correlation time but it does not give a closed equation since the calculation of $\langle\delta q/\delta\xi(t_1)\rangle$ needs a prior knowledge of $P(q,t)$.

A completely different methodology in order to find approximate Markovian evolution equations for the probability density is to start from the Lagrangian-like function (2.7). From the structure of this Lagrangian we see that the non-Markovian character of the process $q(t)$ reflects itself in the presence of terms involving $\ddot{q}$ as well as in the initial condition terms in (2.6). It is then apparent, in order to have a consistent Markovian approximation to the process, to remove such dependences from the action, instead of making "expansions" in τ. The most drastic of all possible Markovian approximations is to neglect the initial boundary term in (2.6) and to take $\ddot{q}=0$ in (2.7). In the additive case ($g=1$) this leads us immediately to a true Fokker-Planck Lagrangian corresponding to an effective multiplicative white-noise SDE. By comparison with the general structure[17] for the Lagrangian of Fokker-Planck operators it coincides with the adiabatic approximation of Ref. 10. This justifies the dynamical contents of the Fokker-Planck equation proposed by Jung and Hänggi. For the multiplicative case (2.7) and in the approximation in which $\ddot{q}=0$, we are still left with a Lagrangian that has $\dot{q}^4$ contributions. These contributions are not allowed in a Lagrangian associated with a Markovian Fokker-Planck equation.[17] The easiest way to recover a desired Fokker-Planck approximation is to get rid of all powers of $\dot{q}$ larger than quadratic [that means to neglect the term $g'(q)\dot{q}^2$ in (2.7)]. The Fokker-Planck that results in such a case is[32]

$$\partial_t P(q,t)=-\partial_q\left[\left[\frac{(1-\tau g'f/g)f}{1-\tau f'+\tau g'f/g}+D'(q)/2\right]P(q,t)\right]$$
$$+\partial_q^2 D(q)P(q,t)\ ,$$
$$D(q)=Dg(q)^2/(1-\tau f'+\tau g'f/g)^2\ . \tag{3.15}$$

It is worth mentioning that it is not the smallness of τ that leads to a Markov approximation. Even in the case of keeping only terms up to first order in τ in the Lagrangian the problem will remain to be non-Markovian due to the presence of the $\ddot{q}$ contribution. Notwithstanding, if we keep all powers in τ but neglect the terms in $\ddot{q}$, we get independent of the magnitude of τ a Markovian approximation without increasing the complexity of the calculation.

Let us now consider the problem discussed at the end of Sec. II, that is, the inclusion of a white noise in (2.1). We have seen that the inverse of the correlation function (2.26) has now all powers in τ and $\delta^{(n)}(s-s')$. The Lagrangian will contain then arbitrary time derivatives of q. This identifies the difficulty of the nontrivial extension of the adiabatic approximation of Ref. 10 to this case. A Markovian Fokker-Planck approximation can be done in the same sense as before, that is, neglecting in the Lagrangian terms containing time derivatives $q^{(m)}$ with m higher than 1, and also all power $\dot{q}^n$ with n higher than 2. The Lagrangian obtained in this way corresponds to the Fokker-Planck equation (3.15) with $g=1$ and τ and D replaced, respectively, by

$$\tilde{\tau}=\tau[D/(\epsilon+D)]^{1/2} \text{ and } \tilde{D}=D+\epsilon .$$

The fact that the stationary distribution of such an equation is not exact for the linear case $f(q)=-aq$ gives an idea of the limitations of this approximation. It is worth remarking that the Fokker-Planck equation of Ref. 10 is also not exact for a linear model in the original case with $\eta=0$, although it gives the correct stationary distribution.

IV. STATIONARY DISTRIBUTION

In this section we address the question of calculating the stationary distribution $P_{\rm st}(q)$ for the process $q(t)$ starting from the Lagrangian (2.7) and without relying on the use of any approximate equation for the time-dependent probability density $P(q,t)$. For simplicity we restrict ourselves here to the additive noise case $g=1$. We anticipate that our result, obtained in the weak noise limit, is

$$\Phi(q)=-\lim_{D\to 0} D\ln P_{\rm st}(q)=-\int f(q)dq+\tau f^2(q)/2 . \tag{4.1}$$

It is worth recalling several alternative paths which lead to this result. Such previous calculations are based on different Fokker-Planck approximations. The result (4.1) has been tested by numerical calculations in several specific models giving quite accurate results, and ones certainly better than those obtained by other proposed approximate stationary distributions. The novelty of our calculation below is to show that (4.1) can be obtained in a natural way without invoking Markovian or Fokker-Planck approximations and without explicit use of τ-expansions. A first already-known way of obtaining (4.1) is starting from a Fokker-Planck approximation obtained in the first order of the τ expansion scheme.[6,12,30] A stationary solution of this equation is consistently searched in the form

$$P_{\rm st}(q)=P_0(q)[1+\tau P_1(q)] , \tag{4.2}$$

where

$$P_0(q)=N\exp\left[-\frac{1}{D}\int f(q)dq\right] \tag{4.3}$$

is the stationary solution obtained in the white-noise limit $\tau=0$, with N a normalization constant. One obtains a normalized $P_{\rm st}(q)$ with

$$P_1(q)=-\left[f'+\frac{1}{2D}f^2-\langle f'+\frac{1}{2D}f^2\rangle_0\right] , \tag{4.4}$$

where $\langle \cdots \rangle_0$ is the average taken with $P_0(q)$. The problem with (4.2)–(4.4) is that $P_{\rm st}(q)$ might become negative for certain range of values of q. This can be avoided invoking the smallness of τ so that $(1+\tau P_1)\approx\exp(\tau P_1)$. In this way one obtains the weak-noise potential given in (4.1). The *ad hoc* exponentiation involved in the above procedure can be better justified as follows.[33] Defining $\Phi(q)$ as in (4.1) and substituting we find that $\Phi(q)$ obeys the equation

$$(\Phi'-2D\partial_q)[2f+(1+\tau f')\Phi'-2D\tau f'']=0 , \tag{4.5}$$

whose solution gives (4.1). Two additional known ways of obtaining (4.1) are through the formal stationary solutions of the Fokker-Planck approximations of Refs. 23 and 10. Although the two equations describe rather different dynamics both have the same stationary distribution

$$P_{\rm st}(q)=N|1-\tau f'(q)|\exp\{-\Phi(q)/D\} , \tag{4.6}$$

with $\Phi(q)$ given by (4.1). We note that the prefactor in (4.6) can be also obtained both from the solution of (4.5) and from (4.2)–(4.4) if one expands the exponential terms independent of D before taking the limit $D\to 0$ involved in the definition of $\Phi(q)$.

The calculation of Φ that we develop here is based in the idea of a nonequilibrium potential[34–35] defined as in (4.1). It can be calculated by a minimum principle written in terms of the path-integral representation of the stochastic process[35]

$$\Phi(q)=\min\int_{q(-\infty)\in A}^{q(0)=q} dt L_0(q(t),\dot{q}(t),\ddot{q}(t)) . \tag{4.7}$$

L_0 is defined as the singular part of the Lagrangian in the limit $D\to 0$:

$$L_0=\lim_{D\to 0} DL(q(t),\dot{q}(t),\ddot{q}(t)) . \tag{4.8}$$

In our case (2.1), $L_0\equiv DL$. We note that the boundary terms coming from the initial condition in (2.6) do not contribute to $\Phi(q)$. In addition the formula (4.7) is explicitly written for the case in which the deterministic dynamics [Eq. (2.3) with $D=0$] has a single attractor. In this case the action integral is minimized over all the paths starting from the attractor A at $t=-\infty$ and reaching the point q at time $t=0$. In the case of several coexisting attractors, which we do not consider explicitly here, (4.7) has to be generalized by matching the local potentials associated with each attractor and finding the absolute minimum of these local potentials.[35]

The smoothness of the potential Φ for a general case is an open question. For Markovian processes characterized by n-variable Fokker-Planck equations it is known[34] that a potential Φ twice differentiable only exists if the Hamiltonian associated with the Lagrangian is completely integrable. This includes the problems which satisfy detailed balance. However, it is also generally accepted[34,35] that a weak-noise stationary distribution of the form implied by (4.1) remains as a useful ansatz in general nonintegrable cases. We follow here this last point of view for the colored noise problem.

In order to clarify the meaning of (4.7) we first examine the white-noise limit. In this case the Euler-Lagrange equations defining the minimizing path and obtained from (2.1) for $\tau=0$ become

$$\ddot{q}-f'(q)\dot{q}=-[\dot{q}-f(q)]f'(q) \ . \tag{4.9}$$

Two integrals of (4.9) are $\dot{q}=\pm f$. The path satisfying $\dot{q}=f$ corresponds to the deterministic motion and it gives a vanishing action: $\int L(q(t),\dot{q}(t))=0$. In fact, for an initial condition in the attractor such path never leaves the attractor. The appropriate path to be used in (4.7) satisfies $\dot{q}=-f$ and with initial condition in the attractor at $t=-\infty$ reaches a point q at time t. The action integral is now

$$\int_{-\infty}^{0} dt\frac{1}{4D}[\dot{q}-f(q)]^2=-\frac{1}{D}\int_{-\infty}^{0} dt\, \dot{q}f(q) \ , \tag{4.10}$$

which reproduces (4.1) for $\tau=0$ and the exact white-noise stationary distribution (4.3).

In the colored-noise case the path minimizing the action integral associated with (2.7) satisfy the Euler-Lagrange-like equations given by

$$\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}}-\frac{d}{dq}\frac{\partial L}{\partial \dot{q}}+\frac{\partial L}{\partial q}=0 \ . \tag{4.11}$$

In our case they explicitly read

$$\tau^2[\ddddot{q}-3f''(q)\dot{q}\ddot{q}-f'''(q)\dot{q}^3-f'(q)^2\ddot{q} -f''(q)f'(q)\dot{q}]-\ddot{q}+f(q)f'(q)=0 \ . \tag{4.12}$$

We are interested in minimizing paths with fixed initial and final value for q at times $-T_1$ and T_2, respectively, and whatever initial and final values of $\dot{q}$. Two of the four boundary conditions required to solve (4.12) are given by the fixed initial and final values of q. The other two boundary conditions follow from the variational problem[36] for the action S in (2.6) with free boundary values of $\dot{q}$,

$$S=-\int L dt-\frac{\tau}{2D}[\dot{q}-f(q)]^2\Big|_{t=-T_1} \ . \tag{4.13}$$

$\delta S=0$ requires in addition of the Euler-Lagrange equation (4.12) the boundary condition at $t=T_2$

$$\frac{\partial L}{\partial \ddot{q}}\Big|_{t=T_2}=\frac{\tau}{2D}\{\tau\ddot{q}+[1-\tau f'(q)]\dot{q}-f(q)\}\Big|_{t=T_2}=0 \tag{4.14}$$

and at $t=-T_1$

$$\frac{\partial L}{\partial \ddot{q}}\Big|_{t=-T_1}-\frac{\tau}{D}[\dot{q}-f(q)]\Big|_{t=-T_1}=0 \ , \tag{4.15}$$

so that

$$\{\tau\ddot{q}-[1+\tau f'(q)]\dot{q}+f\}\big|_{t=-T_1}=0 \ . \tag{4.16}$$

Note that the boundary term coming from the initial condition modifies the boundary condition at the initial time. It is shown in Appendix C that (4.14) and (4.16) coincide with the boundary conditions obtained from the equation for the minimizing path in the phase-space representation of Ref. 4(a).

A particular enlightening case to understand the contents of (4.7) and (4.12) is the linear problem $f(q)=-aq$, $a>0$, $g(q)=1$ for which the exact stationary distribution is known.[6,12] The solution of (4.12) for this case is

$$q(t)=A_1e^{t/\tau}+A_2e^{-t/\tau}+B_1e^{at}+B_2e^{-at} \tag{4.17}$$

with four undetermined constants A_1, A_2, B_1, and B_2. The choice of a path starting at the attractor $q=0$ at $t=T_1=-\infty$ requires $A_2=B_2=0$. This path reaches at $t=T_2=0$ a point $q=A_1+B_1$ with velocity $\dot{q}=A_1/\tau+aB_1$. Calculating the action integral along this path we find

$$\int_{-\infty}^{0} dtL(q,\dot{q},\ddot{q})=\int_{-\infty}^{0} dt\frac{1}{4D}[\tau\ddot{q}+(1+\tau a)\dot{q}+aq]^2 =\frac{1}{D}(1+a\tau)(aq^2/2+\tau\dot{q}^2/2) \ . \tag{4.18}$$

We not choose the free final value of $\dot{q}$ of the path by the minimum action requirement. This is $\dot{q}=0$ in (4.18) which leads to the exact stationary distribution

$$P_{st}=N\exp(-\{[(1+a\tau)/D](aq^2/2)\}) \ .$$

The condition $\dot{q}=0$ can be in fact obtained from the general requirement (4.14) at $t=0$, while at $t=-\infty$ (4.16) is automatically fulfilled in this case. (See Appendix C.)

The simplicity of the white-noise and the linear colored-noise cases is associated with the fact that the Euler-Lagrange equations are in those cases invariant under time reversal, but the Lagrangian is not. The Euler-Lagrange equations admit as a solution the solution of the deterministic equation of motion and a second "antideterministic" solution. The first gives a vanishing action and the second is the path to be chosen. The difference between the Lagrangian and its invariant part under time reversal is what makes the process dissipative. This difference is a total derivative with respect to time of a function which turns out to coincide with the potential Φ (Ref. 21).

We wish to note that the Euler-Lagrange equation (4.12) has been also obtained in Ref. 4(d) using a different formulation with paths defined on the infinite time interval $(-\infty,\infty)$. As a consequence, no boundary terms appear in the action integral used in Ref. 4(d). This formulation is only valid when the system is in the stationary state at the initial time, in contrast with our formulation where any kind of initial conditions can be considered (see Sec. II).

A solution of (4.12) with (4.14) and (4.16) for a general case is certainly not trivial. Solutions for each particular function $f(q)$ can be attempted. A general solution using τ as an expansion parameter is not always well defined. For example, for the linear case above, an expansion in τ of the minimizing path is at best singular. An alternative to find the solution of (4.12) in the stationary state which permits one to obtain general results is to look for the equation satisfied by Φ starting from its definition (4.1). In the white-noise case it is well known that Φ satisfies the Hamilton-Jacobi equation of the Hamiltonian dynamics associated with the Fokker-Planck Lagrangian.[34] In our case L in (4.1) is not properly speaking a Lagrangian function. However, the variational problem solved by the Hamilton-Jacobi equation can be generalized to Lagrangian-like functions which depend on time derivatives of $q(t)$ of order higher than one.[37] In this generalization one obtains Hamilton-Jacobi like equations for variational problems as the one posed by (4.7). The basic idea is the introduction of generalized conjugate momenta associated with the time derivatives of q. In our case one introduces momenta Π^0 conjugate to q and Π^1 conjugate to $\dot{q}$ as

$$\Pi^0=\frac{\partial L}{\partial \dot{q}}-\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}} , \tag{4.19}$$

$$\Pi^1=\frac{\partial L}{\partial \ddot{q}} , \tag{4.20}$$

and the generalized Hamiltonian becomes

$$\begin{aligned}H(q,\dot{q};\Pi^0,\Pi^1)&=\Pi^0\dot{q}+\Pi^1\ddot{q}-L\\&=\dot{q}\Pi^0+\frac{1}{\tau^2}(\Pi^1)^2-\frac{1}{\tau}[(1-\tau f')\dot{q}-f]\Pi^1 .\end{aligned} \tag{4.21}$$

The Hamiltonian-Jacobi-like equation associated with the variational problem (4.13) is now obtained in the usual way replacing Π^0 by $\partial\overline{\Phi}/\partial q$ and Π^1 by $\partial\overline{\Phi}/\partial\dot{q}$ in the Hamiltonian and equating it to zero. Explicitly we have

$$\dot{q}\frac{\partial\overline{\Phi}}{\partial q}+\frac{1}{\tau^2}\left[\frac{\partial\overline{\Phi}}{\partial\dot{q}}\right]^2-\frac{1}{\tau}[(1-\tau f')\dot{q}-f]\frac{\partial\overline{\Phi}}{\partial\dot{q}}=0 . \tag{4.22}$$

Equation (4.22) is for a function $\overline{\Phi}(q,\dot{q})$. The dependence on $\dot{q}$ comes from the fact that (4.22) is associated with a variational problem in which the final velocity $\dot{q}$ is fixed. As explicitly seen in the linear case we are interested in the $\dot{\Phi}(q,\dot{q})$ at the value $\dot{q}_0$, which makes it minimum

$$\Phi(q)=\overline{\Phi}(q,\dot{q}_0),\quad \left.\frac{\partial\dot{\Phi}}{\partial\dot{q}}\right|_{\dot{q}=\dot{q}_0}=0 . \tag{4.23}$$

As an example we note that indeed (4.17) solves Eq. (4.22). The important point to notice is that the structure of (4.22) identifies that $\dot{q}=0$ is precisely the value $\dot{q}_0$ for which $\partial\overline{\Phi}/\partial\dot{q}=0$. This proves that the minimizing path that leaves an attractor at $t=-\infty$ reaches at $t=0$ the prescribed point q with velocity $\dot{q}=0$. An alternative proof of this fact based on the phase-space representation is given in Appendix C. As a consequence of this fact it seems natural to look for a solution of (4.22) in the form

$$\overline{\Phi}(q,\dot{q})=\Phi(q)+\sum_{n=2}^{\infty}\dot{q}^n\Phi_n(q) . \tag{4.24}$$

Replacing (4.24) in (4.22) we obtain

$$\frac{d\Phi}{dq}=-\frac{2}{\tau}f(q)\Phi_2(q) , \tag{4.25}$$

$$\Phi_2^2-\frac{2}{\tau}(1-\tau f')\Phi_2+\frac{3}{\tau}f\Phi_3=0 , \tag{4.26}$$

$$\frac{1}{\tau^2}\sum_{m=2}^{n}m(n+2-m)\Phi_m\Phi_{n+2-m}-\frac{1}{\tau}(1-\tau f')n\Phi_n+\frac{1}{\tau}f(n+1)\Phi_{n+1}+\Phi'_{n-1}=0 . \tag{4.27}$$

The set of equations (4.25)–(4.27) define the solution of (4.22) around the interesting point $\dot{q}=0$. A natural approximation seems to be a quadratic approximation in (4.24) such that $\overline{\Phi}(q,\dot{q})=\Phi(q)+\dot{q}^2\Phi_2(q)$. In this case we obtain from (4.26)

$$\Phi_2=\frac{\tau}{2}(1-\tau f') , \tag{4.28}$$

which substituted in (4.25) leads to the desired result (4.1). The consistency of the quadratic approximation in (4.24) requires $\Phi_2(q)>0$ so that $\dot{q}=0$ is a minimum. Thus it is fulfilled in cases with a single attractor, which we consider here, with $f'<0$.

At this point it is important to remark that the argument used to obtain the potential Φ in the scheme followed here is the one of weak-noise intensity which leads to the variational problem (4.7), and no small τ or Fokker-Planck approximations have been invoked. Also, τ expansions have not been used in the development. However, the question remains of the validity of the quadratic approximation to (4.24). This has no general answer unless we argue using the only free parameter left in (4.25)–(4.27) which is τ. $\Phi(q)$ is known both for $\tau\to 0$ and $\tau\to\infty$. In the white-noise limit $\Phi=-\int f(q)dq$ so that (4.25) implies that $\Phi_2=\tau/2+O(\tau^2)$. Replacing this in (4.26) and taking into account (4.27) one can generally conclude that $\Phi_3=O(\tau^3)$, so that (4.28) is valid for small τ to order τ^2.

In the opposite limit it is known[12] that $\Phi=\tau f^2/2$ so that $\Phi_2=-\tau^2/2f'$. From (4.26) and (4.27) one then concludes that

$$\Phi_3(q)=\frac{\tau^2}{2f''(3f'-6)}+O(\tau),\quad \tau\gg 1 \tag{4.29}$$

and therefore

$$\Phi_2(q)=-\frac{\tau^2}{2}f'+\frac{\tau}{2}+\frac{3}{2}\frac{\tau f f''}{f'(6f'+12)}+O(\tau^0) . \tag{4.30}$$

The quadratic approximation to (4.24) is therefore valid in both limits $\tau\gg 1$, $\tau\ll 1$ whenever the last term in (4.30) can be neglected. In general it can be understood as a useful interpolation between the two limits which is more justified when the last term in (4.30) is small. Nevertheless, we insist that the above discussion aims to give a general justification to the quadratic approximation relying on the parameter τ, but the problem can be analyzed for each particular $f(q)$ without restoring to considerations on the value of τ. In any case (4.25)–(4.27) give an exact interesting formulation of the problem of calculating the potential Φ.

Our calculation develops itself in the q-configuration space of interest following the general line of reasoning of solving non-Markovian problems without enlarging the space of variables to make it Markovian. It is, however, interesting to digress on the relation of our scheme with a Markovian approach starting from the two-variable Fokker-Planck equation (2.5). The stationary distribution of (2.5) in the weak-noise limit can be found through the Hamilton-Jacobi equation associated with the true Hamiltonian (2.12)

$$\frac{1}{\tau^2}\left[\frac{\partial\psi}{\partial\xi}\right]^2-\frac{\xi}{\tau}\frac{\partial\psi}{\partial\xi}+(f+\xi)\frac{\partial\psi}{\partial q}=0\ , \tag{4.31}$$

where $\psi(q,\xi)$ is now a function of (q,ξ). Equation (4.31) is the starting point of other approaches[38,39] to find the stationary distribution of the colored noise problem. In such approaches $P_{st}(q)=\int d\xi \exp[-\psi(q,\xi)/D]$ and (4.31) is solved for ψ invoking again small τ expansions. Once more, the form (4.1) can be reobtained. The connection with (4.22) is, *a posteriori,* rather simple. If a change of variables from ξ to $q=f(q)+\xi$ is done in (4.31) one recovers (4.22) for $\Phi(q,\dot{q})=\psi[q,\dot{q}-f(q)]$. In this context the stationary distribution should be obtained as $P_{st}(q)=\int d\dot{q}\exp[-\Phi(q,\dot{q})/D]$. The equivalence of this with our recipe (4.23) comes from the asymptotic evaluation of the integral over $\dot{q}$ in the limit $D\to 0$ in which everything has been worked out. It is then clear that the τ-expansion solution of $\psi(q,\xi)$ of Schimansky-Geier[39] is equivalent to a τ-expansion solution of (4.22).

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APPENDIX A

In this appendix we obtain the inverse $\bar{R}$ of $\bar{C}$ [see Eq. (2.18)] defined according to

$$\int_{t_0}^{t} ds'\bar{C}(s,s')\bar{R}(s',s'')=\delta(s-s'')\ . \tag{A1}$$

We recall here that $C(s,s')$ admits the following formal expansion[6,12,30] when $s>s'$:

$$C(s,s')=\frac{D}{\tau}\exp(-|s-s'|/\tau)=2D\sum_{n=0}^{\infty}\tau^n\delta^{(n)}(s-s') \tag{A2}$$

where $\delta^{(n)}$ indicates the nth derivative of the δ function with respect to s'. Here the convention $\int_0^\infty dt\,\delta(t)=\frac{1}{2}$ is used. Taking into account (2.18) and (A2), it is then natural to look for an expansion of $\bar{R}$ in the form

$$\bar{R}(s',s'')=\frac{1}{g(q(s'))g(q/s''))}\sum_{n=0}^{\infty}a_n\delta^{(n)}(s'-s'')\ . \tag{A3}$$

When $t_0<s<t$, splitting the integral in (A1) for $s'<s$ and $s'>s$ and substituting (A2) and (A3) one finds by simple coefficient identification that (A3) fulfills (A1) with $a_n=0$ $(n>2)$ and $A_1=1/2D$; $A_2=-\tau^2/2D$. The remarkable result is that although C has contributions of all orders in τ, the series expansion of its inverse is cut at order τ^2.

When $s=t$ or $s=t_0$, the derivation we have made before does not apply. To take into account these two cases, we must add to $\bar{R}$ some "surface terms." It is easy to see that the correct result is given by

$$\bar{R}(s',s'')=\frac{1}{2Dg(q(s'))g(q(s''))}\{\delta(s'-s'')-\tau^2\delta''(s'-s'')+2\tau[\delta(s'-t_0)\delta(s''-t_0)+\delta(s'-t)\delta(s''-t)]$$
$$+2\tau^2[\delta'(s'-t_0)\delta(s''-t_0)-\delta'(s'-t)\delta(s''-t)]\}\ . \tag{A4}$$

Also, at the end of Sec. II we have discussed the situation in which we add a white-noise contribution to (2.1) as shown in Eq. (2.23). To simplify, we consider the case $g=1$. It is possible to repeat the above analysis in order to get the inverse [in the sense of (A1)] of the new correlation function. In this case the series (A3) is not cut at the second term, and we will obtain contributions to all orders in τ.

APPENDIX B

In this appendix we obtain the transition probability density $P(q,\xi,t/q_0,\xi_0,t_0)$ using the discretized path integral in the prepoint discretization. In the continuous limit expression (2.21) is recovered. This result is important, since this expression has a clear meaning only in this way.

The discretized version of (2.10) is given in the prepoint discretization by

$$P(q,\xi,t/q_0,\xi_0,t_0)=\lim_{\epsilon\to 0}\int\prod_{j=1}^{N}\left[\frac{d\hat{q}_j}{2\pi}dq_j\,e^{i\hat{q}_j\epsilon[(q_j-q_{j-1})/\epsilon-f_{j-1}-\xi_j g_{j-1}]}\right]$$
$$\times\prod_{k=1}^{N}\left[\frac{d\hat{\xi}_k}{2\pi}d\xi_k e^{i\epsilon\hat{\xi}_k[(\xi_k-\xi_{k-1})/\epsilon+\xi_{k-1}/\tau]}e^{-D\epsilon\hat{\xi}_k^2/\tau^2}\right]\delta(q_N-q)\delta(\xi_N-\xi)\ , \tag{B1}$$

where $N\epsilon=t$, and $f_i\equiv f(q_i)$, $g_i\equiv g(q_i)$.

We integrate first over $\hat{q}_j$. This results in the appearance of delta functions

$$\delta([(q_j-q_{j-1})/\epsilon-f_{j-1}-\xi_j g_{j-1}]\epsilon)$$

which makes the integration over ξ_j immediate, yielding the expression

$$P(q,\xi,t/q_0,\xi_0,t_0)=\lim_{\epsilon\to 0}\int\prod_{j=1}^{N}\left[\frac{dq_j}{\epsilon g_{j-1}}\right]\prod_{k=1}^{N}\left[\frac{d\hat{\xi}_k}{2\pi}e^{i\epsilon\hat{\xi}_k[(\eta_k-\eta_{k-1})/\epsilon+\eta_{k-1}/\tau]}e^{-D\epsilon\hat{\xi}_k^2/\tau^2}\right]\delta(q_N-q)\delta(\eta_N-\xi)\ , \tag{B2}$$

where

$$\eta_k=[(q_k-q_{k-1})/\epsilon-f_{k-1}]/g_{k-1},\quad k\geq 1,\quad \eta_0=\xi_0\ . \tag{B3}$$

The Gaussian integral over $\hat{\xi}_k$ results in

$$P(q,\xi,t/q_0,\xi_0,t_0)=\lim_{\epsilon\to 0}\int\prod_{j=1}^{N-1}\left[\frac{dq_j}{g_{j-1}}\right]\left[\frac{\tau^2}{4\pi D\epsilon^3}\right]^{N/2}e^{-\epsilon\tau^2/4D\sum_{K=2}^{N}[(\eta_k-\eta_{k-1})/\epsilon+\eta_{k-1}/\tau]^2}$$
$$\times e^{-\tau^2/4D\epsilon\{[(q_1-q_0)/\epsilon-f_0-\xi_0 g_0(1-\epsilon/\tau)]/g_0\}^2}\delta((q_N-q_{N-1})/\epsilon-f_{N-1}-\xi g_{N-1})\ . \tag{B4}$$

In the limit $\epsilon\to 0$, the last exponential in (B4) tends to a δ function. Therefore we recover (2.21) with

$$D_g[q(s)]=\lim_{\epsilon\to 0}\frac{g_0}{\epsilon}\prod_{j=1}^{N-1}\left[\left[\frac{\tau^2}{4\pi D\epsilon^3}\right]^{1/2}\frac{dq_j}{g_{j-1}}\right]\ . \tag{B5}$$

APPENDIX C

In this appendix we obtain some properties of the path minimizing the action integral. We first show that the path defined by (4.12) with fixed initial and final value for q and boundary conditions (4.14) and (4.16) coincide with the one obtained in the phase-space representation of Ref. 4(a). We also show that when the path starts at an attractor at $t=-\infty$, the final value of $\dot{q}$ is zero and condition (4.16) is automatically fulfilled.

The minimizing path obtained in Ref. 4(a) satisfies the following equations:

$$\dot{q}=f(q)-i\int_{-T_1}^{T_2}dt'\,C(t,t')z(t')\ ,$$
$$\dot{z}=-f'z\ , \tag{C1}$$

where C is the noise correlation function, and the initial and final value for q are fixed, i.e., $q(-T_1)=q_0$, $q(T_2)=q$. From (C1) we have

$$\ddot{q}=f'\dot{q}+(i/\tau)\int_{-T_1}^{t}dt'C(t,t')z(t')$$
$$-(i/\tau)\int_{t}^{T_2}dt'\,C(t,t')z(t')\ , \tag{C2}$$

where the exponential correlation function (2.2) has been used. Differentiating (C2) we get

$$\dddot{q}=f''\dot{q}^2+f'\ddot{q}+(\dot{q}-f)/\tau^2+2izD/\tau^2\ . \tag{C3}$$

The auxiliary variable z can then be expressed as a function of q. Eliminating this variable we get (4.12). The boundary conditions (4.14) and (4.16) are obtained from (C2) by taking $t=T_2$ and $t=-T_1$, respectively. This shows the complete equivalence of both approaches.

We consider now the stationary case, when the path starts at an attractor at $t=-T_1=-\infty$ and reaches the point at $t=T_2=0$. Taking into account that z goes to zero when $t\to-\infty$ in an exponential way,[4(a)] we get from (C1), $\dot{q}(-\infty)=f(q(-\infty))=0$. In the same way we obtain from (C2), $\ddot{q}(-\infty)=0$. Condition (4.16) is then automatically fulfilled.

Finally, we show that in the stationary case, the final value of $\dot{q}$ is zero. Multiplying Eq. (4.12) by $\dot{q}$ and integrating over t in the interval $(-\infty,0)$ we get

$$\left[\tau^2(\dot{q}\,\dddot{q}-\frac{1}{2}\ddot{q}^{\,2}-\dot{q}^{\,3}f''-\frac{1}{2}\dot{q}^{\,2}f'^2)-\frac{1}{2}\dot{q}^{\,2}+\frac{1}{2}f^2\right]\Bigg|_{t=0}=0 \tag{C4}$$

since this expression vanishes at $t=-\infty$. Now, using (4.14) to obtain $\dddot{q}$ we have

$$\dot{q}[\tau^2(\ddot{q}-\dot{q}^{\,2}f''-f'^2\dot{q})+\tau(f'\dot{q}-ff')-\dot{q}+f]|_{t=0}=0\ . \tag{C5}$$

From (C3) and (4.14) it is easy to see that (C5) can be written in the form

$$\dot{q}(2izD)|_{t=0}=0\ . \tag{C6}$$

If $z(0)=0$ we have $z(t)=0$ and we get the deterministic motion. Therefore $\dot{q}(0)=0$. This result is in agreement with that obtained in Sec. IV, where it is shown that $\dot{q}_0=0$ makes the action $\bar{\Phi}(q,\dot{q}_0)$ minimum.

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[1]F. Langouche, D. Roekaerts, and E. Tirapegui, *Functional Integration and Semiclassical Expansions* (Reidel, Dordrecht, 1982), see in particular Chap. VII and references therein.

[2](a) R. Phytian, J. Phys. A **10**, 777 (1977); (b) R. Phytian and W. D. Curtis, *ibid.* **13**, 1575 (1980).

[3]L. Pesquera, M. A. Rodríguez, and E. Santos, Phys. Lett. **94A**, 287 (1983).

[4](a) J. F. Luciani and A. Verga, J. Stat. Phys. **50**, 567 (1988); (b)

A. Förster and A. S. Mikhailov, Phys. Lett. **126A**, 459 (1988); (c) P. Hänggi, Z. Phys. B **75**, 275 (1989); (d) A. J. Bray and A. J. McKane, Phys. Rev. Lett. **62**, 493 (1989).

[5]P. Colet, H. S. Wio, and M. San Miguel, Phys. Rev. A **39**, 6094 (1989).

[6]J. M. Sancho and M. San Miguel, in *Noise in Nonlinear Dynamical Systems,* edited by P. V. E. McClintock and F. Moss (Cambridge University Press, Cambridge, England, 1988), Vol. 1, Chap. 3.

[7](a) E. Peacock-López, B. J. West, and K. Lindenberg, Phys. Rev. A **37**, 3530 (1987); (b) G. P. Tsironis and P. Grigolini, Phys. Rev. Lett. **61**, 7 (1988); F. J. de la Rubia, E. Peacock-López, G. P. Tsironis, K. Lindenberg, L. Ramírez-Piscina, and J. M. Sancho, Phys. Rev. A **38**, 3827 (1988); (c) P. Hänggi, P. Jung, and P. Talkner, Phys. Rev. Lett. **60**, 2804 (1988); C. R. Doering, R. J. Bogley, P. S. Hagan, and C. D. Levermore, *ibid.* **60**, 2805 (1988); (d) R. Fox, Phys. Rev. A **37**, 911 (1988); (e) P. Grigolini, L. Lugiato, R. Mannella, P. V. E. McClintock, M. Merri, and M. Pernigo, *ibid.* **38**, 1966 (1988).

[8](a) T. Leiber, F. Marchesoni, and H. Risken, Phys. Rev. A **38**, 983 (1988); (b) P. Jung and P. Hänggi, Phys. Rev. Lett. **61**, 11 (1988).

[9]V. Altares and G. Nicolis, Phys. Rev. A **37**, 3630 (1988).

[10](a) P. Jung and P. Hänggi, Phys. Rev. A **35**, 4464 (1987); (b) J. Opt. Soc. Am. B **5**, 979 (1988).

[11]J. M. Sancho, M. San Miguel, H. Yamazaki, and T. Kawakubo, Physica **116A**, 560 (1982).

[12]J. M. Sancho, M. San Miguel, S. L. Katz, and J. D. Gunton, Phys. Rev. A **26**, 1589 (1982).

[13]L. Arnold, *Stochastic Differential Equations* (Wiley, New York, 1974).

[14]H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1984).

[15]The Lagrangian generally depends on the time discretization used. Throughout this paper we use the prepoint discretization (see Ref. 1).

[16]H. Haken, Z. Phys. B **24**, 321 (1976); R. Graham, *ibid.* **26**, 281 (1977).

[17]F. Langouche, D. Roekaerts, and E. Tirapegui, Nuovo Cimento **536**, 135 (1979).

[18]C. De Dominicis and L. Peliti, Phys. Rev. B **18**, 353 (1978).

[19]P. Martin, E. Siggia, and H. A. Rose, Phys. Rev. A **8**, 423 (1973); K. Kawasaki, Prog. Theor. Phys. **51**, 1064 (1974); **52**, 84 (1974); **52**, 1527 (1974).

[20]L. Garrido and M. San Miguel, Prog. Theor. Phys. **59**, 40 (1978); **59**, 65 (1978).

[21]D. Lurie, L. Garrido, and M. San Miguel, J. Stat. Phys. **21**, 313 (1979).

[22]Preparation effects in the decay of unstable states driven by colored noise have been recently studied. Analog experiments: (a) M. James, F. Moss, P. Hänggi, and C. Van den Broeck, Phys. Rev. A **38**, 4690 (1988); (b) J. Casademunt, J. I. Jiménez-Aquino, J. M. Sancho, C. J. Lambert, R. Mannella, P. Martano, P. V. E. McClintock, and N. G. Stocks (unpublished). Theoretical analysis: (c) J. M. Sancho and M. San Miguel, Phys. Rev. A **39**, 2722 (1989); (d) J. Casademunt and J. M. Sancho, Phys. Rev. A **39**, 4915 (1989); (e) J. Casademunt, J. I. Jiménez-Aquino, and J. M. Sancho (unpublished).

[23]R. F. Fox, Phys. Rev. A **34**, 4525 (1986); J. Stat. Phys. **46**, 1145 (1987).

[24]P. Hänggi, Z. Phys. B **31**, 407 (1978); P. Hänggi, in *Stochastic Processes Applied to Physics,* edited by L. Pesquera and M. A. Rodriguez (World Scientific, Singapore, 1985), p. 69.

[25]R. Kubo, J. Math. Phys. **4**, 174 (1963).

[26]N. G. Van Kampen, Physica **74**, 215 (1974); Phys. Rep. **24C**, 171 (1976).

[27]R. H. Terwiel, Physica **74**, 248 (1974).

[28]J. M. Sancho, F. Sagués, and M. San Miguel, Phys. Rev. A **33**, 3399 (1986).

[29]M. San Miguel and J. M. Sancho, in *Stochastic Nonlinear Systems,* edited by L. Arnold and R. Lefever (Springer-Verlag, Berlin, 1981).

[30]J. M. Sancho and M. San Miguel, Z. Phys. B **36**, 357 (1980).

[31]P. Hänggi, T. J. Mroczkowski, F. Moss, and P. V. E. McClintock, Phys. Rev. A **32**, 695 (1985).

[32]The FPE (3.15) and the approximation made to obtain it comes out directly using the change of variables that leads from a stochastic differential equation with multiplicative noise to one with additive noise.

[33]A. Schenzle and T. Tél, Phys. Rev. A **32**, 596 (1985).

[34]R. Graham and T. Tél, J. Stat. Phys. **35**, 729 (1984).

[35]R. Graham and T. Tél, Phys. Rev. A **31**, 1109 (1985); **33**, 1322 (1986); **35**, 1328 (1987).

[36]C. Lanczos, *The Variational Principles of Mechanics* (University of Toronto Press, Toronto, 1949).

[37]H. A. Buchdahl, J. Math. Phys. **29**, 1122 (1988).

[38]V. Altares and G. Nicolis, J. Stat. Phys. **46**, 191 (1987).

[39]L. Schimansky-Geier, Phys. Lett. **126A**, 455 (1988).