High-energy beam transport in crystal channels

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(Received 19 July 1989)

The transport of high-energy accelerating charged particles in channeling conditions in a crystalline solid is considered by means of a Fokker-Planck model. Multiple scattering on electrons, appropriate to the channeling of positive particles, and radiation damping due to the emission of channeling radiation are also included. Analytic solutions have been obtained for the case of a harmonic channeling potential. Without acceleration, diffusion due to multiple scattering occurs. With acceleration, the adiabatic damping retards this, although the reduction in critical channeling angle with increasing energy eventually competes. For light particles (positrons), the emission of channeling radiation can lead to a steady state. Implications for crystal accelerator schemes are discussed.

I. INTRODUCTION

The phenomenon of channeling[1] of energetic charged particles along axes and planes of high symmetry in crystalline solids has prompted several investigators[2–7] to suggest schemes for accelerating particles in solid matter. The lattice fields are exploited to play the role of the magnetic transport in conventional accelerators. It is in this regard interesting to note that an electric field of 1 V/Å is equivalent to a magnetic field of 30 T. Different schemes have been proposed for obtaining the accelerating gradient. Recent work[2,3] has suggested that gradients from 1 to 100 GeV/cm might be possible. Here, we do not address any particular concept, but consider an idealized generic situation in which the particle is subject to a uniform accelerating field. This could be construed to model part of a synchronous accelerating wave. We wish to understand the limits of possibility by obtaining the basic scaling laws for the particle transport. No scheme is even close to a level of maturity to warrant a more sophisticated treatment. However, interest is stimulated in thinking about acceleration lying beyond the superconducting supercollider (SCC).[8]

Setting the issue of accelerating field sources aside (although it is a formidable one which must be solved—stimulated channeling radiation is interesting but as yet only suggested theoretically[9]), it was recognized that the crucial question for crystal accelerator schemes is that of the multiple scattering of the beam. This is so despite its reduction for positive particles under channeling conditions. For negative particles, channeling increases the multiple scattering over that in the amorphous solid and the application of crystal accelerator schemes to such particles is probably limited. The accelerating gradient has a positive effect as well.[3] The essential question we examine here is the additional effect of the radiation damping due to the emission of channeling radiation. Radiation damping of channeled particles has been considered before and, while early investigations suggested this would occur,[10] subsequent research concluded the radiative damping would always be dominated by the multiple scattering.[11] This result does not apply in the case of an accelerating particle and it seems reasonable to ask if a process analogous to the radiation damping in, for example, a storage ring may take place. We will find this to be so for light particles although the practical consequences are problematic in that it occurs for the most ambitious accelerating gradients. In the absence of acceleration, our results are consistent with the conclusions of Ref. 11.

In the next section, we will describe the basic equation of our model and discuss its solution in general terms. In Sec. III, we will present our results in specific cases. Section IV will comprise a summary of our conclusions.

II. FOKKER-PLANCK TRANSPORT MODEL

We have adopted a one-dimensional Fokker-Planck treatment for the distribution function of the charged particle beam in the crystal accelerator. In the case of planar channeling, the problem is one-dimensional *a priori*. For axial channeling, this describes the evolution of the projected-angle distribution function[12] in a radial plane. All of these are equivalent in an axisymmetric channel. This is justified for particles of very low transverse energy which are confined to the region near the center of the channel; essentially the proper-channeled particles. We will return to this point when we discuss our results in the next section. Furthermore, because we are interested in highly relativistic particles, we make a paraxial approximation, $p_\perp/p_\parallel \ll 1$, $p_z \simeq p \gg 1$, and retain the pitch angle θ only to lowest order: $\theta \simeq p_\perp/p$. With these assumptions, the Fokker-Planck equation becomes

$$0=\frac{1}{c}\frac{\partial f}{\partial t}+\theta\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}+\tilde{E}\frac{\partial f}{\partial p}-\frac{\tilde{U}_x}{p}\frac{\partial f}{\partial \theta}-\beta\frac{\partial f}{\partial p}-\beta_R\frac{\partial(\theta f)}{\partial\theta}-\frac{1}{2}\frac{\partial}{\partial\theta}\left[\frac{\langle\Delta\theta^2\rangle}{\Delta t}\frac{\partial f}{\partial\theta}\right]. \tag{1}$$

Here and throughout, momenta have been expressed in terms of the projectile rest momentum $m_I c$, and energy in units of the rest energy. The first five terms arise from the convective derivative in phase space; the fourth and fifth terms are due to the accelerating field (taken to be in the z direction) and the channel force, respectively. The next term accounts for energy losses such as bremsstrahlung, channeling radiation, or collisional ionization. The remaining terms are due to radiation damping from the emission of channeling radiation and momentum-space diffusion due to multiple scattering. An equation of this kind without the acceleration or energy-loss term has been used to investigate channeling of relativistic particles at small depths in Ref. 11. A boundary-value problem with data independent of time is appropriate, and we therefore look for steady-state solutions, $\partial f/\partial t \equiv 0$.

Analytic solutions can be obtained when the channeling potential is harmonic. In axial channeling, this also implies proper-channeled particles. A weakly anharmonic case has been treated in Ref. 13. The additional phase mixing which occurs would not impact the results we will find here. The potential then takes the form

$$\tilde{U}=\frac{V_0 x^2}{2m_I c^2} \tag{2}$$

and the force

$$\tilde{U}_x=-\frac{V_0 x}{m_I c^2}\equiv -kx \ , \tag{3}$$

where V_0 is the channel well curvature. Its precise value depends on the model one chooses for the ion-atom potential.[1] It is typically on the order of a few times 10^{16} eV/cm^2. An exact number is not very important here either. For this channel force, the radiation damping coefficient β_R takes the form

$$\beta_R=\tfrac{4}{3}r_e k \ , \tag{4}$$

where r_e is the classical electron radius and k is defined in Eq. (3). In the channeling of positive particles, the multiple scattering is dominated by the electrons in the medium. For this case, we use for the momentum-space diffusion coefficient[10]

$$\left\langle \frac{\Delta\theta^2}{\Delta T}\right\rangle=\frac{4\pi}{p^2}r_e^2 N Z_{\rm val}\left[\frac{m_e}{m_I}\right]^2 L_R\equiv\frac{D}{p^2} \ . \tag{5}$$

Logarithmic dependencies on the particle energy are neglected throughout; they are represented by L_R, which is taken to be constant with typical value ≈ 10. The number density of the solid is N and $Z_{\rm val}$ is the number of valence electrons. We are, in general, interested in the case in which the energy losses are dominated by the acceleration and will typically neglect that term in Eq. (1).

We solve Eq. (1) by introducing the characteristics of the first-order operator and then use a result due to Chandrasekhar.[14] The $-\beta_R f$ term can be handled trivially by introducing $\chi=e^{-\beta_R z}f$. This has no effect on the statistical averages over the distribution function. The characteristics are defined by a system of first-order ordinary differential equations that are just the single-particle equations of motion. Parametrizing the solution curves by s, we have

$$\frac{dp}{\alpha}=ds \ , \tag{6}$$

$$\frac{dx}{\theta}=ds \ , \tag{7}$$

$$\frac{d\theta}{-\tilde{U}}=\frac{ds}{p} \ , \tag{8}$$

and

$$dz=ds \ , \tag{9}$$

where $\alpha=\tilde{E}-\beta$. In the general solution of this system of equations, three integration constants appear. Denoting these by ξ,η,ζ, the solution defines a transformation of variables $(z,p,x,\theta)\rightarrow(s,\xi,\eta,\zeta)$. Applying this to the first-order operator of Eq. (1) reduces it to the single partial differential $\partial/\partial s$. The entire equation is cast in the form

$$\frac{\partial\chi}{\partial s}=\frac{1}{2}\left\langle\frac{\Delta\theta^2}{\Delta t}\right\rangle\left[\phi^2(s)\frac{\partial^2\chi}{\partial\eta^2}+2\phi\psi\frac{\partial^2\chi}{\partial\eta\partial\zeta}+\psi^2(s)\frac{\partial^2\chi}{\partial\zeta^2}\right] . \tag{10}$$

With the diffusion coefficient given in Eq. (5) only a function of $p=p(s)$, the Green's-function solution (δ-function initial data) follows immediately from lemma II of Ref. 14:

$$\chi=\frac{1}{2\pi\sqrt{\Delta}}\exp[-(a\eta^2+2h\zeta\eta+b\zeta^2)/2\Delta] \ , \tag{11}$$

where

$$a=2\int_0^s\psi^2(s')ds' \ , \tag{12}$$

$$b=2\int_0^s\phi^2(s')ds' \ , \tag{13}$$

$$h=-2\int_0^s\psi\phi\, ds' \ , \tag{14}$$

and

$$\Delta=ab-h^2 \ . \tag{15}$$

The functions ψ,ϕ depend on the specific problem. The statistical properties of interest such as $\langle x^2\rangle,\langle\theta^2\rangle$ do also. (The angular brackets denote averages over the distribution function.) They are calculated from those of η,ζ. In the next section, we will present the solution for the specific case of a radiation-damped accelerating channeled particle. Some other solutions will be briefly described as well.

III. SOLUTIONS OF THE FOKKER-PLANCK EQUATION

In this section, we will describe the solution of the Fokker-Planck equation, Eq. (1), including radiation damping and acceleration. We are interested in the case in which the acceleration dominates the energy loss β. The equation for p, Eq. (6), decouples and we have

$$p(s)=\alpha s+\xi \ . \tag{16}$$

The remaining pair of equations are equivalent to the second-order equation of motion,

$$\frac{d^2x}{d\bar{s}^2}+\frac{dx}{d\bar{s}}+\frac{\bar{k}x}{\bar{s}}=0 , \tag{17}$$

where

$$\bar{k}=\frac{k}{\alpha\beta_R}, \quad \bar{s}=\beta_R(s+\xi/2) .$$

This equation can be transformed into Whittaker's equation. The solution basis we choose is denoted[15] $M_{\bar{k},1/2}(-\bar{s}), W_{-\bar{k},1/2}(-\bar{s})$. These are linearly independent for $\bar{k}>0$. The solution of the system, Eqs. (6)–(9) becomes

$$z=s , \tag{18}$$

$$p=\alpha s+\xi , \tag{19}$$

$$x=e^{-\bar{s}/2}[\eta M_{\bar{k},1/2}(\bar{s})+\zeta W_{-\bar{k},1/2}(-\bar{s})] , \tag{20}$$

$$\theta=\frac{\beta_R e^{-\bar{s}/2}}{\bar{s}^{1/2}}[\eta M_{-\bar{k}+1/2,0}(\bar{s}) -i(1+\bar{k})\zeta W_{-(\bar{k}+1/2),0}(-\bar{s})] . \tag{21}$$

Attention must be paid to the argument of -1, which we have taken to be π. These define the transformation $(z,p,x,\theta)\rightarrow(s,\xi,\eta,\zeta)$ and the inverse. By taking moments of these, we find the moments of x and θ in terms of moments of ζ and η. Thus

$$\langle x^2\rangle=e^{-\bar{s}}(M^2_{\bar{k},1/2}\langle\eta^2\rangle+2M_{\bar{k},1/2}W_{\bar{k},1/2}\langle\xi^2\rangle) , \tag{22}$$

$$\langle\theta^2\rangle=\frac{\beta_R^3 e^{-\bar{s}}}{\bar{s}}[M^2_{\bar{k},0}\langle\eta^2\rangle-2i(1+\bar{k})M_{\bar{k},0}W_{-(\bar{k}+1/2),0}\langle\zeta\eta\rangle-(1+\bar{k})^2W^2_{-(\bar{k}+1/2),0}\langle\zeta^2\rangle] , \tag{23}$$

$$\langle x\theta\rangle=\frac{\beta_R e^{-s}}{\sqrt{\bar{s}}}\{M_{\bar{k}+1/2,0}W_{\bar{k},1/2}\langle\eta^2\rangle+[W_{-\bar{k}+1/2}M_{\bar{k}+1/2,0}-i(1+\bar{k})M_{\bar{k}+1/2}W_{-(\bar{k}+1/2),0}]\langle\zeta\eta\rangle -i(1+\bar{k})W_{-\bar{k},1/2}W_{-(\bar{k}+1/2),0}\langle\zeta^2\rangle\} \tag{24}$$

and

$$\langle\eta^2\rangle=b=\frac{D}{\beta_R\alpha^2}\Gamma^2(1+\bar{k})\int_{\beta_R\xi/\alpha}^{\bar{s}}du\frac{e^u}{u^2}W^2_{-\bar{k},1/2}(-u) , \tag{25}$$

$$\langle\zeta^2\rangle=a=\frac{D}{\beta_R\alpha^2}\frac{\Gamma^2(1+\bar{k})}{\pi\bar{k}^{3/2}}\int_{\beta_R\xi/\alpha}^{\bar{s}}du\frac{e^u}{u^2}M^2_{-\bar{k},1/2}(-u) , \tag{26}$$

$$\langle\zeta\eta\rangle=-h=\frac{D}{\beta_R\alpha^2}\frac{\Gamma^2(1+\bar{k})e^{\bar{k}}}{\sqrt{2\pi}\,k^{\bar{k}}} \times\int_{\beta_R\xi/\alpha}^{\bar{s}}du\frac{e^u}{u^2}M^2_{-\bar{k},1/2}(-u) . \tag{27}$$

We can obtain explicit analytic expressions for these by noting that typically $\bar{k}\gg1$. The large-parameter asymptotic expansions of the Whittaker's functions can be used. This is essentially the WKB limit of Eq. (17). In more general situations, the asymptotic solution of the differential equation would have to be constructed from scratch. Physically, this reflects the fact that, even at the largest accelerating gradients one might contemplate, the transverse motion in the channel well is adiabatic. The asymptotic expansions of the Whittaker's functions are inserted in Eqs. (21)–(23) and in the integrals in Eqs. (12)–(14). These integrals have been approximately evaluated in two cases: $\beta_R z<1$ and $\beta_R z>1$. The evaluation of the integrals is outlined in an Appendix.

The results we have obtained for the mean-square radius and mean-square angle of scattering are

$$\langle x^2\rangle=\frac{De^{-\beta_R z}}{\alpha k}\left[\left[1+\frac{\alpha z}{p_0}\right]^{1/2}-1\right] \tag{28a}$$

$$\simeq\frac{De^{-\beta_R z}}{2kp_0}z, \quad \alpha z/p_0<1 \tag{28b}$$

$$\langle\theta^2\rangle=\frac{De^{-\beta_R z}}{\alpha p_0}\frac{1}{\left[1+\frac{\alpha z}{p_0}\right]}\left[\left[1+\frac{\alpha z}{p_0}\right]^{1/2}-1\right] \tag{29a}$$

$$\simeq\frac{De^{-\beta_R z}}{2p_0^2}z, \quad \alpha z/p_0<1 \tag{29b}$$

and

$$\langle x^2\rangle=\frac{D}{2\beta_R p_0^2\left[1+\frac{\alpha z}{p_0}\right]^2} \tag{30a}$$

$$\simeq\frac{D}{2\beta_R p_0^2}, \quad \frac{\alpha z}{p_0}\ll1 \tag{30b}$$

$$\simeq\frac{D}{2k\beta_R\alpha z}, \quad \frac{\alpha z}{p_0}\gg1 \tag{30c}$$

$$\langle\theta^2\rangle=\frac{D}{2\beta_R p_0^2\left[1+\frac{\alpha z}{p_0}\right]^2} \tag{31a}$$

$$\simeq\frac{D}{2\beta_R p_0^2}, \quad \frac{\alpha z}{p_0}\ll1 \tag{31b}$$

$$\simeq \frac{D}{2\beta_R \alpha^2 z^2}, \quad \frac{\alpha z}{p_0} \gg 1 \tag{31c}$$

for the cases $\beta_R z < 1$ and $\beta_R z > 1$, respectively. The initial momentum is p_0. To the order of approximation retained, the correlation, $\langle x\theta \rangle$, vanishes in both cases. The factor $e^{-\beta_R z}$ which appears in the results for $\beta_R z < 1$ has been displayed as it stands. One must be careful to note that this does not imply these moments decay exponentially in z. The expressions are only valid to $O(\beta_R z)$ and we are only entitled to retain terms to this order in the small-argument expansion of the exponential; the exponential factor is really a "shorthand" for $(1-\beta_R z)$. This is underscored by the following observation: when $\alpha/\beta_R p_0 \ll 1$, $\alpha z/p_0$ can be less than 1 in both of the cases $\beta_R z < 1$ and $\beta_R z > 1$. Then, the moments in the two cases pass continuously into each other for $\beta_R z \approx 1$ provided $(1-\beta_R z + \ldots) \rightarrow 1$, where the ellipsis represents higher-order terms.

We see then, for small z, the mean-square radius and mean-square angle of scatter increase with z. This is understandable. The solution we have obtained is the Green's function; the solution for a δ-function boundary state for both x and θ. In order for there to be any radiative effects, the particles first must pick up some transverse energy by scattering. The solution for an arbitrarily specified boundary distribution can be constructed from the Green's function in the usual way. However, we can, from our results, infer the behavior of the statistical properties of such a distribution without doing this.

In considering the qualitative behavior of a channeled beam, it is helpful to consider two cases $\alpha/\beta_R p_0 < 1$ and $\alpha/\beta_R p_0 > 1$. We refer to these as the radiation-dominated and acceleration-dominated cases, respectively. In either case, the injected beam is assumed to be sufficiently cool that the rms angle of scatter is much smaller than ψ_c, where ψ_c is the critical angle for channeling.[1,16] In our notation, $\psi_c = a_c(2k/p)^{1/2}$, where a_c is the effective channel size. In the radiation-dominated case, a very cold beam will heat until $z \approx \beta_R^{-1}$, when the rms angle reaches its maximum $\langle \theta^2 \rangle \approx D/2p_0^2\beta_R$. Equating this value of the rms angle to ψ_c defines a critical initial momentum:

$$p_c = \frac{D}{4\beta_R a_c k} . \tag{32}$$

A beam whose initial momentum exceeds p_c and whose rms angle is smaller than ψ_c will propagate with little loss of flux by virtue of the radiation damping and acceleration. Here the rms angle of scatter diminishes with distance faster than does ψ_c, which is decreasing due to the acceleration. Saturation is reached when the channeling radiation in a unit distance carries off as much energy as the accelerating gradient is providing. The channeling radiation rate is a strong function of the particle momentum,[17] proportional to p^2. Altogether, the constraints imposed on the radiation-dominated regime are severe. For example, for positrons in Si, using Eqs. (4), (5), and (32), we find the critical momentum is approximately 250 GeV/c. At this energy, the particle radiates about 5% of its energy per centimeter. Accelerating gradients which exceed this are inconsistent with the radiation-dominated regime, $\alpha/\beta_R p_0 < 1$. However, it is important to recall that α is the net gradient. We have been considering the situation in which the accelerating gradient dominates losses. We can consider the case in which the accelerating gradient just balances the radiative energy loss, storage "ring" fashion. Then $\alpha = 0$ and Eqs. (28)–(30) are valid in this limit. The results just reflect the radiation-dominated case we have been discussing. This is a steady-state configuration with the heating due to scattering balanced by the radiation damping and the energy loss restored by the accelerating gradient. This might be interesting in applications to γ-ray sources or QED effects in solids.[18]

In the acceleration-dominated case, the rms scattering angle initially increases with distance just as in the radiation-dominated case. Here, however, it can reach its extreme value before the radiation has had time to cool the beam. This is further complicated by the reduction in ψ_c as the particle momentum increases during the acceleration. We would like to get into the radiatively cooled regime before the rms angle of scatter exceeds ψ_c. From Eq. (28a) and the definition of ψ_c, we have

$$\frac{\psi_c^2}{\langle \theta^2 \rangle} = \frac{2a_c^2 k\alpha}{D} \frac{1}{(1+\alpha z/p_0)^{1/2}-1} . \tag{33}$$

When $\alpha z/p_0 \gg 1$, this is a decreasing function of z and eventually all the beam particles will dechannel. We will return to a discussion of this situation later. To obtain a criterion that the particles do not appreciably dechannel before they enter the radiative regime, we put $z = \beta_R^{-1}$ in Eq. (33) and equate it to unity. This defines a threshold initial momentum for a given gradient, α:

$$p_T = \frac{\alpha}{\beta_R} \frac{1}{\left[\left[1 + \frac{2a_c^2 k\alpha}{D} \right]^2 - 1 \right]} . \tag{34}$$

Because we also require for consistency $\alpha/\beta_R p_T > 1$, this implies $2a_0^2 k\alpha/D > 0.4$. This sets a lower bound on the accelerating gradient. When these conditions are satisfied, the beam behavior essentially will be as in the radiation-dominated case discussed earlier. Because of the strong dependence of β_R on the particle mass, satisfying these conditions for particles heavier than the electron is beyond contemplation; for muons, for example, the threshold momentum is approximately 8 PeV/c.

For heavy particles, the radiation damping is negligible. If $\beta_R = 0$ in Eq. (29), we recover results obtained previously by direct calculation.[19] (The solution can be expressed in terms of Bessel functions.) In this case, for $\alpha z/p_0 > 1$, we have

$$\frac{\langle \theta^2 \rangle}{\psi_c^2} = \frac{D}{2ka_c^2\alpha}(\alpha z/p_0)^{1/2} . \tag{35}$$

This scaling is more pessimistic than that of Ref. 3. As αz in this approximation is the final momentum, this defined a maximum momentum in terms of the accelerating gradient and channel properties:

$$\frac{p_{\max}}{p_0}=\left[\frac{2ka_c^2\alpha}{D}\right]^2 . \tag{36}$$

Evaluating this for protons at an initial momentum of 20 TeV/c, an accelerating gradient of 100 GeV/cm, and with $k \simeq 3$ eV/Å^2, a reasonable value for proper-channeled particle, we have $p_f \simeq 2000$ TeV/c. It would take about 200 m of crystal to do this.

Insofar as the mean-square radius, Eq. (28) and Eq. (30), is concerned, when its value becomes comparable to or greater than the radius at which the channel potential deviates significantly from a harmonic well, the theory is, strictly speaking, no longer valid. However, as the radius of a proper-channeled particle increases due to scattering, it remains channeled until its rms angle exceeds ψ_c although it may have crossed into an adjacent channel $\langle x^2\rangle^{1/2} > a_c$. Therefore, it might be expected that the expressions we have obtained for the mean-square scattering angle are relatively insensitive to the channeling potential shape. There is some evidence to support this contention. If we put $\beta_R=0$ in Eq. (29) and take $\tilde{E}=0$ as well, we obtain the mean-square scattering angle in the case of no acceleration but energy-loss consistent with that for a relativistic heavy particle. We find, for 15-GeV/c protons channeling in 4 mm of Ge, an rms angle of about 0.03 mrad. This is smaller than observed values of about 0.1 mrad.[20] The discrepancy may be due to the neglect of nuclear scattering. The data on the scattering under channeling conditions in cooled crystals in Ref. 20 appear to support this. For heavy particles in strong gradients such that $\alpha z/p_0 > 1$, this issue is not significant. From Eq. (28), we find, for the case considered earlier of 20-TeV/c protons, the final rms radius of 4 Å. This is consistent with the assumptions made in the theory.

IV. SUMMARY AND CONCLUSIONS

In this paper, we have developed a Fokker-Planck transport model of particle transport in a generic crystal accelerator. The principal motivation was to determine if the channeling radiation damping might be sufficient to overcome the multiple scattering under the condition of particle accleration in analogy with the radiative cooling in conventional accelerators or ionization cooling schemes.[21] For light particles the cooling can occur but a radiation barrier is soon reached. Nevertheless, there may be some interesting applications. For heavy particles, the radiation is not important and the transport does not appear to be a problem to energies of perhaps 2000 TeV although substantial crystal accelerating gradients are required. The results are a consequence of the strong multiple scattering and the stringent requirements on the beam emittance imposed by the channeling. Reducing the multiple scattering relaxes constraints; in particular, the acclerating gradient can be reduced proportionally. To this end, we have been considering the properties of some novel materials in order to access their potential in this application. One of these, porous Si, appears promising, but presently little is known about its properties as they pertain to particle channeling. Research to address some of these issues is in progress.[22] Should these prove to satisfactorily channel particles, they may also be of interest in the problem of the production of stimulated channeling radiation as well. Here it has been demonstrated theoretically that channels with radii somewhat larger than the naturally occurring ones are advantageous.

Lastly, we observe that the theory can be extended in a straightforward way to treat the problem of particle transport in curved crystal channeling.[23] We plan to present our results for this in a subsequent paper.

ACKNOWLEDGMENTS

We wish to thank F. R. Huson, W. W. Mackay, S. Ohnuma, B. C. Covington, M. Tigner, and A. Chao for many helpful discussions. This work has been supported by the Houston Area Research Consortium (HARC), Grant No. 89-T55-1, the U.S. Department of Energy, Contract No. DE-FG05-80ET-53088, and the National Science Foundation (NSF), Grant No. 88-11128.

APPENDIX

It is straightforward to show that all of the integrals appearing in a, b, and h can be obtained as a linear combination of

$$I_{\pm}\equiv\int_{s_0}^{s_1} du\frac{e^{u}e^{\pm 4i(\bar{k}u)^{1/2}}}{u^{3/2}} , \tag{A1}$$

$$s_0=\beta_R z/p_0 , \tag{A2}$$

and

$$s_1=\beta_R/\alpha(\alpha z+p_0) . \tag{A3}$$

We want to evaluate this in two limits: $\beta_R z<1$ and $\beta_R z>1$.

For $\beta_R z>1$, we change variables and then integrate by parts:

$$I_{\pm}=4ie^{4\bar{k}}\int_{w_0}^{w_1} dw\frac{e^{-w^2}}{(w+2\bar{k}^{1/2})^2} \tag{A4}$$

$$=4ie^{4\bar{k}}\int_{w_0}^{w_1}\frac{d}{dw}\left[\frac{1}{w+2k^{1/2}}\right]e^{-w^2/dw} , \tag{A5}$$

where

$$w_0=i\sqrt{s_0}\mp 2\bar{k}^{1/2} \tag{A6}$$

and

$$w_1=i\sqrt{s_1}\mp 2\bar{k}^{1/2} \tag{A7}$$

So

$$I_{\pm}=-4ie^{4\bar{k}}\left[\left.\frac{e^{-w^2}}{w\pm 2\bar{k}^{1/2}}\right|_{w_0}^{w_1}+2\int_{w_0}^{w_1}dw\, e^{-w^2}\mp 2\bar{k}^{1/2}\int_{w_0}^{w_1}dw\frac{e^{-w^2}}{w\pm 2\bar{k}^{1/2}}\right] . \tag{A8}$$

The second term is the difference of two error functions and its asymptotic expansion is known. To evaluate the last integral, rewrite the integrand:

$$\frac{e^{-w^2}}{w\pm 2\bar{k}^{1/2}}=-\frac{1}{2w(w\pm 2\bar{k}^{1/2})}\frac{de^{-w^2}}{dw}\ . \tag{A9}$$

Then integrate by parts and repeat this procedure. In this way, a pair of asymptotic expansions of the form

$$\frac{e^{-(\mp 2\bar{k}^{1/2}+i\sqrt{s_1})^2}}{i\sqrt{s_1}}\left[\frac{a_1}{\bar{k}^{1/2}}+\frac{a_2}{\bar{k}}+\cdots\right] \tag{A10}$$

and

$$\frac{e^{-(\mp 2\bar{k}^{1/2}+i\sqrt{s_1})^2}}{i\bar{k}}\left[\frac{b_1}{s_1}+\frac{b_2}{s_2}+\cdots\right] \tag{A11}$$

are obtained. Because of the exponential factor and $\beta_R z \gg 1$, the contribution from the upper end point is dominant. We have

$$\begin{aligned}I_\pm \simeq\ & -4ie^{4\bar{k}}\exp\{-[4\bar{k}\mp 4(\bar{k}s_1)^{1/2}-s_1]\}\\ &\times\left[\frac{1}{i\sqrt{s_1}}-\frac{1}{i\sqrt{s_1}\mp 2\bar{k}}\right.\\ &\left.-\frac{1}{i\sqrt{s_1}}\frac{1}{1\mp i/2(s_1/\bar{k})^{1/2}}\right]\\ &\pm\frac{2\bar{k}^{1/2}e^{4\bar{k}}\exp\{-[4\bar{k}\mp 4(\bar{k}s_1)^{1/2}-s_1]\}}{4\bar{k}s_1}\ .\end{aligned} \tag{A12}$$

The lowest-order terms cancel and the surviving dominant contribution gives

$$I_\pm=\pm\frac{e^{\pm 4(s_1\bar{k})^{1/2}+s_1}}{2\bar{k}^{1/2}s_1}\ . \tag{A13}$$

The next-order terms are smaller by $(s_1/4\bar{k})^{1/2}$, which is just the small parameter in the expansion of the Whittaker's function. Thus to proceed to higher order would require the inclusion of the higher-order terms of the asymptotic expansion of the Whittaker's function.

For small $\beta_R z$, two integrals must be evaluated:

$$I_\pm=\int_{\beta_R p_0/\alpha}^{\beta_R p_0/\alpha+\beta_R z}du\frac{e^u}{u^{3/2}}e^{\pm 4i(\bar{k}u)^{1/2}} \tag{A1$'$}$$

and

$$I_0=\int_{\beta_R p_0/\alpha}^{\beta_R p_0/\alpha+\beta_R z}du\frac{e^u}{u^{3/2}} \tag{A13$'$}$$

$$=e^{\beta_R p_0/\alpha}\int_0^{\beta_R z}du\frac{(1+u+u^2/2+\cdots)}{u^{3/2}} \tag{A14}$$

$$\simeq 2e^{\beta_R p_0/\alpha}\left[\frac{\alpha}{\beta_R p_0}\right]^{1/2}\left[1-\frac{1}{\left[1+\frac{\alpha z}{p_0}\right]^{1/2}}\right]\ . \tag{A15}$$

Similarly

$$I_\pm\simeq e^{\beta_R p_0/\alpha}\int_{\beta_R/\alpha}^{\beta_R p_0/\alpha+\beta_R z}du\frac{e^{\pm 4i(\bar{k}u)^{1/2}}}{u^{3/2}} \tag{A16}$$

$$=2e^{\beta_R p_0/\alpha}\int_{(\beta_R p_0/\alpha)^{1/2}}^{[\beta_R p_0/\alpha(1+\alpha z/k)]^{1/2}}dy\frac{1}{y^2}e^{\pm 4i(\bar{k}y)^{1/2}}\ . \tag{A17}$$

Writing $y^2=-d(1/y)/dy$ and integrating by parts, we have

$$\begin{aligned}I_\pm\simeq\ & 2\left[\frac{\alpha}{\beta_R p_0}\right]^{1/2}e^{\beta_R p_0/\alpha}\left[e^{\pm 4i(\bar{k}\beta_R p_0/\alpha)^{1/2}}-\frac{e^{\pm 4i[\bar{k}\beta_R p_0/\alpha(1+\alpha z/p_0)]^{1/2}}}{1+\alpha z/p_0}\right]\\ &\pm 8i\bar{k}^{1/2}e^{\beta_R p_0/\alpha}\left[E_1\left[\pm 4i\left[\frac{\bar{k}\beta_R p_0}{\alpha}\right]^{1/2}\right]-E_1\left\{\pm 4i\left[\frac{\bar{k}\beta_R p_0}{\alpha}\left[1+\frac{\alpha z}{p_0}\right]\right]^{1/2}\right\}\right],\end{aligned} \tag{A18}$$

where E_1 is the exponential integral. For all cases of interest $\bar{k}\beta_R p_0/\alpha=kp_0/\alpha^2\gg 1$, so these can be expanded further:

$$I_\pm\simeq\frac{\alpha}{16\bar{k}^{1/2}\beta_R p_0}\left[e^{\pm 4i(\bar{k}\beta_R p_0/\alpha)^{1/2}}-\frac{e^{\pm 4i[\bar{k}\beta_R p_0/\alpha(1+\alpha z/p_0)]^{1/2}}}{(1+\alpha z/p_0)}\right]\ . \tag{A19}$$

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