# Global coordinates for the breather-kink (antikink) sine-Gordon phase space: An explicit separatrix as a possible source of chaos

A. R. Bishop
*Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545*

D. W. McLaughlin
*Program in Applied Mathematics and Department of Mathematics, University of Arizona, Tucson, Arizona 85721*

M. Salerno
*Dipartimento di Fisica Teorica, Università di Salerno, I-84100 Salerno, Italy*

(Received 21 April 1989)

In this paper we present global coordinates for the kink-antikink breather sine-Gordon phase space with a separatrix and singular points. Using these coordinates we derive reduced equations governing perturbed dynamics, and we use them in a Melnikov calculation to establish the presence of chaos in the reduced sine-Gordon system.

## I. INTRODUCTION

In this paper we will introduce global coordinates for the breather-kink (antikink) phase space. These provide collective coordinates that are sufficiently general to describe the kink-antikink to breather transition in the presence of perturbations.

Today one can find a large number of collective coordinate representations in the literature. It is reasonable to ask "why introduce yet another such representation?" There are three reasons that these new collective coordinates interest us: (1) The most immediate and natural source of chaos in the sine-Gordon partial differential equation (PDE) is the breather to kink-antikink transition. Yet an analytical description of this transition under "whole line" boundary conditions has proven elusive because of the lack of a global representation of the phase space including a separatrix which locates the transition. The coordinates introduced herein provide this global representation and Melnikov calculations are then immediate. (2) There is an interesting correspondence between the kink (antikink)-breather phase space and that of the double sine-Gordon equation that one of us (M.S.) discovered with the collective coordinates.[1] (3) The global phase space has a singularity, with a natural geometrical and physical interpretation.

The outline of the paper is as follows: In Sec. II we define global coordinates for the reduced phase space; in Sec. III we describe the connection to the double sine-Gordon equation; in Sec. IV the geometry of the singular points is presented; in Sec. V the equations in the presence of perturbations are derived; in Sec. VI a Melnikov calculation establishes horseshoes in the reduced dynamics; the conclusion discusses dangers in the use of collective coordinates.

## II. BREATHER-KINK (ANTIKINK) REDUCTION

The sine-Gordon equation,

$$\phi_{tt}-\phi_{xx}+\sin\phi=0 \ , \tag{2.1}$$

has a two-parameter family of solutions in the form

$$\phi^{k\text{-}a}(x,t,\theta,\tau)=4\tan^{-1}\left[\frac{\coth\theta\sinh[(t-\tau)\sinh\theta]}{\cosh(x\cosh\theta)}\right] . \tag{2.2}$$

A member of this family is known as a "kink-antikink" state. Each of these states represents the elastic collision of a kink with an antikink. See Fig. 1. In the past ($t \ll 0$), the state separates into an antikink far to the left approaching a kink far to the right with equal but opposite velocities. The relative velocity $v$ is given in terms of the parameter $\theta$ by the formula

$$v=\tanh\theta, \quad 0<\theta<+\infty \ . \tag{2.3}$$

In the distant future ($t \gg 0$), the kink and antikink exchange roles elastically. Representation (2.2) describes the collision in the "center-of-mass frame," with center of mass at $x=0$. The collision occurs at $t=\tau$. Notice that $\phi(x,t)\to 0$ as $x\to\pm\infty$ for any fixed $t$; yet, $\phi(x+ct,t)\to\pm 2\pi$ as $t\to\mp\infty$ for all $0<c<v$. Finally,

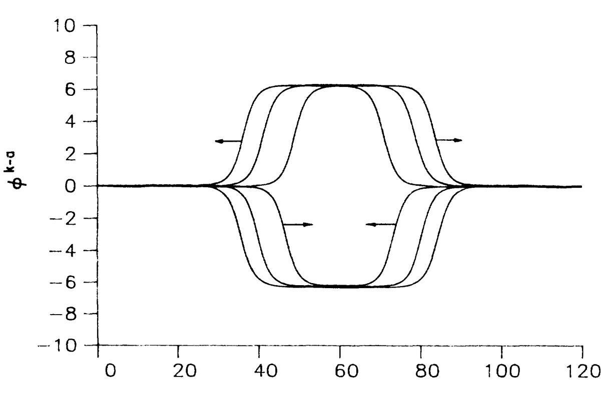


FIG. 1. Kink-antikink collision.

 

the energy of the kink-antikink state is given in terms of $\theta$ by

$$\mathcal{H}(\phi^{k\text{-}a}(\theta,\tau))=16\cosh\theta \ , \tag{2.4}$$

hence the energy exceeds 16. There also exists the symmetric situation of a kink far to the left entering into collision with an antikink far to the right. In our discussions, we will always focus upon the other collision, and then add this symmetric situation for completeness.

At lower energy the kink-antikink components bind into a bound state called a "breather" (see Fig. 2). A two-parameter family of breather solutions has the form

$$\phi^b(x,t,\theta^b,\tau)=4\tan^{-1}\left[\frac{\tan(\theta^b)\sin[(t-\tau)\cos\theta^b]}{\cosh(x\sin\theta^b)}\right] . \tag{2.5}$$

Each member of this family of breathers is (i) even in $x$, (ii) localized in $x$ in the sense that $\phi(x,t)\to 0$ as $x\to\pm\infty$ for any $t$, (iii) periodic in time with temporal period

$$T=\frac{2\pi}{\omega^b},\quad \omega^b=\cos\theta^b,\quad 0\le\theta^b\le\frac{\pi}{2} \ . \tag{2.6}$$

The energy of these breather states is given in terms of the parameter $\theta^b$ by

$$\mathcal{H}(\phi^b)=16\sin\theta^b \ . \tag{2.7}$$

Notice that as $\theta^b=\pi/2$, the breather state has infinite temporal period, and can be interpreted as a kink-antikink state at threshold energy for binding:

$$\mathcal{H}(\phi^b(\theta^b=\pi/2))=\mathcal{H}(\phi^{k\text{-}a}(\theta=0))=16 \ . \tag{2.8}$$

Frequently the breather solution is considered as an analytic continuation of the kink-antikink state through the introduction of complex parameters:

$$\theta^b=\frac{\pi}{2}-i\theta \ .$$

An advantage of the parameters which we are about to introduce is that both the breather and kink-antikink families are realized in the same real phase space, without any analytic continuation.

In both the breather and the kink-antikink cases, we introduce as parameters the waveforms at $x=0$:

$$u\equiv\phi(x=0) \ ,$$
$$\tilde{u}\equiv\phi_t(x=0) \ . \tag{2.9}$$

In both cases it is useful to record the relationship between the $(u,\tilde{u})$ and the $(\theta,\tau)$ parameters: For the kink-antikink case,

$$u=4\tan^{-1}\left[\coth\theta\sinh\left[\frac{\alpha}{16}\right]\right] ,$$
$$\tilde{u}=\frac{4\cosh\left[\frac{\alpha}{16}\right]\cosh\theta}{1+\coth^2\theta\sinh^2\left[\frac{\alpha}{16}\right]} , \tag{2.10a}$$
$$\frac{\alpha}{16}=(t-\tau)\sinh\theta \ ,$$

and for the breather case,

$$u=4\tan^{-1}\left[\tan\theta^b\sin\left[\frac{\alpha^b}{16}\right]\right] ,$$
$$\tilde{u}=\frac{4\cos\left[\frac{\alpha^b}{16}\right]\sin\theta^b}{1+\tan^2\theta^b\sin^2\left[\frac{\alpha^b}{16}\right]} , \tag{2.10b}$$
$$\frac{\alpha^b}{16}=(t-\tau)\cos\theta^b \ .$$

The inverse relationships can be computed explicitly. Notice from (2.2) and (2.5) that the temporal evolution of the parameters $(\theta,\alpha)$ is trivial in both cases: For the kink-antikink case,

$$\alpha_t=16\sinh\theta \ ,$$
$$\theta_t=0 \ , \tag{2.11a}$$

and for the breather case,

$$\alpha_t^b=16\cos\theta^b \ ,$$
$$\theta_t^b=0 \ . \tag{2.11b}$$

For the temporal evolution of the parameters $(u,\tilde{u})$, we consider each case separately. First we take the kink-antikink case and use the transformation formula (2.10a), the temporal evolution (2.11a), and the inverse of transformation (2.10a) to obtain

$$u_t=\tilde{u} \ ,$$
$$\tilde{u}_t=-\sin u-\tan\frac{u}{4}\left[\frac{\tilde{u}^2}{4}+\sin^2\frac{u}{2}\right] . \tag{2.12}$$

Then we perform the same calculation in the breather case using formulas (2.10b) and (2.11b), and obtain the identical expression (2.12).

Parameters $(u,\tilde{u})$ and the one time evolution (2.12) apply to both the kink-antikink and breather cases. We did

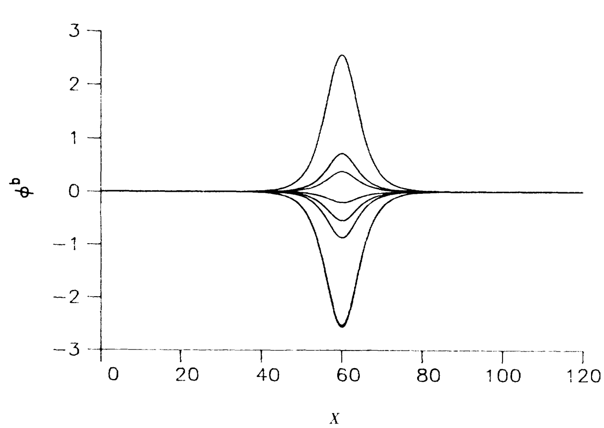

FIG. 2. Oscillations of a breather.

not expect such uniformity. Indeed, that is why we calculated the time evolution (2.12) separately for each case. In retrospect this uniformity could have been anticipated from reduction formalism in Hamiltonian mechanics, but the result remains the same: parameters $(u,\tilde{u})$ with the one time evolution given by (2.12) describe both the kink-antikink and breather cases.

## III. REDUCED PHASE SPACE: GLOBAL COORDINATES

In this section we present in some detail a global description of the kink-antikink breather reduced phase space. First, some preliminaries: (i) From the definition (2.9), restricted to the kink-antikink and breather configurations we see that $u$ ranges from $(-2\pi,2\pi)$, while $\tilde{u}$ from $(-\infty,+\infty)$. (ii) In terms of the $(u,\tilde{u})$ parameters, the energy is given by

$$\mathcal{H}(u,\tilde{u})=\frac{8}{\cos(u/4)}\left[\frac{\tilde{u}^2}{4}+\sin^2\frac{u}{2}\right]^{1/2} \tag{3.1}$$

and ranges from $(0,+\infty)$. (iii) The kink-antikink states are separated from the breather configurations by the $\mathcal{H}=16$ level curve(s):

$$\tilde{u}=\pm 4\cos^2\frac{u}{4}\quad(\mathcal{H}=16)\ . \tag{3.2}$$

Inside the separatrix $(0\leq\mathcal{H}<16)$ the motion is periodic and represents the breather family; on the separatrix $(\mathcal{H}=16)$, the phase point relaxes as $t\to\infty$ toward the points $(u=\pm2\pi,\tilde{u}=0)$; the kink-antikink configuration resides outside the separatrix $(\mathcal{H}>16)$. A numerically generated graph of these level curves of $\mathcal{H}$ is depicted in Fig. 3. If one focuses on the fact that $(u,\tilde{u})$ are the position and velocity of the field configuration $\phi^{k\text{-}a}(x,t)$ at the midpoint (or center of mass) $x=0$, one realizes that the phase point

$$(u(t),\tilde{u}(t))\to(\pm2\pi,0)\quad\text{as } t\to\pm\infty \tag{3.3}$$

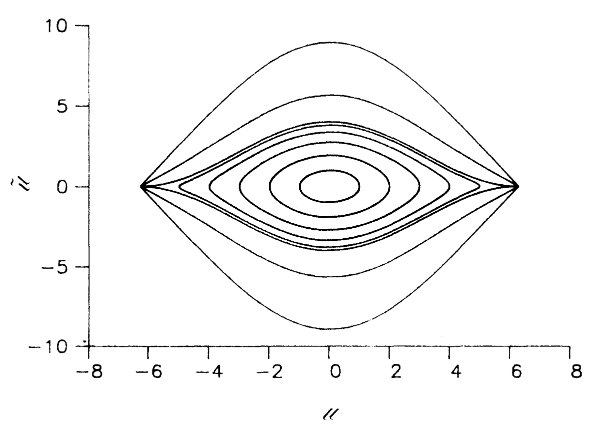


FIG. 3. Phase space of breather-kink (antikinks); level curves of the Hamiltonian $H$. Note the separatrix, the oscillatory breather states, and the kink-antikink states. Note also the singular point.

for all members of the kink antikink family. [The middle pendulum at $x=0$ is in the down configuration $(\phi=-2\pi)$ long before the collision, experiences a $(+4\pi)$ twist during the interaction, and relaxes back to the down position $(\phi=+2\pi)$ long after the collision.] Although physically apparent, the motion of the middle pendulum $[\phi(x=0)]$ misses another physically obvious fact; namely, that as $t\to\pm\infty$, the kink and the antikink carry (translational) energy. Indeed, members of the kink-antikink family are distinguished by the asymptotic value of the separation speed $v$. The way by which the coordinate $(u,\tilde{u})$ detects this asymptotic separation speed is that the Hamiltonian function $\mathcal{H}$ in (3.1) is not defined on the lines $u=\pm2\pi$. Even at the points $(\pm2\pi,0)$ $\mathcal{H}$ is undetermined. In fact, each level curve $\mathcal{H}=\mathcal{H}_c>16$ approaches the singular points $(\pm2\pi,0)$. The angle of approach measures the asymptotic velocity. Clearly, the limit points of the trajectories must be included in the phase space. At this stage in the construction they are not, and the phase space (as depicted in Fig. 3) is

$$(u,\tilde{u})\in(-2\pi,2\pi)\times(-\infty,\infty)\ . \tag{3.4}$$

In order to include these limit points one uses a process known geometrically as "blowing up." In this process one completes the phase space (3.4) by replacing each singular point $(\pm2\pi,0)$ with a line called a "distinguished fiber." The physical information carried by the two distinguished fibers is the asymptotic velocities, together with the kink-antikink or antikink-kink nature of the collision; the geometric information which they carry is the angle at which each level curve $\mathcal{H}=\mathcal{H}_c>16$ approaches the singular points. In this manner the global phase space for both the kink-antikink and breather is the blow-up given by the coordinate patch $u=z-2\pi$, $\tilde{u}=wz$, with

$$\begin{aligned} z_t&=wz\ ,\\ w_t&=-\frac{\sin z}{z}\\ &\quad-\tan\left[\frac{z-2\pi}{4}\right]\left[\frac{w^2z}{4}+\frac{1}{z}\sin^2\left[\frac{z-2\pi}{2}\right]\right]-w^2\ . \end{aligned} \tag{3.5}$$

Dynamical system (2.12) is then extended to a Hamiltonian system on the "blown-up" phase space. Note that, by (3.5), the distinguished fiber $(z=0,w=w)$ is a line of fixed points.

## IV. CONNECTION TO THE DOUBLE PENDULUM

In this section we describe an interesting connection between the $(u,\tilde{u})$ parametrization of the breather-kink (antikink) states and the double pendulum.[1] We begin with the dynamical system (2.12),

$$\begin{aligned} u_t&=\tilde{u}\ ,\\ \tilde{u}_t&=-\sin u-\tan\frac{u}{4}\left[\frac{\tilde{u}^2}{4}+\sin^2\frac{u}{2}\right]\ , \end{aligned} \tag{4.1}$$

which has energy invariant (3.1),

$$\mathcal{H}(u,\tilde{u})=\frac{8}{\cos(u/4)}\left[\frac{\tilde{u}^2}{4}+\sin^2\frac{u}{2}\right]^{1/2} . \tag{4.2}$$

Using this invariant, we rewrite system (4.1),

$$u_t=\tilde{u} ,$$
$$\tilde{u}_t=-\sin u-\frac{\lambda}{2}\sin\frac{u}{2} , \tag{4.3}$$

with $\lambda=\mathcal{H}^2/64$. The last equation (4.3) is that of the double pendulum.

The correspondence between the dynamics of the breather-kink (antikink) states and the double pendulum deserves some discussion. The correspondence is between orbits of (4.1) and orbits for a family of double pendula, indexed by the coupling strength $\lambda=\mathcal{H}^2/64\geq 0$. More precisely, one begins with a level curve of $\mathcal{H}$ in the phase space $(u,\tilde{u})$, and associates to it one double pendulum with coupling strength fixed by $\mathcal{H}$ at $\lambda=\mathcal{H}^2/64$; for this particular double pendulum, there is one level curve of its Hamiltonian $\mathcal{H}_D$,

$$\mathcal{H}_D(u,\tilde{u})=\frac{\tilde{u}^2}{2}-\cos u-\lambda\cos\frac{u}{2},\quad \lambda=\frac{\mathcal{H}^2}{64} ,$$

which is the original level curve of $\mathcal{H}$.

Thus we are led to consider a family of double pendula indexed by coupling strength $\lambda$. The potential energies for members of this family are sketched in Fig. 4. In the "breather sector," $0<\lambda<4$ $(0<\mathcal{H}<16)$, the orbitals are oscillatory [see Fig. 4(a)]. On the separatrix, $\lambda=\mathcal{H}^2/64=4$, and the potential is sketched in both Figs. 4(a) and 4(b). The threshold orbit which separates kink-antikink states from breather states is also their separatrix. The kink-antikink sector has $\lambda=\mathcal{H}^2/64>4$; the potentials are sketched in Fig. 4(b). Notice that each kink-antikink state remains associated to a separatrix, in each case a separatrix for the double pendulum at coupling strength $\lambda=\mathcal{H}^2/64$. We emphasize that in this double pendula description, the breather to kink-antikink transition occurs at a bifurcation at $(\lambda=4)$ in the critical point structure of the family of potentials (see Fig. 4).

One of us (M.S.) has built a mechanical analog for the double pendulum, with two pendula coupled by gears.[2] When one watches the oscillations of this mechanical analog, one is actually watching the vibrations of a breather.

We close this section by recording the analytical formulas for the separatrix, which is the threshold between the kink-antikink and breather states:

$$\mathcal{H}(u,\tilde{u})=16 \Longrightarrow \tilde{u}=\pm 4\cos^2\frac{u}{4} , \tag{4.4}$$

$$u=4\tan^{-1}t , \tag{4.5}$$
$$\tilde{u}=\frac{4}{1+t^2} .$$

## V. REDUCED PHASE SPACE IN THE PRESENCE OF PERTURBATIONS

In this section we will add the physical effects of dissipation and external forcing. These effects will be incorporated with the mathematical formalism, and approximations, of soliton perturbation theory.[3–5]

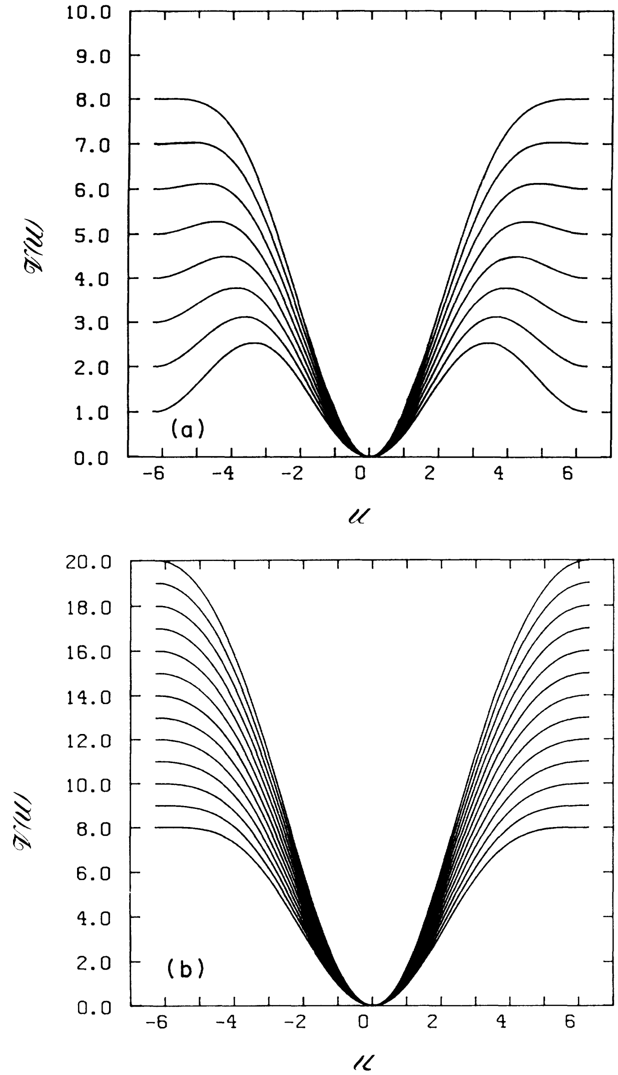


FIG. 4. Potentials for the associate double sine-Gordon equation. (a) The breather case. (b) The kink-antikink case.

We consider the perturbed sine-Gordon equation,

$$\phi_{tt}-\phi_{xx}+\sin\phi=\epsilon[f(t)-\gamma\phi_t],\quad 0<\epsilon\ll 1 \tag{5.1}$$

and ask "how do members of the breather (or kink-antikink) family respond to weak dissipation and driving?" The answer is now well known. Predominantly two effects occur: (1) the parameters of the soliton adjust dynamically, and (2) additional degrees of freedom (such as radiation) are generated. Both types of effects play important roles. It is dangerous to ignore radiation, particularly when studying chaotic behavior of the perturbed PDE (4.1). Nevertheless, we shall do so and focus upon the adjustment of the soliton's parameters in an approximation which neglects any dependence of the soliton on the generated radiation.

The formalism of soliton perturbation theory begins by seeking a solution of (4.1) in the form

$$\phi^{\epsilon}=\phi^{b}+\epsilon\tilde{\phi},\quad 0<\epsilon\ll 1 \tag{5.2}$$

where

$$\phi^{b}=4\tan^{-1}\left[\frac{\xi(t)}{\cosh(x\sin\theta^{b})}\right], \tag{5.3}$$

where

$$\theta^{b}=\theta^{b}(t),$$

$$\xi_t=\sin\theta^{b}\cos\left[\frac{\alpha^{b}(t)}{16}\right].$$

With a projection formalism,[3-5] one derives equations of motion for $(\theta,\phi)=(\theta^{b},\alpha^{b})$:

$$\theta_t=-\epsilon f\frac{\pi}{4}\frac{\cos\frac{\alpha}{16}}{\cos\theta\left[1+\tan^2\theta\sin^2\frac{\alpha}{16}\right]^{1/2}}-\frac{\epsilon\gamma\tan\theta\cos^2\frac{\alpha}{16}}{\left[1+\tan^2\theta\sin^2\frac{\alpha}{16}\right]}\left[1+\frac{\cot\theta\sinh^{-1}\left[\tan\theta\sin\frac{\alpha}{16}\right]}{\sin\frac{\alpha}{16}\left[1+\tan^2\theta\sin^2\frac{\alpha}{16}\right]}\right], \tag{5.4a}$$

$$\alpha_t=16\cos\theta-\epsilon f\frac{4\pi}{\sin\theta}\left[\frac{\sin\frac{\alpha}{16}}{\cos^2\theta\left[1+\tan^2\theta\sin^2\frac{\alpha}{16}\right]^{1/2}}-\cot\theta\sinh^{-1}\left[\tan\theta\sin\frac{\alpha}{16}\right]\right]$$
$$-\frac{\epsilon\gamma}{\cos\theta}\left[\frac{\sin\frac{\alpha}{16}\cos\frac{\alpha}{16}}{\cos\theta\left[1+\tan^2\theta\sin^2\frac{\alpha}{16}\right]}+\frac{\sin\theta\sinh^{-1}\left[\tan\theta\sin\frac{\alpha}{16}\right]}{\left[1+\tan^2\theta\sin^2\frac{\alpha}{16}\right]^{3/2}}\cos^3\frac{\alpha}{16}\right]. \tag{5.4b}$$

Using (2.10b), we change variables from $(\theta^{b},\alpha^{b})\to(u,\tilde{u})$:

$$u_t=\tilde{u}+\epsilon\frac{\tilde{u}\cos^2\frac{u}{4}}{4XY}\left\{4\pi f\left[8\sin\frac{u}{4}\cos^2\frac{u}{4}+Y\sinh^{-1}\left[\tan\frac{u}{4}\right]\right]\right.$$
$$\left.+\gamma\tilde{u}\cos^2\frac{u}{4}\left[17\tan\frac{u}{4}+\sinh^{-1}\left[\tan\frac{u}{4}\right]\left[16\cos^2\frac{u}{4}+\frac{\left[\frac{\tilde{u}}{4}\right]^2}{\cos^3\frac{u}{4}}\right]\right]\right\}, \tag{5.5a}$$

$$\tilde{u}_t=-\sin u-\tan\frac{u}{4}\left[\frac{\tilde{u}^2}{4}+\sin^2\frac{u}{2}\right]$$
$$-\epsilon\frac{\cos\frac{u}{4}}{X}\left\{\pi f\left[\frac{\tilde{u}^2}{4}-\sin^2\frac{u}{2}\right]+4\pi f\frac{\frac{\tilde{u}^2}{4}\sin\frac{u}{4}}{Y}\left[2\sin\frac{u}{4}-\frac{Y^2}{4}\sinh^{-1}\left[\tanh\frac{u}{4}\right]+2\sin\frac{u}{2}\cos\frac{u}{4}X^{1/2}\right]\right.$$
$$-\frac{16\gamma}{Y}\frac{\tilde{u}}{4}\cos^3\frac{u}{4}\left[2\left[\frac{\tilde{u}}{4}\right]^2\frac{4\cos^4\frac{u}{4}-\sin^2\frac{u}{2}-X}{\sin\frac{u}{2}\cos^2\frac{u}{4}}\left[\tan\frac{u}{4}+\cos^2\frac{u}{4}\sinh^{-1}\left[\tan\frac{u}{4}\right]\right]\right.$$
$$\left.\left.-\tfrac{1}{4}\tan\frac{u}{4}\left[\left[\frac{\tilde{u}}{4}\right]^2+\cos^4\frac{u}{4}\right]\left[\tan\frac{u}{4}+\frac{\left[\frac{\tilde{u}}{4}\right]^2\sinh^{-1}\left[\tan\frac{u}{4}\right]}{\cos^3\frac{u}{4}}\right]\right]\right\}, \tag{5.5b}$$

where the factors $X$ and $Y$ are given by

$$X \equiv \left[\frac{\tilde{u}}{2}\right]^2 + \sin^2\frac{u}{2} ,$$

$$Y \equiv \left[\frac{\tilde{u}}{2}\right]^2 - 4\cos^4\frac{u}{4} .$$

Equations (5.5) are the reduced equations for the breather sector. The same reduced equations apply in the kink-antikink sector. In the case of no dissipation ($\gamma=0$) we have derived equations (5.4) using a Hamiltonian canonical formalism. The results of this calculation agree with those in the literature. We have also compared this case with sample numerical calculations of the perturbed *pde*, obtaining good agreement. We did not check the dissipative term against the expressions in the literature, primarily because the behavior of the reduced model should be rather insensitive to the details of the dissipative term.

## VI. MELNIKOV FUNCTION FOR THE REDUCED PERTURBED SYSTEM

In this section we use the Melnikov method to predict the occurrence of chaos in the perturbed reduced system derived in the preceding section, via the Smale-Birkoff theorem.[6] As is well known the method consists in computing the Melnikov function

$$M(t_0) = \int_{-\infty}^{\infty} \mathbf{F}[u(t-t_0)] \wedge \mathbf{X}[u(t-t_0)]dt \tag{6.1}$$

along the homoclinic orbit of the unperturbed system. In (6.1) $\mathbf{F}$ denotes the perturbation vector, $\mathbf{X}$ is the vector field, and $\wedge$ is the usual wedge product. The simple zeros of $M$ (as function of $t_0$) indicate the occurrence of transverse homoclinic intersections which in turn implies the existence of a hyperbolic set (Smale horseshoe) in phase space. To simplify the analysis we consider the reduced system (5.5a) and (5.5b) in the presence only of an ac driver of the type $f=\sin(\omega t)$ (the extension to the case $\gamma\neq 0$ is straightforward). To this end we rewrite system (5.5a) and (5.5b) in the more compact form:

$$\begin{aligned} u_t &= \tilde{u} + \epsilon F_1(u,\tilde{u}) , \\ \tilde{u}_t &= -\sin u - \tan\frac{u}{4}\left[\frac{\tilde{u}^2}{4} + \sin^2\frac{u}{2}\right] \\ &\quad + \epsilon\sin(\omega t) F_2(u,\tilde{u}) , \end{aligned} \tag{6.2}$$

with $\mathbf{F}\equiv(F_1,F_2)$ easily obtained from (5.5a) and (5.5b). The Melnikov function is then computed as

$$M(t_0) = \epsilon[P(u,\tilde{u})\cos t_0 + R(u,\tilde{u})\sin t_0] , \tag{6.3}$$

where

$$\begin{aligned} P(u,\tilde{u}) &= \int_{-\infty}^{\infty} \sin(\omega t) Q(u,\tilde{u}) dt , \\ R(u,\tilde{u}) &= \int_{-\infty}^{\infty} \cos(\omega t) Q(u,\tilde{u}) dt , \end{aligned} \tag{6.4}$$

with $Q(u,\tilde{u})$ given by

$$\begin{aligned} Q(u,\tilde{u}) &= F_1(u,\tilde{u})\left[-\sin u - \tan\frac{u}{4}\left[\frac{\tilde{u}^2}{4} + \sin^2\frac{u}{2}\right]\right] \\ &\quad - F_2(u,\tilde{u})\tilde{u} . \end{aligned} \tag{6.5}$$

Equation (6.3) shows that $M$ has simple zeros at

$$t_0 = \frac{1}{\omega}\tan^{-1}\frac{P(u,\tilde{u})}{R(u,\tilde{u})} \tag{6.6}$$

and therefore the reduced system has chaotic trajectories in phase space.

## VII. CONCLUSION

In this paper we have presented global coordinates for the kink-antikink breather phase space with a separatrix and singular points. A correspondence to the double sine-Gordon equation has been established. Using these coordinates we have derived reduced equations governing a perturbed dynamics and used them in a Melnikov calculation to establish the presence of horseshoe in the reduced system.

In conclusion we feel compelled to emphasize two warnings about collective coordinate reductions.

(1) It can be very dangerous to neglect radiation, particularly when studying bifurcation toward chaos. An $x$-independent background is certainly generated by the homogeneous driver and is frequently very important in the resulting dynamics. Long-wavelength instabilities of this background can inject additional spatial radiation modes into the field, whose interaction with each other and with the original collective modes yields chaos. The background may interact parametrically with the collective modes producing stable oscillations. Such phenomena will never appear in reduced system (5.5a) and (5.5b), because both background and radiation modes are not a part of the ansatz for the field.

(2) It is safest to use soliton perturbation theory when the perturbation is both weak and slowly varying in time. Although it is certainly possible to develop a theory valid for a rapidly oscillating perturbation, to our knowledge this has not been done mathematically.

Of course, the ansatz can be modified to include these additional effects. Our point here is that, in any such modification, coordinates (2.10) provide a global representation of the breather-kink (antikink) component of the field. After such modifications, systems such as (5.5) can be used for both analytical and numerical studies.

## ACKNOWLEDGMENTS

One of us (M.S.) gratefully acknowledges support from the Center for Nonlinear Studies at Los Alamos Scientific Laboratory where this work was initiated. We also acknowledge informative conversations with N. Ercolani on "blow up" geometry.

[1]M. Salerno, Phys. Lett. A **134**, 421 (1989).
[2]M. Salerno, Physica D **17**, 227 (1985).
[3]D. Kaup and A. C. Newell, Proc. R. Soc. London, Ser. A **361**, 413 (1978).
[4]D. W. McLaughlin and A. C. Scott, Phys. Rev. A **18**, 1652 (1978).
[5]V. I. Karpmann and E. M. Maslov, Zh. Eksp. Teor. Fiz. **73**, 537 (1977) [Sov. Phys.—JETP **46**, 281 (1977)]; **74**, 1569 (1978) [**47**, 819 (1978)].
[6]See, for example, J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields,* Vol. 42 of *Applied Mathematical Sciences* (Springer-Verlag, Berlin, 1983).