

## Diffusion theory of electrons in a uniform electric field: Steady-stream analysis

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Diffusion theory of electrons in a drift tube is extended to satisfy approximate energy balance as the electrons migrate radially and axially in a steady, uniform electric field. Previously, it was assumed that the characteristic energy is independent of position in the drift region, leading to the well-known linear diffusion equation for the spatial variation of electron density. This assumption is not made in the present paper. Results for the spatial variation of average energy are close to those obtained from exact solution to the linearized Boltzmann equation. In addition, it is shown that diffusion theory with energy balance gives a relation between characteristic energy and drift-tube geometry which is in close agreement with the empirical relation that has been used for years to analyze steady drift-tube measurements, without invoking the assumption of anisotropic diffusion.

### I. INTRODUCTION

Electrons are said to be in equilibrium with the electric field when the energy imparted to them by a steady, uniform electric field is exactly balanced by energy lost in elastic and inelastic collisions with heavy particles. This definition of the equilibrium state implies steady, uniform values of electron density, average velocity, and average energy, i.e., no temporal or spatial gradients in these quantities. The steady, uniform motion of electrons under equilibrium conditions is customarily described by transport coefficients—mobility  $\mu$  and diffusion coefficient  $D$ —which are characterized by  $E/N$ , the ratio of electric field to gas density. This paper deals with the theory of measurement of the transport coefficient  $D$ , based on the diffusion approximation to the Boltzmann equation governing the energy distribution function of the electrons. Both average velocity and average energy of the electrons are allowed to vary in space. Therefore, the degree to which the equilibrium state defined above obtains is predicted rather than assumed. The reason for this distinction is clarified below.

A fundamental assumption of existing diffusion theory of steady electron migration in a drift tube is that average velocity and average energy of the electrons are independent of position in the region of measurement. This assumption leads to the well-known linear diffusion equation with constant coefficients, which can be solved by standard techniques. This equation, however, does not predict accurately the behavior of electrons in a drift-tube experiment. Modification is required to account for the inescapable fact that average velocity and average energy actually vary in space in a drift-tube experiment. A modification that has been used rather successfully is based on the *assumption* of anisotropic diffusion. A comprehensive review of anisotropic diffusion theory, including that based on the density gradient expansion, plus an excellent historical summary of electron swarms, plus a thorough compilation of measurements of electron transport and rate coefficients is given in Ref. 1.

A different approach to the theoretical description of electron migration in a drift tube is used in this paper. Rather than striving to find an extension of simple diffusion theory with constant average energy, the view is adopted that average energy must obey a conventional conservation equation like that obeyed by electron density. This approach has the advantage that only two additional transport coefficients are needed, and these can be expressed in terms of  $\mu$  and  $D$  when the energy dependence of the momentum-transfer collision frequency  $\nu$  can be expressed as  $\nu(\epsilon) = A_l \epsilon^{(l+1)/2}$ , where  $\epsilon$  is electron energy and  $A_l$  is independent of energy. Furthermore, these two additional transport coefficients have physical interpretations, such as thermal conductivity, etc.

This approach does have a disadvantage; new complexity is introduced by the nonlinear nature of the balance equations which govern the spatial variation of electron density and average energy. However, an approximate solution for density  $n(\mathbf{r})$  and average energy  $\bar{\epsilon}(\mathbf{r})$  can be obtained in the limit of small departure from the equilibrium state, which is defined above as that state with no gradients of any kind. This approximate solution appears to be equivalent to the “lowest-mode solution” of the Boltzmann equation published in 1963.<sup>2</sup>

Based on this approximate solution, it is shown that diffusion theory with energy balance gives results for the spatial variation of density and average energy close to those obtained from exact solution to the Boltzmann equation.<sup>2</sup> In addition, it is shown that diffusion theory with energy balance gives a relation between characteristic energy and drift-tube geometry which is in close agreement with the empirical relation which has been used for years to analyze steady drift-tube measurements, without invoking the assumption of anisotropic diffusion.

### II. THEORY

It is assumed that a steady stream of electrons from a point source located at the origin of a cylindrically symmetric coordinate system with radial coordinate  $\rho$  and

axial coordinate  $z$ , is collected on a plate of infinite radial extent placed perpendicular to the  $z$  axis at a distance  $h$  cm from the point source. Throughout the region between the point source and the collector, there exists parallel to the  $z$  axis a steady, uniform electric field which draws electrons to the collector. The magnitude of this electric field is denoted by  $E$ . It is assumed that the average energy of the electrons in the stream is not large enough to cause significant excitation or ionization of the gas in the drift chamber.

The continuity equation, also called the diffusion equation, for electrons of density  $n$  streaming steadily with current density  $\Gamma$  in an electric field  $\mathbf{E} = -E\mathbf{k}$  is

$$\nabla \cdot \Gamma = 0, \quad \Gamma = -\nabla(nD) - \mu \mathbf{E} n, \quad (1)$$

where  $D$  and  $\mu$  are diffusion and mobility coefficients given by the expressions

$$D = \frac{2e}{3m} \int_0^\infty \frac{\varepsilon^{3/2} f_0}{\nu(\varepsilon)} d\varepsilon, \quad (2)$$

$$\mu = -\frac{2e}{3m} \int_0^\infty \frac{\varepsilon^{3/2}}{\nu(\varepsilon)} \frac{\partial f_0}{\partial \varepsilon} d\varepsilon \quad (3)$$

in which  $m$  is electronic mass,  $e$  is electronic charge,  $\nu(\varepsilon)$  is electron-neutral momentum-transfer collision frequency,  $\varepsilon$  is electron energy, and  $f_0$  is the isotropic part of the electron energy distribution. As emphasized by Allis,<sup>3</sup> the product  $nD$  should appear *under* the gradient operator in Eq. (1) when average energy is space dependent. In this way, thermal effects such as thermal diffusion<sup>4</sup> are accounted for.

The ratio  $D/\mu$  is called characteristic energy  $\varepsilon_k$ .<sup>5</sup> When the collision frequency  $\nu(\varepsilon) = \nu_m$  is independent of energy  $\varepsilon$ , then  $\mu = e/m\nu_m$  and  $D = \mu\Theta$ , where  $\Theta$  is electron temperature expressed in eV. In this special case, the characteristic energy  $\varepsilon_k = \Theta = \frac{2}{3}\bar{\varepsilon}$ , where  $\bar{\varepsilon}$  is average energy of electrons in the stream.

Whereas previous theoretical analyses are based on the assumption that  $\bar{\varepsilon}$  is independent of position, it is assumed in the present work that  $\bar{\varepsilon} = \bar{\varepsilon}(\mathbf{r})$  in order to satisfy approximately the energy balance equation for the streaming electrons. Balance between energy gained by the electrons from the electric field and the energy lost in elastic collisions with neutral ions is expressed by the following energy-balance equation for electrons streaming with heat flow  $\mathbf{H}$ :

$$\nabla \cdot \mathbf{H} = -\Gamma \cdot \mathbf{E} - 2 \frac{m}{M} \nu_m n \bar{\varepsilon}, \quad \mathbf{H} = -\nabla(nG) - B \mathbf{E} n, \quad (4)$$

where  $M$  is atomic mass and it is assumed that  $\bar{\varepsilon} \gg \frac{3}{2}kT_g$ , where  $T_g$  is gas temperature. The quantities  $G$  and  $B$  are additional transport coefficients given by the following expressions:

$$G = \frac{2e}{3m} \int_0^\infty \frac{\varepsilon^{5/2} f_0}{\nu(\varepsilon)} d\varepsilon, \quad (5)$$

$$B = -\frac{2e}{3m} \int_0^\infty \frac{\varepsilon^{5/2}}{\nu(\varepsilon)} \frac{\partial f_0}{\partial \varepsilon} d\varepsilon. \quad (6)$$

When the collision frequency  $\nu(\varepsilon) = \nu_m$  is independent of

energy  $\varepsilon$ , then  $B = \frac{5}{2}D$ ,  $G = B\Theta$ . When these values are substituted in the second of Eqs. 4, the result can be written

$$\mathbf{H} = \frac{5}{2}\Theta\Gamma - \frac{5}{2}Dn\nabla\Theta$$

which is the common form of the heat flow vector  $\mathbf{H}$ . Just as Eq. (1) follows from taking the first moment of Eq. (21), i.e., multiplying by  $x^{1/2}dx$  and integrating over all  $x$ , the corresponding energy-balance equation follows from taking the second moment of Eq. (21), i.e., multiplying by  $x^{3/2}dx$  and integrating over all  $x$ .

In the interest of getting semiquantitative analytic solutions to Eqs. (1) and (4), it is assumed that the latter equation can be approximated by the equation

$$\Gamma \cdot \mathbf{E} = -2 \frac{m}{M} \nu_m n \bar{\varepsilon}. \quad (7)$$

It is suggested below that solutions based on this procedure are equivalent to exact solutions of the Boltzmann equation for the lowest mode found by Parker.<sup>2</sup>

### III. CONSTANT COLLISION FREQUENCY

In general, the momentum-transfer collision frequency  $\nu(\varepsilon)$  is a function of electron energy  $\varepsilon$ . For simplicity, it is customary to express the energy dependence of  $\nu$  by the power law

$$\nu(\varepsilon) = A_l \varepsilon^{(l+1)/2}, \quad (8)$$

where  $A_l$  is independent of  $\varepsilon$ . When  $l = -1$ , then  $\nu(\varepsilon) = \nu_m = \text{constant}$ , and the equations of the previous section take on their simplest form. By the substitutions  $p = n\Theta$ ,  $p_q = n\Theta_q \equiv nD_q/\mu$ ,  $r = (W/D_q)p$ ,  $s = (W/D_q)z$ , where  $W \equiv \mu E$ , and the subscript  $q$  means equilibrium value, Eqs. (1) and (7) become, respectively,

$$0 = \nabla^2 p - \frac{\partial p_q}{\partial s}, \quad 0 = -\frac{\partial p}{\partial s} + p_q - p. \quad (9)$$

Previous diffusion theories of electrons drifting and diffusing in a uniform electric field are based on the assumption that  $p = p_q$ , which is tantamount to ignoring the second of Eqs. (9). Then the first of Eqs. (9) becomes simply

$$0 = \nabla^2 n - \frac{\partial n}{\partial s}. \quad (10)$$

The theory of anisotropic diffusion is based on the assumption that Eq. (10) should be modified to

$$0 = \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial n}{\partial r} \right] + \frac{D_L}{D_q} \frac{\partial^2 n}{\partial s^2} - \frac{\partial n}{\partial s}, \quad (11)$$

where  $D_L$  is called the longitudinal diffusion coefficient.<sup>6</sup> This equation does not appear to be consistent with the spherical harmonic expansion method of solving the Boltzmann equation, as shown below [cf. Eq. (12) for constant collision frequency, and Eq. (39) for collision frequency proportional to  $\varepsilon^{(l+1)/2}$ ].

The form of the diffusion equation when average energy is space dependent and the collision frequency is in-

dependent of electron energy is found by eliminating the variable  $p_q$  from the first of Eqs. (9) by means of the second. The result is

$$0 = \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial p}{\partial r} \right] - \frac{\partial p}{\partial s}, \quad (12)$$

where  $p$  is electron partial pressure. There are two notable differences between this equation and Eq. (11): (1) The natural dependent variable is electron partial pressure, not density; (2) there is no term proportional to  $\partial^2 p / \partial s^2$ .

#### A. Solution

The traditional method of measuring the diffusion coefficient  $D_q$ , or rather the ratio  $D_q/\mu$ , is based on the general result that a beam of electrons streaming steadily from a point source in a uniform electric field spreads in the direction perpendicular to that of the field by an amount that depends on  $D_q$ . The theory of this transverse spreading is examined in this section.

The solution of Eq. (12) for an unbounded gaseous system with a finite plane source of radius  $a$ , such that  $p(\rho, 0) = p_0$ , for  $\rho < a$ , and 0 for  $\rho > a$ , is represented by

$$p(r, s) = p_0 r_a \int_0^\infty J_1(kr_a) J_0(kr) \exp(-k^2 s) dk, \quad (13)$$

where  $r_a \equiv (W/D_q)a$ .<sup>7</sup> By Eq. (7),

$$\Gamma_z(r, a) = 3 \frac{m}{m} \frac{v_m}{E} p(r, s) = \frac{\mu E}{\Theta_q} p(r, s) \quad (14)$$

so that  $p_0$  can be expressed in terms of electron current  $I_0$  from the small disc source by the equation

$$p_0 = \frac{I_0}{e \pi a^2} \frac{\Theta_q}{\mu E}. \quad (15)$$

In the limit  $a \rightarrow 0$ , Eqs. (13) and (15) can be combined to give<sup>7</sup>

$$\begin{aligned} p(r, s) &= \frac{I_0 E}{2 \pi e D_q} \int_0^\infty k J_0(kr) \exp(-k^2 s) dk \\ &= \frac{I_0 E}{2 \pi e D_q} \frac{1}{2s} \exp\left[-\frac{r^2}{4s}\right]. \end{aligned} \quad (16)$$

From Eqs. (9) and (16), the result for spatial variation of electron density is

$$n = \frac{p_q}{\Theta_q} = \frac{1}{\Theta_q} \left[ p + \frac{\partial p}{\partial s} \right] = \frac{p}{\Theta_q} \left[ 1 + \frac{r^2}{4s^2} - \frac{1}{s} \right] \quad (17)$$

and that for characteristic energy  $\Theta$  is

$$\Theta = \frac{p}{n} = \Theta_q \left[ 1 + \frac{r^2}{4s^2} - \frac{1}{s} \right]^{-1}. \quad (18)$$

The current  $I_{bh}$  collected on a disc of radius  $b$  centered on and perpendicular to the  $z$  axis at  $z = h$  is

$$\begin{aligned} I_{bh} &= 2\pi e \int_0^b \Gamma_z(r, s_h) \rho dr \\ &= I_0 \int_0^b \frac{1}{2s_h} \exp\left[-\frac{r^2}{4s_h}\right] r dr \\ &= I_0 \left[ 1 - \exp\left[-\frac{Wb^2}{4D_q h}\right] \right], \end{aligned} \quad (19)$$

where  $s_h \equiv (W/D_q)h$ . Therefore, the ratio  $R$  of current collected by a disc of finite radius  $b$  to the total current from the point source is

$$R = 1 - \exp\left[-\frac{\lambda b^2}{2h}\right] \quad (20)$$

where, by custom,  $\lambda \equiv \mu E / 2D_q \equiv W / 2D_q$ . According to Eq. (20),  $D_q$  or  $D_q/\mu$ , can be measured by measuring the fraction of current collected by a disc of radius  $b$  located a distance  $h$  from the point source.

#### B. Relation to solution of Boltzmann equation

Exact solution of the steady-state, space-dependent Boltzmann equation for electrons streaming from a point source into unbounded space with constant collision frequency was published in 1963.<sup>2</sup> In the dimensionless notation of the present paper, the equation solved in Ref. 2 is

$$\begin{aligned} \frac{\partial}{\partial x} \left[ x^{3/2} \left[ f_0 + \frac{\partial f_0}{\partial x} + \frac{\partial f_0}{\partial s} \right] \right] \\ + x^{3/2} \left[ \frac{\partial}{\partial s} \left[ \frac{\partial f_0}{\partial x} + \frac{\partial f_0}{\partial s} \right] + \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial f_0}{\partial r} \right] \right] = 0, \end{aligned} \quad (21)$$

where  $x \equiv \varepsilon / \Theta_q$ . According to Ref. 2, the lowest-mode solution of Eq. (21), denoted by  $f_0^0$ , is given by

$$f_0^0 = \frac{A}{s-x} \exp\left[-\left[x + \frac{r^2}{4(s-x)}\right]\right] \quad (22)$$

for  $x$  less than  $s$  and by  $f_0^0 = 0$  for  $x$  greater than  $s$ , where  $A$  is a constant determined by the condition  $\int_0^\infty x^{1/2} f_0^0 dx = n$ .

By substitution, it can be shown that Eq. (22) satisfies the following subsidiary equations:

$$f_0^0 + \frac{\partial f_0^0}{\partial x} + \frac{\partial f_0^0}{\partial s} = 0,$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial f_0^0}{\partial r} \right] = \frac{\partial f_0^0}{\partial s}.$$

By inspection, it can be seen that the  $f_0^0$  which satisfies these equations also satisfies Eq. (21). Taking the energy moment of these two equations, i.e., multiplying by  $x^{3/2} dx$  and integrating over all  $x$ , and assuming that  $f_0^0 \approx f_0$ , gives

$$p - p_q + \frac{\partial p}{\partial s} = 0,$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial p}{\partial r} \right] = \frac{\partial p}{\partial s},$$

which are identical to Eqs. (9) above. Therefore, the solutions for  $p(r, s)$  and  $n(r, s)$  given in Sec. III A correspond to the lowest-mode solution of the Boltzmann equation given in Ref. 2.

In the notation of the present paper, results from Ref. 2 for electron density and characteristic energy far from the source are

$$n \propto \frac{1}{s} \left[ 1 + \frac{3}{2s} \right] \left[ 1 + \frac{r^2}{4s^2} \right]^{-3/2} \exp \left[ -\frac{r^2}{4s} \right], \quad (23)$$

$$\Theta = \Theta_q \left[ 1 + \frac{1}{s} \right] \left[ 1 + \frac{r^2}{4s^2} \right]^{-1}. \quad (24)$$

Comparison with Eqs. (17) and (18) shows certain similarities, especially between Eqs. (18) and (24) for  $\Theta$ . However, Eqs. (17) and (18) satisfy Eq. (1) with  $D = \mu\Theta$ , whereas Eqs. (23) and (24) do not.

#### IV. COLLISION FREQUENCY PROPORTIONAL TO $\epsilon^{(l+1)/2}$

This section deals with the extension of the results of the previous section to the case of energy-dependent collision frequency expressible as  $\nu(\epsilon) = A_l \epsilon^{(l+1)/2}$ . For this purpose, it is assumed that isotropic part of the electron energy distribution is given by the relation

$$f_0 = \frac{n(l+2)}{\Theta^{3/2} \Gamma(3/2(l+2))} \exp \left[ -\left[ \frac{\epsilon}{\Theta} \right]^{(l+2)} \right], \quad (25)$$

where  $n = n(\rho, z)$ ,  $\Theta = \Theta(\rho, z)$ , and  $\Gamma(u)$  is the  $\Gamma$  function defined by the equation

$$\Gamma(u) = \int_0^\infty x^{u-1} \exp(-x) dx.$$

Equation (25) is exact in the limit of zero gradients in density and temperature,<sup>8</sup> provided that the quantity  $\Theta$  is defined properly, and is approximately true otherwise. Equation (25) is normalized so that  $\int_0^\infty \epsilon^{1/2} f_0 d\epsilon = n$ . Evidently, average energy  $\bar{\epsilon} \equiv n^{-1} \int_0^\infty \epsilon^{3/2} f_0 d\epsilon$  is related to  $\Theta$  by the equation

$$\bar{\epsilon} = \frac{\Gamma(5/2(l+2))}{\Gamma(3/2(l+2))} \Theta. \quad (26)$$

According to Eqs. (2) and (3),  $D$  and  $\mu$  are given by the equations

$$D = \frac{2}{3} \frac{e}{m A_l} \frac{\Gamma((4-l)/2(l+2))}{\Gamma(3/2(l+2))} \Theta^{(1-l)/2}, \quad (27)$$

$$\mu = \frac{2}{3} \frac{e}{m A_l} (l+2) \frac{\Gamma((l+6)/2(l+2))}{\Gamma(3/2(l+2))} \Theta^{-(l+1)/2}. \quad (28)$$

Therefore, the ratio of  $D$  to  $\mu$  is

$$\frac{D}{\mu} = \frac{\Theta}{l+2} \frac{\Gamma((4-l)/2(l+2))}{\Gamma((l+6)/2(l+2))} \equiv \beta_l \Theta. \quad (29)$$

For completeness, corresponding expressions for  $B$  and

$G$ , calculated according to Eqs. (5) and (6), are given below:

$$G = \frac{2}{3} \frac{e}{m A_l} \frac{\Gamma((6-l)/2(l+2))}{\Gamma(3/2(l+2))} \Theta^{(3-l)/2}, \quad (30)$$

$$B = \frac{2}{3} \frac{e}{m A_l} (l+2) \frac{\Gamma((l+8)/2(l+2))}{\Gamma(3/2(l+2))} \Theta^{(1-l)/2}. \quad (31)$$

Thus, the ratios  $G/\mu$  and  $B/\mu$  are given by the relations

$$\frac{G}{\mu} = \frac{\Theta^2}{l+2} \frac{\Gamma((6-l)/2(l+2))}{\Gamma((l+6)/2(l+2))}, \quad (32)$$

$$\frac{B}{\mu} = \Theta \frac{\Gamma((l+8)/2(l+2))}{\Gamma((l+6)/2(l+2))}. \quad (33)$$

When electrons are in equilibrium with the field, i.e., when there are no gradients, then the energy-balance equation becomes

$$\begin{aligned} \mu_q E^2 &= -2 \frac{m}{M} n^{-1} \int_0^\infty \epsilon \frac{\partial}{\partial \epsilon} [\epsilon^{3/2} \nu(\epsilon) f_0] d\epsilon \\ &= 2 \frac{m}{M} A_l \frac{\Gamma((l+6)/2(l+2))}{\Gamma(3/2(l+2))} \Theta_q^{(1+3)/2}, \end{aligned} \quad (34)$$

where the subscript  $q$  refers to the equilibrium state. According to Eq. (26), this equation can be written

$$\mu_q E^2 = 2 \frac{m}{M} A_l \frac{\Gamma((l+6)/2(l+2))}{\Gamma(5/2(l+2))} \Theta_q^{(l+1)/2} \bar{\epsilon}_q.$$

Likewise,  $\Theta_q$  is related to the field by the equation

$$\Theta_q^{l+2} = \frac{M e E^2}{3 m^2 A_l^2} (l+2).$$

Based on the relations and definitions given above, Eqs. (9) can be written

$$\begin{aligned} 0 &= \nabla^2 \left[ p \left[ \frac{\Theta}{\Theta_q} \right]^{-(l+1)/2} \right] \\ &\quad - \frac{\partial}{\partial s} \left[ p_q \left[ \frac{\Theta}{\Theta_q} \right]^{-(l+1)/2} \right], \end{aligned} \quad (35)$$

$$\begin{aligned} 0 &= -\frac{\partial}{\partial s} \left[ p \left[ \frac{\Theta}{\Theta_q} \right]^{-(l+1)/2} \right] \\ &\quad + p_q \left[ \frac{\Theta}{\Theta_q} \right]^{-(l+1)/2} - p \left[ \frac{\Theta}{\Theta_q} \right]^{(l+1)/2}, \end{aligned} \quad (36)$$

where  $r = (W/D_q)\rho$ , and  $s = (W/D_q)z$ , as before, and  $D = \beta_l \mu \Theta$ . Note that these equations are identical to Eqs. (9) when  $l = -1$  i.e., for a constant collision frequency. By making the substitutions  $Y \equiv \Theta/\Theta_q$  and  $P \equiv p Y^{-(l+1)/2}$ , and noting that  $p_q = p Y^{-1}$ , these equations can be written

$$0 = \nabla^2 P - \frac{\partial}{\partial s} (P Y^{-1}), \quad (37)$$

$$0 = -\frac{\partial P}{\partial s} + P(Y^{-1} - Y^{l+1}).$$

To linearize these equations, it is assumed that  $y = 1 + \Delta Y$ , where  $\Delta Y \ll 1$ . Then the second of Eqs. (37)

can be solved for  $\Delta Y$  to give

$$Y \approx 1 - \frac{1}{l+2} \frac{\partial \ln P}{\partial s}. \quad (38)$$

By Eq. (38), the following relation holds:

$$PY^{-1} \approx P + \frac{1}{l+2} \frac{\partial P}{\partial s},$$

so that the first of Eqs. (37) becomes

$$0 = \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial P}{\partial r} \right] + \frac{l+1}{l+2} \frac{\partial^2 P}{\partial s^2} - \frac{\partial P}{\partial s} \quad (39)$$

which is the same as Eq. (12) when  $l = -1$ . It is shown in the Appendix that the approximate solution of this equation is

$$P(\bar{r}, \bar{s}) = -\frac{1}{2} P_0 \bar{r}_0^2 \exp \left\{ \frac{\bar{s}}{2} \right\} \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[ \frac{\bar{s} \exp[-\frac{1}{2}(\bar{s}^2 + \bar{r}^2)^{1/2}]}{(\bar{s}^2 + \bar{r}^2)^{1/2}} - \frac{(2\bar{s}_h - \bar{s}) \exp(-\frac{1}{2}[(2\bar{s}_h - \bar{s})^2 + \bar{r}^2]^{1/2})}{[2\bar{s}_h - \bar{s})^2 + \bar{r}^2]^{1/2}} \right],$$

where

$$\bar{s} \equiv \frac{l+2}{l+1} s \quad \text{and} \quad \bar{r} \equiv \left[ \frac{l+2}{l+1} \right]^{1/2} r.$$

By Eq. (7) current density is proportional to the quantity

$$PY^{l+1} \approx P - \frac{l+1}{l+2} \frac{\partial P}{\partial s} = P - \frac{\partial P}{\partial \bar{s}},$$

so that the ratio of current collected by a finite disc of radius  $b$  located at  $z = h$ , to the total current from the point source is given by the equation

$$R = 1 - \frac{\bar{s}_h^2 (\bar{s}_h^2 + \bar{r}_b^2)^{1/2} - 2\bar{r}_b^2}{(\bar{s}_h^2 + \bar{r}_b^2)^{3/2}} \times \exp \left\{ -\frac{1}{2} [(\bar{s}_h^2 + \bar{r}_b^2)^{1/2} - \bar{s}_h] \right\}. \quad (40)$$

This equation is approximately the same as Eq. (20) when  $b^2/h^2 \ll 1$ , as occurs in most experiments. Therefore, measurement of  $D_q$  by the steady-state current-ratio method is independent of  $l$  to a very good approximation.

The temptation to conclude that the coefficient of  $\partial^2 P / \partial s^2$  in Eq. (39)—namely, the factor  $(l+1)/(l+2)$ —represents the ratio of longitudinal diffusion coefficient to transverse diffusion coefficient must be avoided, as the case  $l = -1$  for constant collision frequency would have zero longitudinal diffusion coefficient.

## V. DISCUSSION

### A. Constant collision frequency

To the extent that Eq. (7) is valid, the equations derived in Sec. III are exact, and correspond to the lowest-mode solution of the Boltzmann equation for constant collision frequency discussed in Ref. 2. There are two main differences between results derived here and those derived previously from diffusion theory: (1) The natural variable is electron partial pressure, not density; (2) there is no term in the diffusion equation—Eq. (12)—which is proportional to  $\partial^2 P / \partial s^2$ . These two differences change considerably the form of the solution for the steady-state situation, as exemplified by the solution for the current ratio—Eq. (20).

### B. Collision frequency proportional to $\epsilon^{(l+1)/2}$

Due to increased complexity of diffusion theory with energy balance when the collision frequency depends on electron energy, simplification is needed in order to obtain analytic solutions. The simplification used here is based on the assumption that the difference between actual value of average energy of the electrons and the equilibrium value of average energy is small compared with the equilibrium value. This simplification leads to a modified form of the diffusion equation—Eq. (39)—which is approximately valid for energy-dependent collision frequency  $\nu(\epsilon) \propto \epsilon^{(l+1)/2}$ . This modified form of the diffusion equation does not have a term proportional to a quantity which can be readily identified with a longitudinal diffusion coefficient. This modified form does, however, lead to steady-state solutions which agree with observation. The main result—current ratio observed in steady-state experiments—is discussed further below.

### C. Current ratio theory

An empirical equation similar to Eq. (40) has been used for years to interpret measurements of  $D_q$  based on the fraction of current collected on a disc of radius  $b$  located at  $z = h$ .<sup>1</sup> Huxley<sup>9</sup> deduced an equation for  $R$  which is similar to Eq. (40). Huxley's original equation for the current ratio  $R_e$  is

$$R_e = 1 - \frac{h}{d} \exp[-\lambda(d-h)], \quad (41)$$

where  $d \equiv (h^2 + b^2)^{1/2}$ . Because no theory gives this equation, which fits all measurements rather well, Eq. (41) is called the *empirical* result. In words written some 34 years after Huxley's original paper on the subject:<sup>10</sup> "So successful was the formula that no error in it was suspected. With the discovery of the error, therefore, came the disconcerting result that the apparently soundly based formula of equation 11.7 did not lead to a satisfactory interpretation of the experiments, whereas the 'incorrect' formula of equation 11.8 did."<sup>11</sup> In this quote, equation 11.8 refers to Eq. (41) above, and equation 11.7 refers to the corrected version, which is

$$R_h = 1 - \frac{h^2 d - b^2 / \lambda}{d^3} \exp[-\lambda(d-h)]. \quad (42)$$

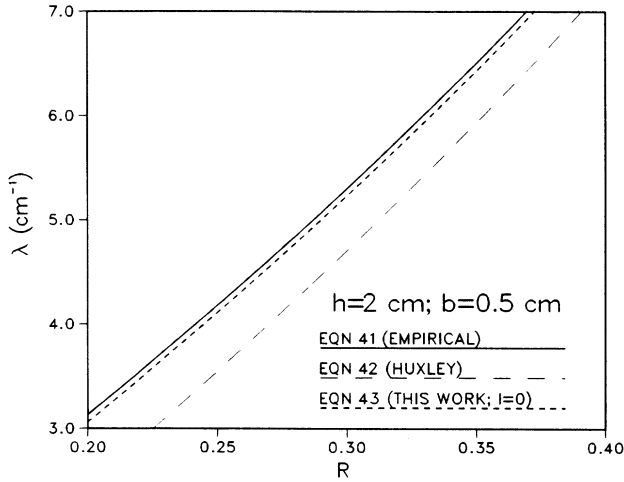


FIG. 1. Comparison of different formulas for the current ratio  $R$ , showing that the formula of the present work is very close to the empirical formula, while the Huxley formula is in error by a significant amount, in the case  $h=2$  cm,  $b=0.5$  cm, and  $l=0$ .

In the same notation, Eq. (40) is

$$R = 1 - \frac{hd_l^2 - f^2b^2/\lambda}{d_l^3} \exp[-\lambda(d_l - h)/f], \quad (43)$$

where  $d_l \equiv \sqrt{h^2 + fb^2}$  and  $f \equiv (l+1)/(l+2)$ . Note that this equation is reduced to Eq. (20) when  $l = -1$ , i.e., for constant collision frequency. Note further that this equation is identical to Eq. (42) when  $l \rightarrow \infty$ , making Eq. (39) identical to Eq. (10), which was solved by others to give Eq. (42).

It has been pointed out that Eqs. (41) and (42) give significantly different values of  $\lambda$  for the same values of  $h$ ,  $b$ , and  $R$  when  $b/h$  is not small.<sup>11,12</sup> For comparison, Eqs. (41), (42), and (43) are shown plotted in Fig. 1 as  $\lambda$  versus  $R$ . In this comparison, values of  $h$  and  $b$  are taken to be 2 and 0.5 cm, respectively, as in the measurements of Refs. 11 and 12, and the value of  $l$  is taken to be 0, corresponding, for example, to helium. This graph shows that Eq. (43) is in close agreement with the empirical results given by Eq. (41), whereas Eq. (42) gives significantly lower values of  $\lambda$  for small values of  $R$ . According to Refs. 11 and 12, the error incurred by using Eq. (42) is larger than the experimental error due to uncertainties in the exact determination of  $h$ ,  $b$ , and  $R$ .

## VI. SUMMARY AND CONCLUSION

Diffusion theory of electrons streaming steadily in a drift tube is extended to satisfy approximate energy balance as the electrons migrate radially and axially in a steady, uniform electric field. It is shown that diffusion theory with energy balance gives results for the spatial variation of density and average energy close to those obtained from exact solution to the Boltzmann equation. In

addition, it is shown that diffusion theory with energy balance gives a relation between characteristic energy and drift-tube geometry which is in close agreement with the empirical relation which has been used for years to analyze steady drift-tube measurements, without invoking the assumption of anisotropic diffusion.

It is concluded that diffusion theory with energy balance is in accord with Boltzmann theory for electrons streaming steadily in a uniform electric field.

## APPENDIX

The solution of Eq. (39) for a gaseous system with a finite plane source of radius  $a$  centered at the origin (0,0) and a plane collector centered at (0, $h$ ), such that  $P(\rho,0)=P_0$  for  $\rho < a$  and 0 for  $\rho > a$ , and  $P(\rho,h)=0$ , is represented by

$$P(\bar{r},\bar{s}) = P_0 \bar{r}_a \exp\left[\frac{\bar{s}}{2}\right] \times \int_0^\infty J_1(k\bar{r}_a) J_0(k\bar{r}) \times \frac{\sinh[-\frac{1}{2}(\bar{s}_h - \bar{s})(1+4k^2)^{1/2}]}{\sinh[-\frac{1}{2}\bar{s}_h(1+4k^2)^{1/2}]} dk,$$

where

$$\bar{s} \equiv \frac{l+2}{l+1} s \quad \text{and} \quad \bar{r} \equiv \left[\frac{l+2}{l+1}\right]^{1/2} r,$$

which becomes

$$P(\bar{r},\bar{s}) = \frac{1}{2} P_0 \bar{r}_a^2 \exp\left[\frac{\bar{s}}{2}\right] \times \int_0^\infty k J_0(k\bar{r}) \times \frac{\sinh[-\frac{1}{2}(\bar{s}_h - \bar{s})(1+4k^2)^{1/2}]}{\sinh[-\frac{1}{2}\bar{s}_h(1+4k^2)^{1/2}]} dk$$

in the limit of small  $\bar{r}_a$ . The fraction

$$\frac{\sinh[-\frac{1}{2}(\bar{s}_h - \bar{s})(1+4k^2)^{1/2}]}{\sinh[-\frac{1}{2}\bar{s}_h(1+4k^2)^{1/2}]}$$

can be written

$$\{\exp[-\frac{1}{2}\bar{s}(1+4k^2)^{1/2}] - \exp[\frac{1}{2}(\bar{s} - 2\bar{s}_h)(1+4k^2)^{1/2}]\} \times \{\exp[-\bar{s}_h(1+4k^2)^{1/2}] + \exp[-2\bar{s}_h(1+4k^2)^{1/2}] + \dots\}.$$

Following others,<sup>1</sup> only the first two terms in this series are retained. Then integration by parts gives

$$\begin{aligned}
 P(\bar{r}, \bar{s}) &= P_0 \bar{r}_a^2 \exp \left[ \frac{\bar{s}}{2} \right] \frac{1}{\bar{r}} \int_0^\infty k^2 J_1(k\bar{r}) dk \frac{\bar{s} \exp[-\frac{1}{2}\bar{s}(1+4k^2)^{1/2}] - (2\bar{s}_h - \bar{s}) \exp[(\frac{1}{2}(\bar{s} - 2\bar{s}_h)(1+4k^2)^{1/2}]}{(1+4k^2)^{1/2}} \\
 &= -P_0 \bar{r}_a^2 \exp \left[ \frac{\bar{s}}{2} \right] \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \int_0^\infty k J_0(k\bar{r}) dk \frac{\bar{s} \exp[-\frac{1}{2}\bar{s}(1+4k^2)^{1/2}] - (2\bar{s}_h - \bar{s}) \exp[(\frac{1}{2}(\bar{s} - 2\bar{s}_h)(1+4k^2)^{1/2}]}{(1+4k^2)^{1/2}} .
 \end{aligned}$$

The integral in this equation can be evaluated<sup>7</sup> to give

$$P(\bar{r}, \bar{s}) = -\frac{1}{2} P_0 \bar{r}_a^2 \exp \left[ \frac{\bar{s}}{2} \right] \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left[ \frac{\bar{s} \exp[-\frac{1}{2}(\bar{s}^2 + \bar{r}^2)^{1/2}]}{(\bar{s}^2 + \bar{r}^2)^{1/2}} - \frac{(2\bar{s}_h - \bar{s}) \exp\{-\frac{1}{2}[(2\bar{s}_h - \bar{s})^2 + \bar{r}^2]^{1/2}\}}{[(2\bar{s}_h - \bar{s})^2 + \bar{r}^2]^{1/2}} \right]$$

which is the desired solution of Eq. (39). Others have proposed similar solutions to equations of the same form as Eq. (39).<sup>9,13</sup>

<sup>1</sup>L. G. H. Huxley and R. W. Crompton, *The Diffusion and Drift of Electrons in Gases* (Wiley, New York, 1974).

<sup>2</sup>J. H. Parker, *Phys. Rev.* **132**, 2096 (1963).

<sup>3</sup>W. P. Allis, in *Motions of Ions and Electrons*, Vol. 21 of *Handbuch der Physik* (Springer, Berlin, 1956), edited by S. Flugge, p. 431.

<sup>4</sup>S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases*, 2nd. ed. (Cambridge University Press, Cambridge, 1960), p. 399.

<sup>5</sup>L. S. Frost and A. V. Phelps, *Phys. Rev.* **127**, 1621 (1962).

<sup>6</sup>L. G. H. Huxley and R. W. Crompton, Ref. 1, Chap. 4.

<sup>7</sup>G. N. Watson, *A Treatise on the Theory of Bessel Functions*

(Cambridge University Press, Cambridge, 1952), 2nd. ed., Chap. XIII.

<sup>8</sup>L. G. H. Huxley and R. W. Crompton, Ref. 1, p. 75.

<sup>9</sup>L. G. H. Huxley, *Philos. Mag.* **30**, 396 (1940).

<sup>10</sup>L. G. H. Huxley and R. W. Crompton, Ref. 1, p. 379.

<sup>11</sup>L. G. H. Huxley and R. W. Crompton, *Proc. Phys. Soc. London, Sect. B* **68**, 381 (1955).

<sup>12</sup>R. W. Crompton and R. L. Jory, *Aust. J. Phys.* **15**, 451 (1962).

<sup>13</sup>J. J. Lowke, in *10th International Conference on Phenomena in Ionized Gases* edited by R. N. Franklin (Donald Parsons, Oxford, 1971), p. 5.