

## Statistical physics of temporal intermittency

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The thermodynamic formalism for dynamical systems is applied to a class of mappings of "laminar-turbulent" temporal intermittency. The corresponding statistical system is shown to be a lattice gas with many-body interactions of clustering type. This one-dimensional system bears a close analogy with the Fisher-Felderhof droplet model of condensation. The abnormal dynamic fluctuations give rise to a phase transition. The critical behaviors, which depend solely on the characteristic exponent  $z$  of the original map, are studied analytically, and a number of unexpected results are obtained. In the pressure-temperature plane, the intermittent state is located on a critical line that separates the chaotic ("turbulent") state from the periodic ("laminar") state. The transition from one phase to the other may be of first order if  $z < 2$ . On the other hand, for  $2 \leq z$ , the "sporadic state" introduced by Gaspard and Wang [Proc. Natl. Acad. Sci. U.S.A. **85**, 4591 (1988)] is existent and corresponds to a codimension-two point on the critical curve.

### I. INTRODUCTION

#### A. Prelude

Dissipative dynamical systems that may display aperiodic behaviors such as deterministic chaos have relatively few excited degrees of freedom, and the asymptotic states appear as attractors in a phase space of finite dimension. On the other hand, their temporal evolution is in a sense irregular and unpredictable, and unfolds indefinitely in time. Therefore, a statistical description is needed to understand the probabilistic behaviors in these systems, on the basis of the ergodic theory.<sup>1</sup> To this end, one wishes to be able to encode the trajectories using the symbolic dynamics,<sup>2</sup> then to consider the probabilistic measure on the space of all possible long trajectories. In this way an analogy may be established with a statistical-mechanical system in one dimension (corresponding to the time axis), and the well-developed theoretical tools of equilibrium thermodynamics and statistical mechanics may be applied for this purpose.<sup>3</sup> Indeed, in the case of the Axiom-A systems, the invariant measures are known to be given as the Gibbs states of the associated statistical-mechanical systems.<sup>3-6</sup> The theory may also be generalizable to spatially extended (spatio-temporal) far-from-equilibrium processes, which would correspond to statistical systems of dimension  $1+d$ , where  $d$  is the spatial dimension of the system.<sup>7</sup> Such a system would not be an isotropic one, however, since it acquires a semi-group character in the particular dimension of time, as a consequence of the irreversible and dissipative nature of the system.

#### B. Posing the problem

In the case of a hyperbolic system, there exists a *finite* Markovian partition of the phase space, and the statistical counterpart has a Hamiltonian with an interaction potential that decreases exponentially with the distance.

Now, it is well known that one-dimensional statistical systems can not exhibit a phase transition unless the interaction potentials decay slowly enough with the distance.<sup>8-10</sup> Consequently, all the thermodynamic functions of a hyperbolic system are analytic, and no phase transition is expected. Much less is known for the nonhyperbolic systems where a thermodynamic phase transition becomes possible; and virtually no example of physical interest is available, for which the construction of the corresponding statistical system has been done.

The present paper is devoted to such a task. We intend to develop a statistical description of temporal intermittency, on the basis of the Sinai-Ruelle-Bowen thermodynamic formalism. We shall be concerned with the intermittent system of Manneville and Pomeau:<sup>11,12</sup>

$$x_{n+1} = f(x_n; z) = x_n + cx_n^z \pmod{1}, \quad (1.1)$$

with  $z > 1$ ,  $c > 0$ . The system is not hyperbolic, because the slope of the map tends to 1 as  $x_n \rightarrow 0$  (Fig. 1).  $x = 0$  is a marginally stable fixed point, and the system (1.1) is at the criticality of a transition from a periodic state to a chaotic one (a universal "intermittent" scenario to chaos).

Our motivations are threefold. First, we would like to determine the physical invariant measure governing the remarkable intermittent dynamics in (1.1), to construct explicitly the Hamiltonian of its statistical counterpart, and to discover the underlying structure of the interactions.

Second, it is well known that this map displays long-range temporal correlation and abnormal fluctuations (cf. Refs. 12-14, also see below). This implies a thermodynamic phase transition in this system, in the sense that its thermodynamic functions, such as the pressure  $P(\beta)$ ,<sup>15</sup> are not analytic. More specifically, we have shown previously<sup>16</sup> that if  $1 < \alpha < 2$ ,

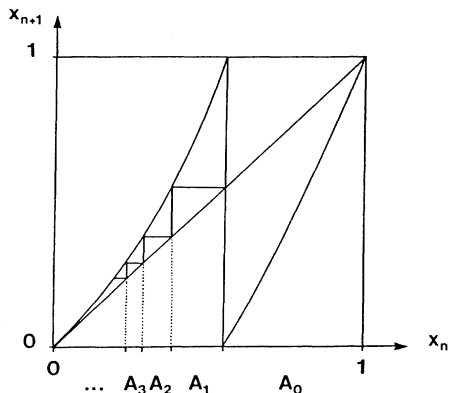


FIG. 1. The Manneville-Pomeau map, with a countable partition  $\{A_k\}$  of the unit interval  $[0,1]$  (from Ref. 17).

$$P(\beta) \simeq h_{KS}(1-\beta) + \text{const} \times (1-\beta)^\alpha \text{ as } \beta \rightarrow 1- , \tag{1.2a}$$

$$P(\beta) = 0, \quad \beta \geq 1$$

and if  $0 < \alpha < 1$ ,

$$P(\beta) \simeq \text{const} \times (1-\beta)^{1/\alpha} \text{ as } \beta \rightarrow 1- , \tag{1.2b}$$

$$P(\beta) = 0, \quad \beta \geq 1$$

where  $\alpha = 1/(z-1)$  and  $h_{KS}$  is the Kolmogorov-Sinai entropy (Fig. 2). One naturally wishes to understand this critical phenomenon at a more microscopic level of statistical mechanics.

Third, it has been revealed by Gaspard and Wang<sup>17</sup> that an intermittent dynamics in (1.1) may or may not possess a positive Liapounov exponent. In the latter case (corresponding to  $2 \leq z$ ), the algorithmic complexity  $K_n$  of Kolmogorov and Chaitin<sup>18,19</sup> increases as

$$K_n \sim n^{\nu_0} (\ln n)^{\nu_1} \text{ with } 0 < \nu_0 < 1 \text{ or } \nu_0 = 1, \nu_1 < 0 , \tag{1.3}$$

which is intermediate between the periodic ( $\nu_0=0$ ) and chaotic ( $\nu_0=1$ ) cases. The dynamic instability is stretched exponential, rather than exponential or a power

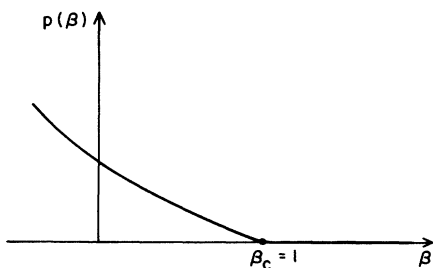


FIG. 2. Schematic drawing of the pressure function  $\mathcal{P}(\beta)$  as expressed by Eq. (1.2). It is convex, and vanishes for all  $1 \leq \beta$ .

law. Such systems are called *sporadic*. It would be of great interest to see to what these intermittent or sporadic states correspond in the related statistical system, and how they compare with the chaotic state and the periodic state (the fixed point  $x=0$ ).

### C. The two Banach spaces $\tilde{B}$ and $B$

It turns out that the Hamiltonian we shall be dealing with has certain curious features and belongs to a marginally known class of interaction potentials, under the notation of  $B \setminus \tilde{B}$ . Let us explain. Two Banach spaces of interaction potentials may be defined regardless of the dimension of the physical space:  $\tilde{B}$  is defined as the space of interaction potentials for which the total interaction energy of one given particle (or spin) with all the others is finite:

$$\sum_{0 \in X} |\Phi(X)| < \infty . \tag{1.4}$$

On the other hand, the space  $B$  is composed of interaction potentials for which the energy per particle (or spin) is finite

$$\sum_{0 \in X} |\Phi(X)|/|X| < \infty . \tag{1.5}$$

Obviously  $\tilde{B} \subset B$ . An example of  $\tilde{B}$  is given by the one-dimensional spin system with pairwise coupling  $J(|i-j|)$ , which exhibits a phase transition if  $J(n) \simeq n^{-\nu}$ , with  $1 < \nu < 2$ .<sup>10</sup> On the other hand, Fisher<sup>8</sup> devised circa 1965 a class of one-dimensional models of condensation (gas-liquid phase transition) that are exactly solvable. Such models have been studied extensively by Fisher and Felderhof.<sup>20</sup> One peculiar property of the Fisher-Felderhof model is that the interaction energy of one particle with the rest of the system may be infinite, hence it belongs to  $B \setminus \tilde{B}$ . This actually was the first example of  $B \setminus \tilde{B}$  from which originated later studies, in particular by Israel,<sup>21</sup> of the Banach space  $B$ . A number of unexpected phenomena have been uncovered for this ultimately large space  $B$ , and for the systems in  $B \setminus \tilde{B}$ . For instance, the pressure of such a system need no longer be continuous as a function of the density,<sup>22,23</sup> a phenomenon called the antiphase transition. Also, it has been proven by Lanford and Ruelle<sup>24</sup> that the systems in  $\tilde{B}$  can not have metastable states. This may no longer be true for  $B \setminus \tilde{B}$ .<sup>25</sup>

As we shall see below, the invariant state of the *non-equilibrium* intermittent processes discussed here turns out to fall into this category  $B \setminus \tilde{B}$ . The Manneville-Pomeau intermittency thus presents a physically observable example (from *nonlinear* systems) for the Fisher-Felderhof-like model. One may ask whether this finding has a wider significance. In this respect, it is worth noting that the interaction potentials of Fisher-Felderhof type have been proven<sup>21</sup> to be dense in the Banach space  $B$ , and even possess certain generic properties in this sense.

### D. Outline of the paper

In order to provide an exactly solvable system, we shall consider a piecewise linear version of the Manneville-

Pomeau map (1.1). We shall see that the corresponding statistical system is a lattice system with many-body clustering interactions. Since the intermittent dynamics is characterized by long trains of "laminar phases" interspersed with "turbulent phases," it is of interest to identify the "laminar" ("turbulent") state with the presence (absence) of a particle on a given lattice site. This naturally leads to the introduction of a "chemical potential" associated to the number of laminar states of a finite lattice of size  $n$  as  $n \rightarrow \infty$ . In this way the thermodynamic formalism is extended to the *grand canonical ensemble*.

The layout of the paper is as follows. In Sec. II we shall present the piecewise linear map, convert our system into a Markovian chain, and review some important properties of this system. In Sec. III the corresponding statistical mechanical system is constructed, and the Hamiltonian is explicitly given. The occurrence of a phase transition is discussed in terms of the nonanalyticity of thermodynamic functions. The critical curve in the pressure-temperature plane is obtained. In Sec. IV the method of Fisher and Felderhof is then applied to clarify the mathematical nature of the phase transition. The critical behavior is studied in detail, and is shown to depend only on the exponent  $z$  of the original mapping. We

shall see that  $z < 2$  and  $z \geq 2$  represent two qualitatively different cases. For  $z < 2$ , the phase transition may be of first order with a plateau portion of the density-pressure diagram (the isotherm), whereas for  $2 \leq z$ , the sporadicity may occur at a single point of the pressure-temperature plane, thus as a codimension-two phenomenon. Some concluding remarks are given in Sec. V.

## II. TEMPORAL INTERMITTENT SYSTEMS

### A. Piecewise linear model

The following piecewise linear model, as an approximation of the Manneville-Pomeau system (1.1), was initially conceived by Gaspard, and first appeared in Ref. 17. Let the value  $a$  be [cf. (1.1)] given by

$$1 = a + ca^z, \quad 0 < a < 1 \quad (2.1)$$

and let us define the interval  $A_0 = [a, 1)$ . Then, the successive preimages of  $A_0$  generate a countable number of intervals  $A_k$ ,  $k = 0, 1, 2, \dots$  (Fig. 1). We suppose that the mapping within each interval is linearized, the resulting system can be easily seen to be (cf. Fig. 3)

$$x_{n+1} = f(x_n; z) = \begin{cases} \frac{\xi_{k-2} - \xi_{k-1}}{\xi_{k-1} - \xi_k} (x_n - \xi_k) + \xi_{k-1} & \text{if } \xi_k \leq x_n < \xi_{k-1} \\ \frac{x_n - a}{1 - a} & \text{if } a \leq x_n < 1, \end{cases} \quad (2.2)$$

with  $\alpha = 1/(z-1)$ , and

$$\xi_k = \frac{a}{(1+k)^\alpha}, \quad k = 1, 2, 3, \dots, \quad \xi_0 = a, \quad \xi_{-1} = 1.$$

The interval length is

$$\Delta_n = \xi_{n-1} - \xi_n \sim \frac{a\alpha}{(1+n)^{\alpha+1}} \quad (2.3)$$

and the slope within each interval is constant,

$$s_n = \frac{\Delta_{n-1}}{\Delta_n} \sim 1 + \frac{\alpha+1}{n}, \quad \lambda_n \equiv \ln s_n \quad (2.4)$$

with  $\Delta_{-1} = 1 - a$ . All these asymptotic features are shared by the original map (1.1), thus one expects that the piecewise linear system (2.2) presents a fairly accurate description of the Manneville-Pomeau system. It is noted, however, that the system (2.2) is equivalent to a countable Markov chain (as we shall see shortly), whereas this latter property has not been proven for the original map (1.1).

### B. Countable Markov chain

The symbolic dynamics associated to the system (2.2) is a Markovian subshift defined on a noncompact space  $\Omega = \{0, 1, 2, \dots\}^{\mathbb{N}}$ ,

$$\sigma: \Omega \rightarrow \Omega, \quad \sigma(\omega_0 \omega_1 \omega_2 \dots) = (\omega_1 \omega_2 \omega_3 \dots) \quad (2.5)$$

with the transition matrix illustrated by the graph of Fig. 4. We shall see that the correspondence is not merely topological, but also in the probabilistic sense. The invariant measure density  $\rho_n$  of the system (2.2) is determined by Perron-Frobenius equation which in the present discrete case takes the form

$$\rho_{n-1} = \rho_n / s_n + \rho_0 / s_0. \quad (2.6)$$

By recurrence relations, and making use of the expressions for  $\Delta_k$  (2.3) and for  $s_k$  (2.4), we can write

$$\begin{aligned} \rho_n &= s_n (\rho_{n-1} - \rho_0 / s_0) = \rho_0 \left[ \prod_{k=0}^n s_k - \sum_{l=1}^n \prod_{k=l}^n s_k \right] / s_0 \\ &= \rho_0 \left[ \frac{1}{\Delta_n} - \sum_{k=0}^{n-1} \frac{\Delta_k}{\Delta_n} \right] / s_0 = \frac{(1-a)\rho_0 \xi_{n-1}}{\Delta_n}, \end{aligned} \quad (2.7)$$

or

$$\rho_n = \frac{(1-a)\rho_0}{1 - [n/(n+1)]^\alpha}, \quad n \geq 1. \quad (2.8)$$

One notices that since this probability density presents discontinuity only on a countable set of points in  $[0, 1]$ , it is absolutely continuous with respect to the Lebesgue measure. Furthermore, the invariant measure is given as

$$\mu(A_n) = \rho_n \Delta_n = \frac{a\mu_0}{n^\alpha}, \tag{2.9a}$$

and for any cylinder  $(\omega_0\omega_1 \cdots \omega_{n-1})$  of size  $n$ ,

$$\mu(\omega_0\omega_1 \cdots \omega_{n-1}) = \mu(A_{\omega_0}) I(\omega_0\omega_1 \cdots \omega_{n-1}), \tag{2.9b}$$

where  $I(\omega_0\omega_1 \cdots \omega_{n-1})$  is the length of the interval coded by  $(\omega_0\omega_1 \cdots \omega_{n-1})$ . The constant  $\mu_0$  in (2.9a) may be specified if the invariant measure is normalizable. Since

$$\sum_{n=0}^{\infty} \mu_n = \mu_0 \left[ 1 + a \sum_{n=1}^{\infty} n^{-\alpha} \right], \tag{2.10}$$

the sum is convergent if  $\alpha > 1$ , or  $z < 2$ , in which case

$$\mu_0 = \frac{1}{1 + a \sum_{n=1}^{\infty} n^{-\alpha}}. \tag{2.11}$$

On the other hand, the invariant measure is not normalizable if  $\alpha \leq 1$ , or  $z \geq 2$ , as is also the case in the original Manneville-Pomeau system. Nevertheless, conditional probabilities are well defined for this kind of stochastic process.<sup>26</sup> The invariant measure is Markovian, as it can be checked by examples, e.g., [cf (2.9b)],

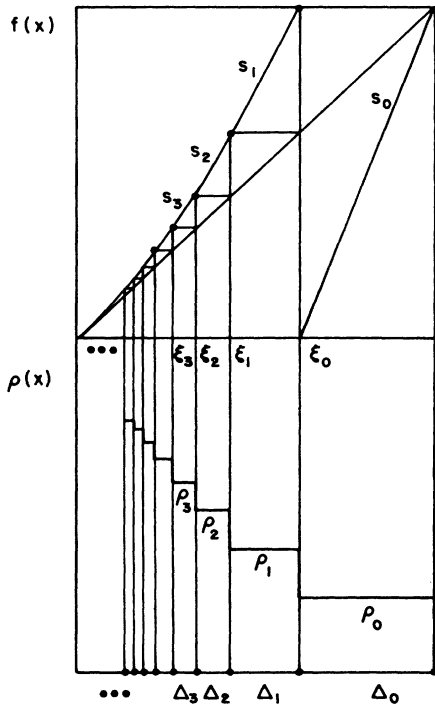


FIG. 3. In the upper part is displayed a piecewise linear version of the Manneville-Pomeau map.  $\Delta_k, k=0, 1, \dots$  are interval lengths of the cells  $\{A_k\}$ , within each the slope  $s_k$  is constant. The lower part of the figure shows the invariant probability density of a staircase form. It diverges at  $x=0$ .

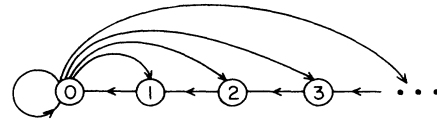


FIG. 4. The graph of our Markov chain with denumerable symbols.

$$\mu(10101) = \mu_1 \mu^2(1|0) = \mu_1 \left[ \frac{\mu(01)}{\mu(0)} \right]^2. \tag{2.12}$$

The transition probability matrix takes the form

$$p_{ij} = \begin{pmatrix} p_{00} & p_{01} & p_{02} & p_{03} & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.13}$$

The fact that  $\mu_n$  is invariant is expressed by the equation

$$\sum_n \mu_n p_{nm} = \mu_m, \tag{2.14}$$

from which it follows that

$$p_{0n} = (\mu_n - \mu_{n+1}) / \mu_0 \equiv \Delta_n, \tag{2.15}$$

with  $\sum_{n=0}^{\infty} p_{0n} = \sum_{n=0}^{\infty} \Delta_n = 1$ . We conclude that the piecewise linear mapping (2.2) is isomorphic to a countable Markov chain with the transition probability given by (2.13) and (2.15).

### C. Non-Gaussian dynamical fluctuations

Let  $(\omega_0\omega_1 \cdots \omega_{n-1})$  be an arbitrary orbit of length  $n$ . We denote by  $N_n$  the number of passages by the state  $A_0$  during  $n$  units of time. If this state is referred to as chaotic (“turbulent”), in contrast to all the other periodic (“laminar”) states, then  $N_n$  is the portion of the time lapse  $n$  during which the signal of the system is chaotic (above the threshold  $a$ ). This is a key quantity to describe the intermittent dynamics.

As discussed in Ref. 17, the probability distribution for  $N_n$  may be obtained using the Feller’s theory of recurrent events.<sup>27</sup> Let  $X$  be the recurrent time for the state  $A_0$ . Then, the probability  $\mathcal{P}(X=n) = p_{0(n-1)}$ , hence its distribution function is

$$F(n) = \sum_{k=0}^{n-1} \Delta_k = 1 - \xi_{n-1} = 1 - an^{-\alpha}. \tag{2.16}$$

Let  $\tau(\sigma^2)$  be the expectation (variance) of the recurrent time  $X$ , then three distinct cases will be identified: (a) if  $\alpha > 2$  ( $1 < z < \frac{3}{2}$ ),  $\tau = \text{finite}$  and  $\sigma^2 = \text{finite}$ ; (b) if  $1 < \alpha < 2$  ( $\frac{3}{2} < z < 2$ ),  $\tau = \text{finite}$  and  $\sigma^2 = \infty$ ; (c) if  $0 < \alpha < 1$  ( $2 < z$ ),  $\tau = \infty$  and  $\sigma^2 = \infty$ .

Let  $G_\alpha(x)$  be the distribution function of the Lévy stable law<sup>28,29</sup> of parameter  $\alpha$ , with  $0 < \alpha \leq 2$ ,  $\alpha=2$  corre-

sponds to Gaussian distribution. For  $\alpha \neq 2$ ,  $G_\alpha(x)$  is defined by the two-sided Laplace transform  $\hat{g}_\alpha(s)$  [ $\text{Re}(s) \geq 0$ ] (Ref. 30)

$$\hat{g}_\alpha(s) = \begin{cases} \exp[-\Gamma(1-\alpha)s^\alpha] & \text{if } \alpha \neq 1, \\ \exp(s \ln s) & \text{if } \alpha = 1. \end{cases} \quad (2.17)$$

The following results can be proved.

(i) If  $1 < z < \frac{3}{2}$ , the fluctuations are Gaussian,

$$\mathcal{P} \left[ N_n \geq \frac{n}{\tau} - x \frac{\sigma}{\tau^{3/2}} n^{1/2} \right] \xrightarrow{n \rightarrow \infty} G_2(x), \quad (2.18a)$$

and

$$E(N_n) \simeq n/\tau, \quad \text{var}(N_n) \simeq \sigma^2 n / \tau^3. \quad (2.18b)$$

In the critical case  $z = \frac{3}{2}$ , the fluctuations are still Gaussian, but

$$E(N_n) \simeq n/\tau, \quad \text{var}(N_n) \simeq n \ln n. \quad (2.19)$$

(ii) If  $\frac{3}{2} < z < 2$  ( $1 < \alpha < 2$ ),

$$\mathcal{P} \left[ N_n \geq \frac{n}{\tau} - x \left( \frac{An}{\tau^{\alpha+1}} \right)^{1/\alpha} \right] \xrightarrow{n \rightarrow \infty} G_\alpha(x), \quad (2.20a)$$

where  $A$  is a constant, and

$$E(N_n) \simeq n/\tau, \quad \text{var}(N_n) \sim n^{3-\alpha}. \quad (2.20b)$$

(iii) If  $z = 2$  ( $\alpha = 1$ ),

$$\mathcal{P}(N_{An(\ln n + x)} \geq n) \xrightarrow{n \rightarrow \infty} G_1(x), \quad (2.21a)$$

and

$$E(N_n) \simeq n/\ln n, \quad (\ln n/n)^2 \text{var}(N_n) \simeq O(1). \quad (2.21b)$$

(iv) If  $2 < z$  ( $0 < \alpha < 1$ ),

$$\mathcal{P} \left[ N_n \geq \frac{n^\alpha}{Ax^\alpha} \right] \xrightarrow{n \rightarrow \infty} G_\alpha(x), \quad (2.22a)$$

and

$$E(N_n) \sim n^\alpha, \quad \text{var}(N_n) \sim n^{2\alpha}. \quad (2.22b)$$

All these results, except (2.21a), may be found in Ref. 17. The proof of the Eq. (2.21a) is given in Appendix A.

#### D. Long-range correlation

Given  $x = (\omega_0 \omega_1 \omega_2 \dots)$ , let us define the characteristic function of the state  $A_0$  as

$$I(x) = \begin{cases} 1 & \text{if } \omega_0 = 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.23)$$

then, obviously

$$N_n(x) = \sum_{k=0}^{n-1} I(f^k(x)). \quad (2.24)$$

We would like to show that the autocorrelation  $\phi(n)$  for this random variable obeys a power law. In fact, the variance of fluctuations  $\text{var}(N_n)$  may be written as<sup>31</sup>

$$\text{var}(N_n) = \int_{-\pi}^{+\pi} \left[ \frac{\sin \frac{\omega n}{2}}{\sin \frac{\omega}{2}} \right]^2 S_\phi(\omega) d\omega, \quad (2.25)$$

where  $S_\phi$  is the Fourier transform of the autocorrelation function  $\phi(n)$ . If

$$\text{var}(N_n) \underset{n \rightarrow \infty}{\sim} n^{2-\nu}, \quad (2.26a)$$

with  $0 < \nu < 2$ ,  $\nu \neq 1$ , it follows from (2.25) that

$$S_\phi(\omega) \underset{\omega \rightarrow 0}{\sim} \omega^{\nu-1}, \quad (2.26b)$$

and the Tauberian theorem<sup>29</sup> implies that

$$\phi(n) \underset{n \rightarrow \infty}{\sim} n^{-\nu}. \quad (2.26c)$$

Therefore, according to the Eqs. (2.20)–(2.22), we have, if  $0 < \alpha < 1$ ,

$$\phi(n) \sim n^{-2(1-\alpha)} \quad \text{with } 0 < \nu = 2(1-\alpha) < 2, \quad (2.27)$$

and if  $1 < \alpha < 2$ ,

$$\phi(n) \sim n^{-(\alpha-1)} \quad \text{with } 0 < \nu = \alpha - 1 < 1. \quad (2.28)$$

For  $2 < \alpha$ ,  $\text{var}(N_n) \sim n$  and the local fluctuations are Gaussian (2.18), the matter is more subtle since Eq. (2.25) can no longer be used to derive the asymptotic form of  $S_\phi$ . However, as we show explicitly in Appendix B, we still have a power law with  $\nu = \alpha - 1$ , which is now larger than the unity. Long-range dynamic correlation is therefore intrinsic to the intermittent system, and is present for all the three cases (a)–(c) listed at the beginning of Sec. II C.

#### E. “Fractal time” (Ref. 32) and the stretched exponential instability

We remark that for  $0 < \alpha \leq 1$  ( $2 \leq z$ ),

$$E(N_n) \sim n^{\nu_0} (\ln n)^{\nu_1} \quad \text{with } 0 < \nu_0 < 1, \text{ or } \nu_0 = 1, \nu_1 < 0. \quad (2.29)$$

In other words, the “turbulent” state occurs on a fractal subset of the time axis with a fractional dimension  $\nu_0 = \alpha < 1$ .

Similar behavior as (2.29) can be found for other important observables, such as the algorithmic complexity of Kolmogorov and Chaitin<sup>18,19</sup> or the logarithm of the separation of nearby orbits,

$$\Lambda_n = \sum_{k=0}^{n-1} \lambda_k. \quad (2.30)$$

As a result, the dynamic instability is stretched exponential,<sup>17,33</sup>

$$\delta x_n \sim \delta x_0 \exp[cn^{\nu_0} (\ln n)^{\nu_1}]. \quad (2.31)$$

This new, *sporadic* dynamic régime is quite different in nature from those cases more familiar to us.

### III. STATISTICAL MECHANICS AND PHASE TRANSITIONS

#### A. Review of the thermodynamic formalism

We shall only recall what will be directly relevant to our present study. Assume that a mapping  $x_{n+1}=f(x_n)$  admits a generating partition  $\{A_i\}$  so that  $x=(\omega_0\omega_1\omega_2\cdots)$  if  $f^{(k)}(x)\in A_{\omega_k}$ ,  $k=0,1,2,\dots$ . According to the thermodynamic formalism,<sup>3,6</sup> given any continuous function, a topological pressure can be accordingly defined either by a variational principle or in terms of the Ruelle-Perron-Frobenius operator theorem.<sup>34</sup> If the particular observable  $-\beta|f'(x)|$  depending linearly on  $\beta$  is chosen, then a thermodynamic pressure function can be expressed as<sup>3,15</sup>

$$P(\beta)=\lim_{n\rightarrow\infty}\frac{1}{n}\ln Z_n(\beta)$$

$$=\lim_{n\rightarrow\infty}\frac{1}{n}\ln\sum_{(\omega_0\cdots\omega_{n-1})}\exp[-\beta U_n(\omega_0\cdots\omega_{n-1})],$$
(3.1a)

where

$$U_n(\omega_0\omega_1\cdots\omega_{n-1})=\inf_{x\in(\omega_0\cdots\omega_{n-1})}\sum_{k=0}^{n-1}\ln|f'(f^{(k)}(x))|$$
(3.1b)

plays the role of a Hamiltonian. Each choice of the value of  $\beta$  induces a particular invariant measure

$$\mu_\beta(\omega_0\cdots\omega_{n-1})\sim\exp[-nP(\beta)-\beta U_n(\omega_0\cdots\omega_{n-1})].$$
(3.2)

The “natural measure” is given by  $\beta=1$  (the reasons for which this measure is especially interesting are explained in Refs. 35 and 36). For attracting states, the Pesin's equality is usually satisfied which implies  $P(\beta=1)=0$ . The parameter  $\beta$ , formally similar to the inverse of a temperature, is associated by a Legendre transform to the spectrum of all possible values of  $\Lambda\sim U_n(\omega_0\omega_1\cdots\omega_{n-1})/n$ , the Liapounov exponent for long (albeit finite) orbits.<sup>15</sup> The pressure function  $P(\beta)$  may be interpreted in terms of the large deviations<sup>37-39</sup> of the Liapounov exponent. The connection of the function  $P(\beta)$  to the generalized entropy  $h(q)$  of Rényi, and the generalized Liapounov exponent  $L(q)$ , is straightforward.<sup>15,40</sup>

Furthermore, for the resulting one-dimensional statistical-mechanical system, the interaction potential may, at least in principle, be deduced as follows.<sup>41</sup> Let

$$h_k(\omega_0\omega_1\cdots\omega_{k-1})=\inf_{x\in I(\omega_0\omega_1\cdots\omega_{k-1})}\ln|f'(x)|,$$
(3.3)

where  $I(\omega_0\omega_1\cdots\omega_{n-1})$  is again the interval of the phase space coded by  $(\omega_0\omega_1\cdots\omega_{n-1})$ . The interaction energy of a sublattice  $(\omega_0\omega_1\cdots\omega_{k-1})$  is

$$\phi_k(\omega_0\omega_1\cdots\omega_{k-1})=h_k(\omega_0\omega_1\cdots\omega_{k-1})$$

$$-h_{k-1}(\omega_0\omega_1\cdots\omega_{k-2})$$
(3.4)

with  $h_{-1}=0$ . The total energy of a configuration  $\omega=(\omega_0\omega_1\omega_2\cdots)$  defined on a finite subset  $\Gamma$  is

$$U_\Gamma(\omega)=\sum_{X\subset\Gamma}\phi_X(\omega|X).$$
(3.5)

If  $\Gamma=\{0,1,2,\dots,n-1\}$ , then

$$U_n(\omega_0\omega_1\cdots\omega_{n-1})=\sum_{k=0}^{n-1}\phi_k(\omega_k)+\sum_{k=0}^{n-2}\phi_{k,k+1}(\omega_k\omega_{k+1})$$

$$+\cdots+\phi_n(\omega_0\omega_1\cdots\omega_{n-1}).$$
(3.6)

By time invariance,  $\phi_k=\phi_0=h_0$ ,  $\phi_{k,k+1}=\phi_{0,1}=h_1-h_0,\dots$ . We obtain

$$U_n(\omega_0\omega_1\cdots\omega_{n-1})$$

$$=\sum_{k=1}^n h_k(\omega_{n-k}\omega_{n-k+1}\cdots\omega_{n-1}).$$
(3.7)

Since Eq. (3.7) is in fact identical to Eq. (3.1b), the definition of  $\phi_k(\omega_0\omega_1\cdots\omega_{k-1})$  is consistent.

#### B. Lattice-gas model of the intermittent system

We shall now apply the aforementioned scheme to our intermittent system (2.2). For convenience, we shall henceforth use a binary coding of the orbits in the system (2.2). This is possible since the binary partition is also a generating one [cf. Fig. 3(a)], although the resulting probabilistic process is no longer Markovian in the metric sense. The construction of the interaction potential is direct in this case, because the slope of the mapping is constant in each interval  $A_k$ ,  $k=0,1,\dots$ . Since

$$h_k(\omega_0\omega_1\cdots\omega_{k-1})=\begin{cases} \lambda_l & \text{if } \omega_0,\omega_1,\dots,\omega_{l-1}=1, \\ \omega_l=0, & l\leq k-1 \\ 0 & \text{otherwise,} \end{cases}$$
(3.8)

we have the following:

- (a)  $\phi_k(\omega_0\omega_1\cdots\omega_{k-1})=0-0=0$   
if  $\omega_0,\omega_1,\dots,\omega_{k-1}=1$ ,
- (b)  $\phi_k(\omega_0\omega_1\cdots\omega_{k-1})=\lambda_l-\lambda_l=0$   
if  $\omega_0,\omega_1,\dots,\omega_{l-1}=1, \omega_l=0, l\leq k-2$ ,
- (c)  $\phi_k(\omega_0\omega_1\cdots\omega_{k-1})=\lambda_{k-1}-0=\lambda_{k-1}$   
if  $\omega_0,\omega_1,\dots,\omega_{k-2}=1, \omega_{k-1}=0$ .

We shall say that  $(\omega_0\omega_1\cdots\omega_{l-1})$  is an isolated cluster of “laminar” states if  $\omega_0,\omega_1,\dots,\omega_{l-2}=1, \omega_{l-1}=0$  (thus ended by a “turbulent” state). Then, the interaction potential is zero if the sublattice under consideration  $(\omega_0\omega_1\cdots\omega_l)$  does *not* belong to a unique cluster; or if all the sites are laminar. On the other hand, for an isolated

cluster the energy is

$$u_l = \sum_{i=0}^{l-1} \lambda_i \sim (\alpha+1) \ln l. \quad (3.9)$$

which may also be rewritten as a sum of pairwise interactions  $\tilde{\phi}_{l,2}(\omega_{l-1-i}=1, \omega_{l-1}=0) = \lambda_i$ ,

$$\phi_l(\omega_0 \omega_1 \cdots \omega_{l-1}) = \sum_{i=0}^{l-1} \tilde{\phi}_{l,2}(\omega_i \omega_{l-1}) = \sum_{i=0}^{l-1} \lambda_i, \quad (3.10)$$

we emphasize that  $\tilde{\phi}_{l,2}$  is  $l$  dependent. It is to be remarked that there is no "bulk" contribution to the energy, since  $u_l$  is essentially the "surface" energy  $W_l$  of an isolated cluster of size  $l$ .

For an arbitrary sublattice  $(\omega_0 \omega_1 \cdots \omega_{n-1})$ , the number  $N_n$  of the "turbulent" sites in this sublattice is also the number of clusters defined as above. These clusters do not interact with each other, and the energy  $U_n(\omega_0 \omega_1 \cdots \omega_{n-1})$  is a sum of energies of individual clusters.

The interaction potential we have obtained is similar to the Fisher-Felderhof model,<sup>20</sup> in that the interaction energy of one given site with the rest of the lattice is not bounded from above. In fact, the energy of an infinite cluster is also the interaction energy  $u_l$  of the last "turbulent state" with all the preceding laminar ones, which diverges logarithmically with  $l$ , according to (3.9). Therefore, our system belongs to the space  $B \setminus \bar{B}$  mentioned in Sec. I. In Ref. 20, three classes of models were investigated:

$$W_l \sim \begin{cases} l^\sigma, & 0 < \sigma < 1 \text{ type A} \\ \exp[c(\ln l)^\nu], & \nu < 1, 0 < c \text{ type B} \\ \ln l, & \text{type C.} \end{cases} \quad (3.11a)$$

$$\exp[c(\ln l)^\nu], \quad \nu < 1, 0 < c \text{ type B} \quad (3.11b)$$

$$\ln l, \text{ type C.} \quad (3.11c)$$

In each case, the surface energy  $W_l$  increases more slowly than  $l$ , so that  $\lim_{l \rightarrow \infty} W_l/l = 0$ . Clearly our intermittent system is of type C, or corresponds to the logarithmic model which was analyzed by Fisher and Felderhof in the second article of their serial publication (cf. Ref. 20. The discrete lattice was considered in Ref. 22).

In various aspects, this system is quite unusual. For instance, letting  $m_l$  denote the number of clusters of length  $(l+1)$  in a sublattice  $(\omega_0 \omega_1 \cdots \omega_{n-1})$ , we have

$$\sum_{l=0}^{\infty} m_l = N_n, \quad \sum_{l=0}^{\infty} (l+1)m_l = n. \quad (3.12)$$

Then, the mean total energy of sublattices of size  $n$  may be written as

$$\begin{aligned} E(U_n(\omega_0 \omega_1 \cdots \omega_{n-1})) &= \sum_l E(m_l) u_l \\ &= (\sum_l p_{0l} u_l) E(N_n), \end{aligned} \quad (3.13)$$

with  $E(m_l) = p_{0l} E(N_n)$ . The sum is finite, thus  $E(U_n)$  behaves similarly as  $E(N_n)$ . In particular, for  $2 \leq z$  ( $0 < \alpha \leq 1$ ), the mean energy will not be proportional to the size of the system [cf. (2.21b), (2.22b)]. Rather, the mean energy per unit of volume vanishes.

In other terms, the sporadic dynamic processes correspond to a kind of statistical system with energies distributed on a fractal subset of the space, with the consequence that the *mean* energy density is zero. This phenomenon has *not* been foreseen in the original work by Fisher and Felderhof, but perhaps is also present in their own model system. Other characteristics of the Fisher-Felderhof-like model will become apparent below.

In what follows we shall be concerned with the statistical mechanics of this lattice-gas model derived from our system of temporal intermittency. We start by observing that besides the quantity  $U_n(\omega_0 \omega_1 \cdots \omega_{n-1})$ , the laminar time, or the number  $N = n - N_n$  of the laminar sites in a lattice of size  $n$ , as  $n \rightarrow \infty$ , is of equal importance here. Therefore we would like to introduce a "chemical potential"  $\mu$ , to be associated with  $n - N_n$ ; just as  $\beta$  was introduced in connection to the energy  $U_n(\omega_0 \omega_1 \cdots \omega_{n-1})$ . This definition is mostly natural, with the consequence that the thermodynamic conjugate variable of  $\mu$ , the density  $\rho$ , will be the unity when in average almost all the sites are laminar, and we have a sort of condensed phase.

Then, the grand canonical partition function  $\Xi(\beta, z, n)$  is

$$\Xi(\beta, z, n) = \sum_{N=0}^n z^N \sum_{\substack{(\omega_0 \omega_1 \cdots \omega_{n-1}) \\ N \text{ fixed}}} e^{-\beta U_n(\omega_0 \omega_1 \cdots \omega_{n-1})}, \quad (3.14)$$

where  $z = \exp(\beta\mu)$  is the fugacity (it should not be confused with the exponent  $z$  of the mapping (1.1), which fortunately will rarely appear in the rest of the paper).

Since  $\beta=1$  corresponds to the natural measure, we observe that

$$\Xi(\beta=1, z, n) \simeq \langle e^{\mu N} \rangle, \quad (3.15)$$

where we see that  $\mu$  is a parameter for describing fluctuations of the random variable  $N = n - N_n$  (see also Appendix C). The pressure is given by

$$\beta p(\beta, z) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \Xi(\beta, z, n). \quad (3.16)$$

Recalling what we have said at the beginning of this section, the pressure (3.16) may be thought of as defined in correspondence with the function

$$\beta \mu [1 - I(x)] - \beta \ln |f'(x)|,$$

which depends on *two* parameters.  $\beta$  and  $\mu$  determine a two-parameter family of invariant measures of the system.

### C. Existence of phase transition

We shall follow the methodology of Ref. 20, and consider the signature of a phase transition as the occurrence of nonanalyticity of the thermodynamic quantities.<sup>42</sup> Let us consider the generating function (the discrete version of Laplace transform) of the grand partition function  $\Xi(\beta, z, n)$ :

$$\Psi(\beta, z, s) = \sum_{n=0}^{\infty} e^{-sn} \Xi(\beta, z, n). \quad (3.17)$$

We observe that if we let  $s_0 = \beta p(\beta, z)$ , then

$$\Psi(\beta, z, s) = \begin{cases} \infty & \text{if } \text{Re}(s) < s_0 \\ 0 & \text{if } \text{Re}(s) > s_0. \end{cases} \quad (3.18)$$

In other words, the  $\beta p(\beta, z)$  may be obtained as a singularity of

$$\Psi(\beta, z, s) = \sum_{n=0}^{\infty} e^{-sn} \sum_{N=0}^n z^N \sum_{\{m_l\}} \exp(-\beta \sum_l m_l u_l). \quad (3.19)$$

Let

$$H(\beta, z, s) = \sum_{l=0}^{\infty} z^l e^{-sl - \beta u_l} = \sum_{l=0}^{\infty} u^l e^{-\beta u_l}, \quad u \equiv z e^{-s} \quad (3.20)$$

be the transformed grand partition function of a single cluster (of all possible sizes). Then it can be checked that

$$\Psi(\beta, z, s) = \sum_{N=0}^{\infty} H^N(\beta, u) e^{-sN}. \quad (3.21)$$

This sum can become divergent in two distinct ways: (a) if  $H(\beta, u) = \infty$ , or the *internal condition*:

$$u e^{-\beta u} = 1, \quad (3.22)$$

where  $u_{\infty} = \lim_{l \rightarrow \infty} (u_l/l) = 0$ ; and (b) if  $H(\beta, u) e^{-s} = 1$ , or the *external condition*:

$$H(\beta, u) = \sum_{l=0}^{\infty} u^l e^{-\beta u_l} = e^s. \quad (3.23)$$

It is convenient to consider  $s$  instead of the pressure  $p$ , and rather than regarding  $s$  as function of  $\beta$  and  $\mu$ ,  $\mu$  is to be considered as function of  $\beta$  and  $s$ , determined by either (3.22) or (3.23). Thus we see already that this function will not be analytic. Evidently,  $H(\beta, u)$  is a monotonically increasing function of  $u$ . Let  $\beta$  and  $s$  be fixed. If, as  $u$  increases from zero, there exists a value of  $u$  less than the unity,  $u^* < 1$ , such that  $H(\beta, u^*) = e^s$ , then the external condition is realized at  $u = u^*$ , before the internal condition  $u = 1$  is attained. In other words, Eq. (3.23) is to be used to obtain the thermodynamic function  $\mu(\beta, s)$ .

On the other hand, if as  $u$  approaches the unity from below,  $H(\beta, u)$  remains smaller than  $e^s$ , then the divergence of  $\Psi(\beta, z, s)$  is to be attributed to the occurrence of the internal condition, and  $\mu(\beta, s)$  is given by (3.22). Figure 5 illustrates these two possibilities. The changeover between the two is determined by the equality of the Eqs. (2.22) and (2.23),

$$H(\beta, u = 1) = e^s \quad \text{or} \quad s = \ln \sum_{l=0}^{\infty} e^{\beta u_l}. \quad (3.24)$$

It determines a critical line in the  $(\beta, s)$  plane, as displayed in Fig. 6. We note that this critical curve  $s(\beta)$  is a monotonically decreasing function that diverges at

$$\beta^* = 1/(\alpha + 1). \quad (3.25)$$

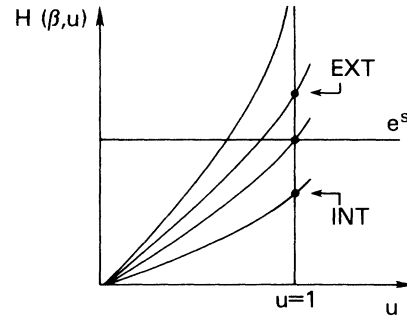


FIG. 5.  $H(\beta, u)$  curve for fixed  $\beta$ . Depending on  $H(\beta, u = 1)$  less than or greater than  $e^s$ , the internal or external condition is selected. The crossover between the two determines a critical line in the  $(\beta, s)$  plane, on which the thermodynamic functions are not analytic.

One has thus proved the existence of a phase transition crossing the critical line (3.24). Section IV is devoted to a detailed analysis in the critical region.

#### IV. STUDY OF THE CRITICAL PHENOMENA

##### A. State equation and thermodynamics

According to Fisher and Felderhof, we may write from the definition of  $u = e^{-s} z$

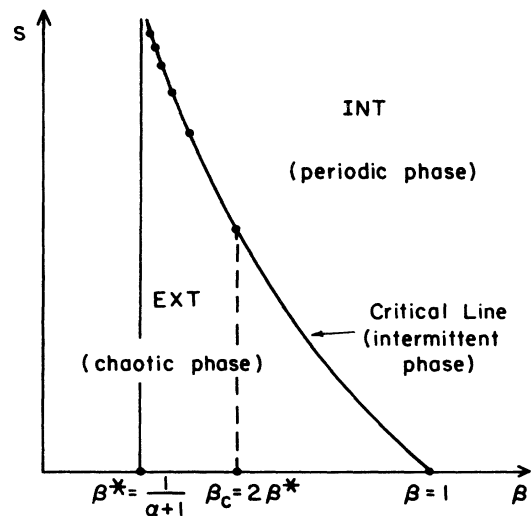


FIG. 6. Phase diagram in the  $(\beta, s)$ . It is representative for  $1 < \alpha$ , then  $\beta_c < 1$ . The critical curve  $s_c(\beta)$  divides the plane into two parts, corresponding respectively the periodic and chaotic states. A series of points of  $s_c(\beta)$  are drawn to indicate a subdivision of the curve into a countable number of pieces, each of them has a different order of phase transition. The intermittent state with long-range correlations and abnormal fluctuations is located on this curve of criticality. For  $0 < \alpha < 1$ , still another phase, the *sporadic* state with *stretched* exponential instability is present at the unique point  $\beta = 1, s = 0$ .



$$\ln z = \ln u + s. \quad (4.1)$$

This is a basic state equation of our system, where  $u$  is now determined by either the internal condition  $u = 1$  (3.22), or the external condition (3.23). By the standard thermodynamic calculus, we list the important quantities of our interest: the specific volume

$$v = \frac{1}{\rho} = \frac{\partial \ln z}{\partial s}, \quad (4.2)$$

the isothermal compressibility

$$\frac{\kappa_T}{\beta} = -\frac{1}{v} \frac{\partial^2 \ln z}{\partial s^2}, \quad (4.3)$$

and the specific heat per unit volume

$$c_V = v \frac{\partial^2 p}{\partial T^2} - \frac{\partial^2 \mu}{\partial T^2} = v \beta^3 \frac{\partial^2 s}{\partial \beta^2} - \beta^3 \frac{\partial^2 \ln z}{\partial \beta^2}. \quad (4.4)$$

### B. The order of phase transition

We need to study the change of the density as the criticality is crossed. On one hand, the ordered phase is given by the internal condition, with a fixed specific volume

$$v_l = \frac{\partial \ln z}{\partial s} = 1. \quad (4.5)$$

This corresponds to the periodic ("laminar") phase, for which

$$\left\langle \frac{N}{n} \right\rangle = 1 \quad \text{or} \quad \left\langle \frac{N_n}{n} \right\rangle = 0, \quad (4.6)$$

while on the other hand the disordered phase is given by the external condition, and corresponds to the chaotic ("turbulent") phase. Let us make use of the external condition to calculate the specific volume, and see if it tends to the unity when the critical curve in the  $(\beta, s)$  plane is approached (with fixed  $\beta$ , for instance). By differentiating both sides of Eq. (3.23), with respect to  $\ln z$ , one obtains

$$v = \frac{\partial \ln z}{\partial s} = 1 + \frac{e^s}{\sum_{l=0}^{\infty} l u^l e^{-\beta u_l}}. \quad (4.7)$$

Let

$$u = e^{-\omega}, \quad \omega \rightarrow 0+ \quad \text{as} \quad u \rightarrow 1-, \quad (4.8)$$

then, the denominator of the second term in (4.7) can be rewritten as

$$\begin{aligned} \Sigma(\omega) &\equiv \sum_{l=0}^{\infty} l u^l e^{-\omega l - \beta u_l} \\ &\sim C_1 + C_2 \int_1^{\infty} dx e^{-x\omega} x^{1-(\alpha+1)\beta}, \end{aligned} \quad (4.9)$$

where  $C_1$  and  $C_2$  are constant. Then,  $v \rightarrow_{\omega \rightarrow 0} 1$  if this integral with  $\omega = 0$  is convergent, i.e., if

$$\beta > \frac{2}{\alpha+1} \equiv \beta_c. \quad (4.10)$$

Since the critical curve ends at  $\beta = 1$  (Fig. 6), (4.10) cannot be realized for  $0 < \alpha < 1$ . We conclude that if

$0 < \alpha < 1$ , then

$$\lim_{u \rightarrow 1^-} v \equiv v_g = 1, \quad v_g - v_l = 0 \quad (4.11)$$

and if  $1 < \alpha$ , then

$$\begin{aligned} v_g = 1, \quad v_g - v_l = 0 \quad \text{for} \quad \beta^* < \beta < \beta_c, \\ v_g < 1; \quad v_g - v_l \neq 0 \quad \text{for} \quad \beta_c < \beta < 1. \end{aligned} \quad (4.12)$$

Hence, we see that the phase transition may be of first order, in the case  $1 < \alpha$ . For fixed  $\beta$  satisfying  $\beta_c < \beta < 1$ , the function  $s(v)$  (the isotherm) presents a plateau at the critical value of  $s = s_c(\beta)$ . This function also has a discontinuity at  $v = 1$  [in fact, the inverse function  $v(s)$  has a plateau at  $v = 1$  for all the values  $s > s_c(\beta)$ ]. See Fig. 7.

When  $\beta$  is decreased to  $\beta_c$ , the phase transition will no longer be of first order. For  $\beta^* < \beta < \beta_c$  (with  $1 < \alpha$ ), as well as for  $0 < \alpha < 1$ , both comprised as  $(\alpha+1)\beta < 2$ ,  $\partial \ln z / \partial s$  is continuous, and we shall consider higher-order derivatives of  $\ln z$  with respect to  $s$ . To this end, we observe that

$$\Sigma(\omega) \simeq C_1 + C_2 \Gamma(2 - (\alpha+1)\beta) \omega^{(\alpha+1)\beta-2} \quad \text{as} \quad \omega \rightarrow 0+. \quad (4.13)$$

Consequently,

$$\frac{\partial \ln z}{\partial s} \equiv v \simeq 1 + \frac{e^s}{C_2 \Gamma(2 - (\alpha+1)\beta)} (\omega)^{2 - (\alpha+1)\beta}. \quad (4.14)$$

Moreover,

$$e^s \sim e^{s_c} + \int_0^{\infty} dx e^{-\omega x} x^{-(\alpha+1)\beta} + C_0 \omega + O(\omega^2), \quad (4.15)$$

where  $C_0$  is a constant. Therefore

$$\begin{aligned} (s - s_c) &\sim \omega^{(\alpha+1)\beta-1} \Gamma(1 - (\alpha+1)\beta) \\ &\quad + C_0 \omega + O(\omega^2). \end{aligned} \quad (4.16)$$

We have

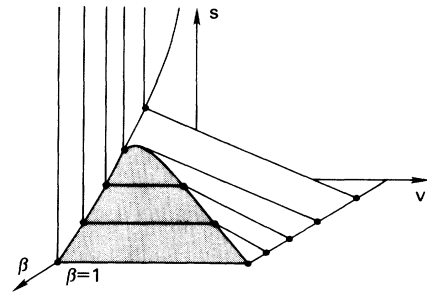


FIG. 7. Three-dimensional sketch of the state equation for the intermittent system with  $1 < \alpha$ . The origin of the axes is  $(\beta = \beta^*, v = 1, s = 0)$ . For  $\beta_c < \beta$ , the pressure function versus density (the isotherms) presents a plateau at the criticality. On the other hand, beyond the critical value of  $s$ , the density remains constant, as indicated here by a vertical line at  $v = 1$ .

$$\begin{aligned} \frac{\partial(\omega)}{\partial s} &= \frac{\partial(-\ln z)}{\partial s} + 1 \\ &\simeq \frac{-e^{s_c}}{C_2 \Gamma(2 - (\alpha + 1)\beta)} \\ &\quad \times (s_c - s)^{[2 - (\alpha + 1)\beta]/[(\alpha + 1)\beta - 1]}. \end{aligned} \quad (4.17)$$

By definition,

$$\begin{aligned} \frac{\kappa_T}{\beta} &= -\frac{1}{v} \frac{\partial^2 \ln z}{\partial s^2} \\ &\sim (\omega)^{1 - (\alpha + 1)\beta} \left[ \frac{\partial(-\ln u)}{\partial s} \right] \\ &\sim (s_c - s)^{[3 - 2(\alpha + 1)\beta]/[(\alpha + 1)\beta - 1]}, \end{aligned} \quad (4.18)$$

which will diverge if  $3 < 2(\alpha + 1)\beta$ , i.e., either  $1 < \alpha < 2$  with  $3/[2(\alpha + 1)] < \beta < 2/(\alpha + 1)$ ; or  $\frac{1}{2} < \alpha < 1$  with  $3/[2(\alpha + 1)] < \beta < 1$ . More generally, it can be checked that if

$$\frac{n+2}{n+1} < (\alpha + 1)\beta < \frac{n+1}{n}, \quad n \geq 1 \quad (4.19)$$

then

$$\left. \frac{\partial^k \ln z}{\partial s^k} \right|_{u=1} = 0, \quad k = 0, 1, \dots, n, \quad \left. \frac{\partial^{n+1} \ln z}{\partial s^{n+1}} \right|_{u=1} = \infty. \quad (4.20)$$

Observing that  $\partial \ln z / \partial s$  is constant in the periodic ("laminar") phase, thus its further derivatives are all zero, one concludes from (4.20) that  $\partial^n \ln z / \partial s^n$  is continuous as the critical point is crossed. Nevertheless, its next derivative is infinite. This completes our consideration on the order of the phase transition.

### C. Asymptotic behavior in the critical region

We shall list a number of results, with necessary indications of their derivations. The method of analysis leans essentially on the work of Fisher-Felderhof.

(i)  $v - v_g$  as  $s \rightarrow s_c$ . Let  $\beta$  be fixed. The specific volume  $v$  approaches to  $v_g$  according to a certain power law,

$$(s_c - s) \sim (v - v_g)^{\delta(\beta)}. \quad (4.21)$$

One would like to determine the index function  $\delta(\beta)$ . The argument is due to Fisher and Felderhof.<sup>20</sup> Two cases are to be distinguished depending on the asymptotic behavior of the sum  $\Sigma(\omega)$  (4.9), as  $\omega \rightarrow 0+$ . For the first case  $2 < (\alpha + 1)\beta$  this sum does converge, hence

$$\Sigma(\omega) \sim \Sigma(\omega=0) + C_2 \Gamma(2 - (\alpha + 1)\beta) \omega^{(\alpha + 1)\beta - 2}, \quad (4.22)$$

and

$$v - v_g \sim e^{s_c} \left[ \frac{1}{\Sigma(\omega)} - \frac{1}{\Sigma(0)} \right] \sim \omega^{(\alpha + 1)\beta - 2}, \quad (4.23)$$

which in combination with (4.16) yields

$$\delta(\beta) = 1/[(\alpha + 1)\beta - 2]. \quad (4.24)$$

On the other hand, in the second case  $(\alpha + 1)\beta < 2$  the sum  $\Sigma(\omega)$  diverges at  $\omega=0$ , and one has

$$\Sigma(\omega) \sim \omega^{(\alpha + 1)\beta - 2}, \quad v - v_g \sim \omega^{2 - (\alpha + 1)\beta}, \quad (4.25)$$

which in combination with (4.16) yields

$$\delta(\beta) = [(\alpha + 1)\beta - 1]/[2 - (\alpha + 1)\beta]. \quad (4.26)$$

One remarks that this index is *not bounded* as  $(\alpha + 1)\beta \rightarrow 2$ .

(ii)  $(v_g - v_l)$  as  $\beta \rightarrow \beta_c^+$ . From (4.9) we have

$$\Sigma(\omega) \sim \frac{1}{(\alpha + 1)\beta - 2} + O(\omega^{(\alpha + 1)\beta - 2}, \omega),$$

so that

$$v_g \sim 1 + e^{s_c} [(\alpha + 1)\beta - 2], \quad (4.27)$$

or

$$v_g - v_l \sim (\alpha + 1)e^{s_c(\beta - \beta_c)}(\beta - \beta_c). \quad (4.28)$$

Thus, the critical exponent is 1.

(iii) At  $\beta = \beta_c$ ,  $(v_g - v_l)$  as  $s \rightarrow s_c$ . Since<sup>43</sup>

$$\Sigma(\omega) \sim \int_1^\infty dx \frac{e^{-\omega x}}{x} = -Ei(-\omega) \underset{\omega \rightarrow 0+}{\sim} -\ln \omega, \quad (4.29)$$

we have, with  $\partial \ln z / \partial s \sim 1 + e^{s_c} / (-\ln \omega)$ ,

$$v_g - v_l \sim \frac{e^{s_c}}{\ln(s_c - s)}, \text{ or } s_c - s \sim \exp \left[ \frac{e^{s_c}}{v_g - v_l} \right], \quad (4.30)$$

which is *not* a power law.

(iv) *Two-phase specific heat*. By definition (4.3), combined with the fact that the chemical potential is identical at the coexisting pressure, we have, as  $\omega \rightarrow 0+$ ,

$$c_V(g) - c_V(l) = (v_g - 1)\beta^2 \frac{\partial^2 s}{\partial \beta^2}, \quad (4.31)$$

which remains finite, although the higher derivatives may be divergent.

(v) *Entropy function*. It follows from

$$s(\beta, \mu) = \rho\mu\beta - \beta\Lambda + G(\Lambda, \rho), \quad \text{with } \Lambda(\beta, \rho) = \rho\mu - \frac{\partial s}{\partial \beta} \quad (4.32)$$

that the entropy function may be written as

$$G(\Lambda, \rho) = s(\beta, \mu) - \beta \frac{\partial s}{\partial \beta}. \quad (4.33)$$

It is evident that for the periodic ("laminar") phase,

$$s = \ln z = \beta\mu, \quad G(\Lambda, \rho) \equiv 0. \quad (4.34)$$

Whereas for the chaotic ("turbulent") phase, differentiating the external condition (3.23) with respect to  $\beta$  yields

$$\frac{\partial s}{\partial \beta} = \frac{\sum_{l=0}^{\infty} e^{-sl - \beta u_l} z^l(-u_l)}{\sum_{l=0}^{\infty} e^{-sl - \beta u_l} z^l(l+1)} \underset{u \rightarrow 1-}{\rightarrow} \frac{\sum_{l=0}^{\infty} e^{-\beta u_l}(-u_l)}{\sum_{l=0}^{\infty} e^{-\beta u_l}(l+1)}. \quad (4.35)$$

This is a nonpositive quantity. It is thus obvious that so far as  $s_c > 0$ , the entropy function (4.34) will not vanish as the criticality is approached from below. Consequently, there is a finite jump in  $G(\Lambda, \rho)$ .

For the particular case  $s_c = 0$  (this happens with  $\mu = 0$  so that  $u \rightarrow 1 -$  implies  $s \rightarrow 0$  and  $\beta \rightarrow 1$ ), recalling that

$$e^{-\beta u_l} = \Delta_l^\beta \sim l^{-(\alpha+1)\beta}, \quad (4.36)$$

we see that the denominator of the expression (4.36) will diverge for  $0 < \alpha < 1$ , in which case the entropy will vanish at the codimension two critical point  $\beta = 1, s = 0$ . This corresponds to the *sporadic state* discussed in the previous sections. For  $1 < \alpha$ , on the other hand, it is easy to see that

$$G(\beta = 1, \mu = 0) = \frac{-\sum_{l=0}^{\infty} \Delta_l \ln \Delta_l}{\sum_{l=0}^{\infty} (l+1)\Delta_l}. \quad (4.37)$$

which is just the Kolmogorov-Sinai entropy of the origi-

$$e^s \sim (1-a)^\beta + (a\alpha)^\beta \int_{l=1}^{\infty} dl e^{-sl} l^{-(\alpha+1)\beta} \\ \sim (1-a)^\beta + (a\alpha)^\beta \left[ \frac{e^{-s}}{(\alpha+1)\beta-1} - \frac{1}{(\alpha+1)\beta-1} s^{(\alpha+1)\beta-1} \Gamma(2-(\alpha+1)\beta, s) \right]. \quad (4.40)$$

Let  $\epsilon = 1 - \beta$ , one can show that as  $s \rightarrow 0$  and  $\epsilon \rightarrow 0$ , (4.40) yields

$$\sum_{m,n} A_{m,n} \epsilon^m s^n + (a\alpha) \Gamma(1-\alpha) s^\alpha + O(\epsilon s^\alpha \ln s), \epsilon s^\alpha = 0, \quad (4.41)$$

where  $A_{m,n}$  are coefficients depending on  $a$  and  $\alpha$ . Based on this equation, it is easy to check that (a) if  $0 < \alpha < 1$ ,

$$s = \epsilon^\nu (b + c\epsilon + \dots) \quad \text{with } \nu = 1/\alpha, \quad (4.42a)$$

and (b) if  $k < \alpha < (k+1)$ ,  $k = 1, 2, \dots$ ,

$$s = \sum_{i=1}^k a_i \epsilon^i + a_{k+1} \epsilon^\nu \quad \text{with } \nu = \alpha, \quad (4.42b)$$

and the coefficient  $a_1 = h_{KS}$ .

For  $0 < \alpha < 2$ ,  $\alpha \neq 1$ , Eq. (4.42) recovers the results of Eq. (1.2), which have been first derived in a very different way.<sup>16</sup> On the other hand, for  $2 < \alpha$ , it should be emphasized that the *local* fluctuations are known to be Gaussian [cf. (2.18)]. However, because of the existence of long range correlation, the large or *global* fluctuations described by the  $P(\beta)$  function are peculiar, and the pressure function is still not analytic with respect to  $\beta$ .

## V. CONCLUDING REMARKS

Let us summarize.

(i) In this paper we have made an explicit construction for the associated statistical process of a class of temporal

nal dynamical system. We would have obtained the same expression, had we applied the formula of the entropy of a Markov chain<sup>44</sup>

$$h_{KS} = -\sum_i \mu_i \sum_j p_{ij} \ln p_{ij} \quad (4.38)$$

to our systems (2.9) and (2.15).

### D. The case $\mu = 0$

It is of interest to discuss a little further the particular case  $\mu = 0$ , since that corresponds to the more familiar situation discussed at the beginning of Sec. III. The pressure function (3.16) is reduced to (3.1a), and the external condition (3.23) to

$$e^s = \sum_{l=0}^{\infty} e^{-sl - \beta u_l}. \quad (4.39)$$

Evidently,  $s(\beta = 1) = 0$ . On the other side of the critical point, the internal condition (3.22) gives  $s(\beta) \equiv 0$ ,  $\beta > 1$ . We would like to see how  $s(\beta)$  approaches to zero as  $\beta \rightarrow 1 -$ . Using (3.23) one can write

intermittent systems. We first converted the mapping into a countable Markov chain and obtained the invariant measure. The turbulent time  $N_n$  is shown to play an important role, and its dynamical fluctuations, both local and global, are studied at length.

(ii) The resulting system is shown to belong to the same category as that of an abstract model already existing in the literature. The latter model has been used by Gallovotti<sup>45</sup> as an example of which the  $\zeta$  function is not meromorphic; and by Hofbauer<sup>46</sup> as an example with two equilibrium states. These authors pointed out that the statistical counterpart of their example is the Fisher's droplet model of condensation in one dimension. We have endeavored to establish the explicit equivalence of our intermittent mapping (2.2) to a lattice gas with *many-body* clustering interactions. It is indeed akin to a special class of models proposed by Fisher and Felderhof, with the surface energy of logarithmic type.

(iii) In the spirit of the thermodynamic formalism, a "chemical potential"  $\mu$  is introduced, in addition to the "inverse of temperature"  $\beta$ , and the phase transition of our system is analyzed in the grand canonical formalism. In particular, one shows that on the critical curve of the pressure-temperature plane, there are a *denumerable* number of points which divide the critical curve into separate segments according to the distinct order of phase transition. With  $1 < \alpha$ , the phase transition is of first order for  $\beta_c < \beta < 1$ . The asymptotic behavior (e.g., critical indices) near  $\beta = \beta_c$  is *independent* of  $\alpha$ .

(iv) The critical line in the pressure-temperature plane

TABLE I. Classification of the temporal intermittent behaviors.

Characteristics	$\alpha = 1/(z-1)$		
	$2 < \alpha$	$1 < \alpha < 2$	$0 < \alpha < 1$
Dynamic instability	chaos	chaos	sporadic
Local fluctuation	Gauss	Lévy	Lévy
Large deviations	abnormal	abnormal	abnormal

separates the two phases corresponding to *chaotic* and *periodic* states, respectively, in the original dynamic system. The *intermittent* state, on the other hand, is located on the codimension-one critical curve. In this latter state, the unusual *global* fluctuations and long range correlation present manifestations of a phase transition, disregarding whether the local fluctuations are Gaussian or not. Besides, for  $0 < \alpha < 1$ , the *sporadic* state may occur as a codimension-two phenomenon, at a unique point of the pressure-temperature plane. Such a state has its energy distributed on a fractal subset of the lattice, with the consequence that the mean energy density *per site* vanishes.

(v) The three types of temporal inermittency are summarized in Table I. It is concluded that the thermodynamic formalism describing large deviations allows a remarkably fine understanding of the intermittent system, and makes a universal classification possible. One may ask next if this approach would be also useful for characterizing the *spatio-temporal* intermittency, thus extending the statistical mechanics to such intriguing nonequilibrium processes.

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APPENDIX A

This and the next two appendixes contain some results on the probabilistic properties of the turbulent time  $N_n$ , which is of major importance for the understanding of the intermittent system. Here we shall prove the limit theorem of  $N_n$  for  $\alpha=1$ , Eq. (2.21a). The other cases with  $\alpha \neq 1$  may be treated similarly.

The stable law with its two-sided Laplace transform

$$\hat{g}_{\alpha=1}(s) = \exp(s \ln s) \tag{A1}$$

has the Fourier transform  $\Phi(z)$  given by<sup>47</sup>

$$\ln \Phi(z) = -\frac{\pi}{2} |z| - iz \ln |z| . \tag{A2}$$

Let us denote by  $S_k$  the time lapse up to and including the  $k$ th occurrence of the state  $A_0$ :

$$S_k = X_1 + X_2 + \dots + X_k , \tag{A3}$$

where  $X$  is the recurrent time for  $A_0$ . We observe that

$$\mathcal{P}(N_n \geq k) = \mathcal{P}(S_k \leq n) . \tag{A4}$$

$S_k$  is the sum of mutually independent random variables  $X_k$ , with a common probability distribution [cf. (2.16)],

$$1 - F(x) \underset{x \rightarrow \infty}{\sim} A(x+1)^{-1} . \tag{A5}$$

Let  $b_n$  be defined by

$$1 - F(b_n) \sim n^{-1}, \quad b_n \sim An \tag{A6a}$$

and  $a_n$  be defined by

$$a_n = \int_0^\infty \sin \left[ \frac{x}{b_n} \right] dF(x) . \tag{A6b}$$

The Fourier transform of  $F(x)$  is

$$\varphi(z) = \int_0^\infty e^{izx} dF(x) , \tag{A7}$$

and we have

$$\begin{aligned} \varphi(z/b_n) - 1 &= \int_0^\infty (e^{iz(x/b_n)} - 1) dF(x) \\ &= \int_0^\infty (e^{izy} - 1) dF(b_n y) , \end{aligned} \tag{A8}$$

then

$$\begin{aligned} n[\varphi(z/b_n) - 1 - iza_n] &= n \int_0^\infty (e^{izy} - 1 - iz \sin y) dF(b_n y) \\ &\simeq \int_0^\infty \frac{e^{izy} - 1 - iz \sin y}{y^2} dy . \end{aligned} \tag{A9}$$

The last integral can be readily shown to coincide with  $\log \Phi(z)$  as given in (A2). Consequently,

$$n[\varphi(z/b_n) - 1 - ia_n z] \underset{n \rightarrow \infty}{\sim} \log \Phi(z) , \tag{A10}$$

which implies

$$[\varphi(z/b_n) e^{-iza_n}]^n \sim \Phi(z) . \tag{A11}$$

Therefore, we have proved that

$$\mathcal{P}[S_n \leq b_n(na_n + x)] \underset{n \rightarrow \infty}{\rightarrow} G_1(x) . \tag{A12}$$

It now remains to determine  $a_n$ . According to (A6b),

$$\begin{aligned} a_n &= \int_0^\infty \sin x dF(b_n x) \\ &= \int_0^\infty [1 - F(b_n x)] \cos x dx \\ &\sim A \int_0^\infty \frac{1}{1 + b_n x} \cos x dx \sim \frac{A}{b_n} \int_0^\infty \frac{1}{x + 1/b_n} \cos x dx , \end{aligned} \tag{A13}$$

since<sup>43</sup>

$$\int_0^\infty \frac{1}{x+c} \cos x \, dx = -\sin(c) \operatorname{Si}(c) - \cos(c) \operatorname{Ci}(c) \sim -(C + \ln c), \quad (\text{A14})$$

where  $C$  is the Euler constant, we conclude that

$$a_n \sim \frac{A}{b_n} \ln b_n \sim \frac{\log n}{n}. \quad (\text{A15})$$

Equation (2.21a) follows from (A4) and (A12), with  $a_n$  and  $b_n$  specified by (A6) and (A15).

APPENDIX B

We would like to calculate the autocorrelation of the random variable  $I(x)$  (2.23), which is the indicator of the “turbulent” phase. One is only concerned with the case  $2 < \alpha$  and would like to show that the correlation function decays in a power law, with the exponent  $\nu = (\alpha - 1)$  [cf. discussion following Eq. (2.28)].

By definition,

$$\phi_n = E(I(x)I(f^n(x))) - E^2(I(x)). \quad (\text{B1})$$

Then, we have

$$E(I(x)) = \mu_0, \quad E(I(x)I(f^n(x))) = \mu_0 P_n, \quad (\text{B2})$$

where  $P_n$  is the probability that  $\omega_n = 0$  under the condition  $\omega_0 = 0$ , in an orbit  $x = (\omega_0 \omega_1 \cdots \omega_n \cdots)$ . The following discussion on the asymptotic behavior of  $P_n$  is largely based on Ref. 27. Observe that

$$P_n = p_{0n-1} P_0 + p_{0n-2} P_1 + \cdots + p_{00} P_{n-1}, \quad (\text{B3})$$

with  $P_0 = 1$ . Then, it can be proved<sup>27</sup> that

$$P_n \xrightarrow{n \rightarrow \infty} \frac{1}{\tau}, \quad \tau = \sum_{k=0}^\infty k p_{0k-1}. \quad (\text{B4})$$

It can be readily checked that for our Markov chain,

$$\mu_0 = \frac{1}{\tau}. \quad (\text{B5})$$

Hence, combining (B2) with (B4) and (B5), we see that the correlation function does indeed decrease to zero. We now would like to show that it decays in a power law. Let

$$U(s) = \sum_{n=0}^\infty s^n P_n, \quad P(s) = \sum_{n=1}^\infty s^n p_{0n-1}, \quad (\text{B6})$$

$$q_k = \sum_{l=k}^\infty p_{0l}, \quad Q(s) = \sum_{k=0}^\infty q_k s^k, \quad (\text{B7})$$

$$r_k = \sum_{l=k+1}^\infty q_l = \sum_{l=k+1}^\infty (l-k) p_{0l}, \quad R(s) = \sum_{k=0}^\infty r_k s^k.$$

Then, we have

$$\begin{aligned} U(s) &= 1/[1 - P(s)], \\ 1 - P(s) &= (1-s)Q(s), \quad P'(1) = Q(1) = \tau, \\ \tau - Q(s) &= (1-s)R(s), \quad P''(1) = 2Q'(1) = 2R(1). \end{aligned} \quad (\text{B8})$$

The last quantity  $R(1)$  is finite for  $2 < \alpha$ .

Then, it can be shown that

$$U'(s) - \frac{1}{\tau(1-s)^2} = \left[ \frac{R(s)}{\tau Q(s)} \right]'. \quad (\text{B9})$$

Clearly  $n(P_n - 1/\tau)$  is the coefficient of  $s^{n-1}$  in (B9). As  $n \rightarrow \infty$ , it is given by the coefficient of  $s^{n-1}$  of

$$\frac{1}{\tau^3} [R'(s)Q(s=1) - R(s=1)Q'(s)]. \quad (\text{B10})$$

Furthermore, it can be easily checked that, the coefficient of  $s^{n-1}$  in  $Q'(s)$  is

$$n \sum_{k=n}^\infty p_{0k} \sim n^{1-\alpha}, \quad (\text{B11a})$$

and the coefficient of  $s^{n-1}$  in  $R'(s)$  is

$$n \sum_{k=n+1}^\infty (k-n) p_{0k} \sim n^{(2-\alpha)}. \quad (\text{B11b})$$

From (B11) we conclude that

$$P_n - 1/\tau \sim n^{-(\alpha-1)} \quad \text{with } 1 < (\alpha-1), \quad (\text{B12})$$

which leads to our conclusion.

APPENDIX C

Let us define a global function  $\Theta(q)$

$$\Theta(q) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \langle e^{-qN_n} \rangle, \quad (\text{C1})$$

which describes large deviations of the random variable  $N_n$ . In Ref. 16,  $\Xi(q)$  instead of  $\Theta(q)$  was used for the same quantity. Here the notation is changed in order to avoid possible confusion with the grand partition function  $\Xi(\beta, z, n)$  (3.14). In fact, comparing (C1) with (3.15), we see that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \Xi(\beta=1, z=e^q, n) = \Theta(q) + q. \quad (\text{C2})$$

In this appendix, we would like to show that for  $|q| \ll 1$ ,

(i)  $1 < \alpha < 2$

$$\Theta(q) = \begin{cases} |q|/\tau + A|q|^\alpha \Gamma(1-\alpha)/\tau^{\alpha+1} & \text{if } q < 0 \\ 0 & \text{if } q > 0, \end{cases} \quad (\text{C3a})$$

(ii)  $0 < \alpha < 1$

$$\Theta(q) = \begin{cases} |q|^{1/\alpha} / [A \Gamma(1-\alpha)]^{1/\alpha} & \text{if } q < 0. \\ 0 & \text{if } q > 0. \end{cases} \quad (\text{C3b})$$

To this end the idea is to make a connection between the behavior of  $\Theta(q)$  near  $q = 0$ , and the local fluctuations described by the Lévy laws (cf. Sec. II C). Then the asymptotic properties of the Lévy stable distributions (see Fig. 8) will be sufficient to prove Eq. (C3).

Notice that it has been argued in Ref. 16 that  $P(1-\beta)$  behaves qualitatively in a similar manner as (C3). Then, Eq. (1.2) follows from (C3). Let us consider first case (i). According to the local theorem (2.20a), we may write

$$\begin{aligned} \langle e^{-qN_n} \rangle &= \int_0^\infty dPr(N_n) e^{N_n q} \\ &\simeq \int_{x_m}^{x_M} dG_\alpha(x) \\ &\times \exp \left\{ - \left[ \frac{n}{\tau} - x \left( \frac{An}{\tau^{\alpha+1}} \right)^{1/\alpha} \right] q \right\}, \end{aligned} \quad (C4)$$

where the bounds of the integral are imposed by  $0 \leq N_n < n$ , with

$$x_M = \left( \frac{\tau}{A} \right)^{1/\alpha} n^{(\alpha-1)/\alpha}, \quad x_m = (1-\tau)x_M. \quad (C5)$$

It can be seen that  $x_M$  and  $x_n$  may be extended to the infinity, provided that  $|q| \ll 1$ , and the integral (C4) be convergent. Noting that<sup>48</sup>

$$g_\alpha(x) \sim \begin{cases} x^{-(1+\alpha)} & \text{if } x \rightarrow +\infty \\ e^{-c|x|^{\alpha/(\alpha-1)}} & \text{if } x \rightarrow -\infty, \end{cases} \quad (C6)$$

with  $c > 0$ .  $\lim_{x \rightarrow -\infty} g_\alpha(x)$  decreases faster than exponentially. Therefore, for  $q < 0$ , it is indeed legitimate to write

$$\begin{aligned} \langle e^{-qN_n} \rangle &\simeq e^{-(n/\tau)q} \int_{-\infty}^{+\infty} dG_\alpha(x) \\ &\times \exp \left\{ \left[ x \left( \frac{An}{\tau^{\alpha+1}} \right)^{1/\alpha} \right] q \right\} \\ &= \exp \left\{ \frac{n}{\tau} |q| + |\Gamma(1-\alpha)| \left[ \left( \frac{An}{\tau^{\alpha+1}} \right)^{1/\alpha} |q| \right]^\alpha \right\}. \end{aligned} \quad (C7)$$

The last equality follows from the definition of the Laplace transform (2.17). Consequently, we obtain, for  $q < 0$ ,

$$\Theta(q) = |q|/\tau + A |q|^\alpha |\Gamma(1-\alpha)|/\tau^{\alpha+1}. \quad (C8)$$

On the other hand, for  $q > 0$ , taking the limit  $x \rightarrow +\infty$  would lead the integral to diverge, since the long tail of  $g_\alpha(x)$  cannot counterbalance the exponentially increasing factor of the integrand. Nevertheless, let us define

$$\hat{F}(c, x_M) = \int_{-\infty}^{x_M} dG_\alpha(x) e^{cx}, \quad c = \left( \frac{An}{\tau^{\alpha+1}} \right)^{1/\alpha}. \quad (C9)$$

The asymptotic behavior of (C9) is related to a degenerate hypergeometric function, and one can show that

$$\hat{F}(c, x_M) \sim \frac{1}{c} x_M^{-(\alpha+1)} e^{cx_M}, \quad cx_M = \frac{n}{\tau} q. \quad (C10)$$

The exact cancellation of the exponential factor by the prefactor  $e^{-(n/\tau)q}$  yields

$$\langle e^{-qN_n} \rangle \sim \frac{A}{n^\alpha q} \quad (C11)$$

and

$$\Theta(q) = 0 \quad \text{for } q > 0. \quad (C12)$$

Therefore, Eq. (C3a) is proved.

As for Eq. (C3b), we shall use a slightly different approach, since in the case  $0 < \alpha < 1$ , the stable distribution

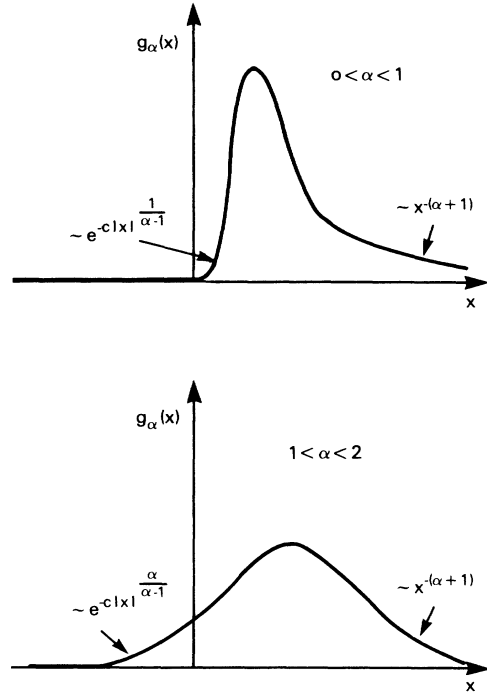


FIG. 8. (Asymmetric) Lévy stable probability density. The critical case  $\alpha = 1$  is similar to the cases  $1 < \alpha$ , with  $g_1(x) \sim 1/x^2$  as  $x \rightarrow +\infty$ , and  $g_1(x) \sim \exp[-\exp(-x)]$  as  $x \rightarrow -\infty$ .

density  $g_\alpha(x)$  vanishes for all  $x < 0$ , with

$$g_\alpha(x) \sim \begin{cases} x^{-(1+\alpha)} & \text{if } x \rightarrow +\infty, \\ e^{-cx^{1/(\alpha-1)}} & \text{if } x \rightarrow 0+, \end{cases} \quad (C13)$$

with  $c > 0$ . According to Ref. 27, we can write

$$\langle e^{-qN_n} \rangle = \sum_{k=0}^\infty \frac{A_n^k}{\Gamma(1+\alpha k)}, \quad A_n = \frac{-n^\alpha q}{A \Gamma(1-\alpha)}, \quad (C14)$$

which is the Mittag-Leffler function  $E_\alpha(x)$  with argument  $x = A_n$ . It is known that<sup>49</sup>

$$E_\alpha(x) \underset{x \rightarrow +\infty}{\sim} \frac{1}{\alpha} e^{x^{1/\alpha}}, \quad (C15)$$

from which we see that

$$\langle e^{-qN_n} \rangle \sim \exp \left[ \frac{n |q|^{1/\alpha}}{[A \Gamma(1-\alpha)]^{1/\alpha}} \right], \quad (C16)$$

and the first part of (C3b), for  $q < 0$ , is proved. On the other hand, when  $q > 0$ , we have  $A_n < 0$ . We can make use of the formula for  $x < 0$ ,

$$E_\alpha(x) = - \sum_{k=1}^\infty \frac{1}{\Gamma(1-\alpha k) x^k}, \quad (C17)$$

to conclude that, for  $q > 0$ ,

$$\langle e^{-qN_n} \rangle \sim \frac{A}{n^\alpha q}, \quad (C18)$$

and the second part of (C3b) follows from it.

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- $$\ln \Psi(z) = \begin{cases} -|cz|^\alpha [1 - \lambda(\operatorname{sgn} z) \tan(\pi\alpha/2)] & \text{if } \alpha \neq 1 \\ -|cz| [1 + i\lambda(\operatorname{sgn} z)(2/\pi) \ln|z|] & \text{if } \alpha = 1 \end{cases}$$
- with three parameters (the Lévy exponent  $\alpha$ , the symmetry index  $\lambda$ , and the constant  $c$ ). If  $\lambda=0$ , the probability density is symmetric with respect to  $z \rightarrow -z$ . The law given in terms of the Laplace Transform (2.17) corresponds to the totally asymmetric case ( $\lambda=1$ ), and with
- $$c = \begin{cases} [\Gamma(1-\alpha)\cos(\pi\alpha/2)]^{1/\alpha} & \text{if } \alpha \neq 1 \\ \pi/2 & \text{if } \alpha = 1 \end{cases}.$$
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