Generalized hydrodynamics approach to the Knudsen problem

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It is shown that the Knudsen problem ("paradox") can be resolved by the flow-rate formula for a non-Newtonian fluid flowing under a pressure gradient in a circular tube, as reported elsewhere. The flow-rate formula exhibits a minimum in the low-pressure regime but gives the Hagen-Poiseuille linear dependence on pressure in the high-pressure regime. Since the generalized hydrodynamic equations used for the derivation of the flow-rate formula are derived from the Boltzmann equation, the present solution of the Knudsen problem is well founded on the kinetic theory of gases. The nonlinear transport processes, which increasingly manifest themselves as the gas density decreases, are the cause for the emergence of a minimum in the flow rate for rarefied gases in circular tube flow. The entropy production and also the drag coefficient are calculated as a function of pressure difference, Reynolds number, and other parameters characteristic of the system.

I. INTRODUCTION

Influenced by the Kundt-Warburg experiment,[1] which provided insight into the nature of gases in rarefied conditions, Knudsen[2] performed various experiments, and in 1909 reported an experimental investigation[3] into the validity of the Hagen-Poiseuille volume flow rate for rarefield gases flowing in a long circular tube. Through the experiment he discovered that, although the volume flow rate of the gases investigated follows the prediction[4] by the well-known Hagen-Poiseuille velocity profile when the mean pressure is in the range of normal gas pressure, it does not vanish as the mean pressure approaches to zero, but increases, thereby exhibiting a minimum at a low pressure. See Fig. 1 in which Knudsen's original figure is reproduced. In fact, he was able to fit his data for the volume flow rate per unit pressure difference Q_k to the following empirical formula:[3]

$$Q_k = ap + b\frac{1+c_1 p}{1+c_2 p} , \tag{1}$$

where

$$a = \pi R^4/8L\eta_0 ,$$

$$b = 4\sqrt{2\pi}R^3/3L\sqrt{\rho_1} ,$$

and c_1 and c_2 are numerical constants that depend on the nature of the molecule comprising the gas, with R and L denoting the radius and length of the circular tube, η_0 the shear viscosity of the gas, and ρ_1 the specific density of the gas at temperature T when the pressure is equal to 1 dyn/cm^2, and p the mean pressure. The first term on the right-hand side of (1) is the Hagen-Poiseuille law prediction and the second term is the one giving rise to the aforementioned minimum since it yields a finite value of Q_k which is higher than the value of the first term as p approaches zero. Knudsen obtained the parameter b by using the cosine law for scattering off the surface by the molecules, but b, c_1, and c_2 may be treated as adjustable parameters. Because of the second term in (1) Q_k has a minimum at a nonzero value of p, and this phenomenon is referred to as the Knudsen paradox in the rarefied gas dynamics literature.[5] Since it is really not a paradox from the viewpoint of a broader hydrodynamic theory, we will refer to it as the Knudsen problem. The presence of such a minimum was later confirmed by Gaede.[6]

Much later, the Knudsen problem was studied theoretically in the case of plane Poiseuille flow.[7,8] Since experiments were performed in a circular-tube flow geometry, it was not possible to compare the aforementioned theoretical results with experimental results, but their results predicted the existence of a minimum in the volume flow rate. The theoretical analyses in Refs. 7 and 8 are made with the Bhatnagar-Gross-Krook (BGK) kinetic equation[9] adapted to the plane Poiseuille flow geometry and subjected to the specular and reflective boundary conditions leading to slip boundary conditions.[5] There is, however, no theoretical study available for the problem in a circular-tube flow geometry. In this connection we remark that the cylindrical coordinates necessary for description of circular-tube flow do not yield a kinetic equation in a sufficiently simple form that is easily amenable to mathematical analysis.

In this paper we report on a simple analytic result for the volume flow rate for a rarefied gas flowing in a circular tube subject to a pressure gradient. We take an approach that applies generalized hydrodynamic equations developed previously. In previous papers[10,11] on generalized hydrodynamics there are reported the velocity profiles and volume flow rate of a fluid flowing in plane[10] and circular-tube[11] flow geometry when the constitutive equation for shear stress is non-Newtonian, namely, nonlinear with respect to the shear stress. Such constitutive equations for stresses and other macroscopic variables have been derived from the Boltzmann equation by means of the modified moment method.[12] This is a method for solving the Boltzmann equation or other kinetic equations in a way consistent with the thermodynamic laws so that the macroscopic properties predict-

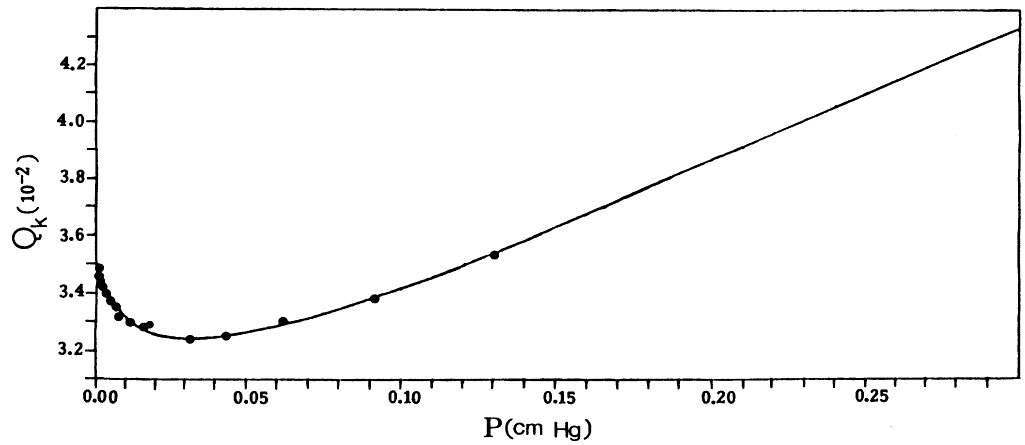

FIG. 1. Reproduction of Knudsen's data on CO_2.

ed by the evolution equations so obtained are in conformation with the thermodynamic laws. In a series of papers[13,14] on rarefied gas dynamics, which are in many ways related to the present study, it is shown that the constitutive equation for the shear stress used here gives rise to correct flow profiles and effective transport coefficients. Therefore the constitutive equation used here is not only well founded on the kinetic theory of gases but also known to be reliable in describing transport and fluid dynamic properties of dilute gases, and, consequently, the volume flow-rate results obtained therewith should be interesting to examine in connection with the Knudsen problem. We also calculate the entropy production associated with the flow and the drag coefficient as a function of pressure difference, Reynolds number, aspect ratio, and molecular parameters characteristic of the gas of interest. Since the analysis leading to the volume flow-rate formula is reported elsewhere, we shall present only the final results except for a brief summary of the equations involved so that the paper is self-contained and the reader can work out the results with ease.

II. GENERALIZED HYDRODYNAMICS AND THE VOLUME FLOW RATE

We assume that a fluid is *laminarly* flowing in a circular tube of length L and radius R, subject to a longitudinal pressure gradient. The pressure difference between the entrance and exit of the tube is denoted by $\Delta p \equiv p_i - p_f$ where p_i and p_f are the pressure at the entrance and exit of the tube, respectively. The fluid is maintained at a constant uniform temperature and therefore *there is no heat flux*. We also assume that *the normal stress differences are negligible*. Since there is axial symmetry in the system, the appropriate coordinates are cylindrical coordinates which we denote by (r,θ,z), the direction of flow being parallel to the z axis. Since the length of the tube is assumed sufficiently long so that the end effects are negligible, the flow properties, and in particular, the fluid velocity components, are independent of z. Moreover, they are independent of the angle θ because of the axial symmetry. The flow is assumed to be in the positive direction of z, the angular component u_θ of the velocity $\mathbf{u}$ is equal to zero, and the radial velocity u_r is also equal to zero for the following reason.

Since we are interested in a steady flow, the steady-state equation of continuity is

$$\nabla\cdot\rho\mathbf{u}=0 , \tag{2}$$

which in the cylindrical coordinates takes the form

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho u_\theta)+\frac{\partial}{\partial z}(\rho u_z)=0 . \tag{3}$$

Since the density ρ and velocity components do not depend on z and θ, we obtain from (3)

$$\frac{\partial}{\partial r}(\rho r u_r)=0 ,$$

or on integration

$$\rho r u_r=\text{const} .$$

Since $u_r=0$ at the boundary $r=R$ (tube wall), we conclude that $u_r=0$ everywhere since $\rho\neq0$. Since $u_\theta=0$, we conclude that

$$\mathbf{u}=(0,0,u_z) .$$

Let us denote the traceless symmetric part of stress tensor by $\mathbf{\Pi}$ and the normal stress differences by

$$N_1=\Pi_{zz}-\Pi_{rr},\quad N_2=\Pi_{rr}-\Pi_{\theta\theta} , \tag{4}$$

where Π_{rr}, etc., are the normal components of $\mathbf{\Pi}$. We will denote the (rz) component of $\mathbf{\Pi}$ by Π

$$\Pi\equiv\Pi_{rz}=\Pi_{zr} , \tag{5}$$

which is the relevant shear stress since all other off-diagonal components are equal to zero for the present flow problem. Then, the momentum balance equations are

$$-\frac{\partial p}{\partial r}-\frac{1}{3r}\frac{\partial}{\partial r}r(N_2-N_1)-\frac{1}{3r}(2N_2+N_1)=0\ , \quad (6)$$

$$-\frac{\partial p}{\partial z}-\frac{1}{r}\frac{\partial}{\partial r}r\Pi=0\ . \quad (7)$$

We emphasize that (6) and (7) are, respectively, the exact r and z component of the steady-state momentum balance equation for the problem. Since the normal stress differences are assumed to be negligible, setting $N_1=N_2=0$, we obtain from (6)

$$\frac{\partial p}{\partial r}=0\ ,$$

and thus the pressure is seen to be independent of r. Since we also assume that the pressure gradient in the z direction is constant, we find

$$p=-\frac{\Delta p}{L}z+p_i\ . \quad (8)$$

The shear stress Π can be shown to obey the equation[12,14,17]

$$\Pi q_e(\Pi)=2\eta_0\gamma\ , \quad (9)$$

where

$$q_e(\Pi)=\sinh\kappa/\kappa\ , \quad (10)$$

$$\kappa=\tau\Pi/\eta_0\ , \quad (11)$$

$$\tau=[2\eta_0(m_r k_B T/2)^{1/2}]^{1/2}/\sqrt{2}n k_B T\sigma\ , \quad (12)$$

$$\gamma=-2^{-1}(\partial u_z/\partial r)\ , \quad (13)$$

with η_0 denoting the Chapman-Enskog (Newtonian) shear viscosity of the gas, m_r and σ the reduced mass and the size parameter of the molecule, respectively, and n the number density. The constitutive equation (9) is obtained from the steady-state evolution equation for Π by setting $N_1=N_2=0$ for the present flow geometry. The evolution equations for stress tensors Π_{rz},Π_{zz}, etc., are derived from the Boltzmann equation by using the modified moment method,[12] which makes sure that the thermodynamic laws are fully satisfied by the constitutive equations derived from the kinetic equation. The factor q_e enters into the theory because in the modified moment method[12] the collision term in the Boltzmann equation is computed in a cumulant expansion which is equivalent to a resummation of an infinite series in the Knudsen number of the distribution function. Since the Chapman-Enskog[15] series may be considered a series in Knudsen number, the q_e factor compensates for the inadequacy of the first-order Chapman-Enskog solution for the distribution function which is the first-order term in the Knudsen number series mentioned. In this connection it must be mentioned that the first-order Chapman-Enskog solution gives rise to the constitutive equation (9) for Π with $q_e=1$ and, consequently, the Navier-Stokes equation at the hydrodynamic level. Note that the latter is the hydrodynamic equation used to obtain the Hagen-Poiseuille velocity profile shown below (Q_{HP}). The constitutive equation (9) has been tested for various problems in rheology[16] and fluid dynamics[17] in the past. Therefore its kinetic theory and experimental foundations are firm.

The velocity component u_z can be determined from the pair of equations (7) and (9). By using (8) and (9) in (7), we can easily find the velocity profile

$$u_z(r)=(R/\tau\delta)[\cosh\delta-\cosh(\delta r/R)] \quad (14)$$

subject to the boundary conditions

$$u_z(R)=0\ ,\qquad (\partial u_z/\partial r)_{r=R}=0\ . \quad (15)$$

We note that because of the second condition in (15) the shear stress Π is given by $\Pi=r\Delta p/2L$. This follows from (7). In (14)

$$\delta=\tau R\,\Delta p/2L\eta_0\ . \quad (16)$$

The velocity profile (14) reduces to the well-known Hagen-Poiseuille velocity profile

$$u_z(r)=\frac{\Delta p}{4L\eta_0}(R^2-r^2) \quad (17)$$

as the parameter δ gets small.

The number of particles flowing in the tube in unit time is given by the formula

$$Q=2\pi n\int_0^R dr\, r u_z(r)\ . \quad (18)$$

By substituting the velocity profile (14) and performing the integration, we obtain from (18)

$$\begin{aligned}Q&=n(2\pi/\tau\delta)[\tfrac{1}{2}\cosh\delta+\delta^{-2}(\cosh\delta-1)-\delta^{-1}\sinh\delta]\\&=\frac{n\pi R^4\Delta p}{8L\eta_0}(1+\Delta Q)\\&\equiv Q_{\mathrm{HP}}(1+\Delta Q)p\end{aligned} \quad (19)$$

where

$$\begin{aligned}\Delta Q&=8\delta^{-2}[\tfrac{1}{2}\cosh\delta+\delta^{-2}(\cosh\delta-1)-\delta^{-1}\sinh\delta]-1\\&=4\delta^{-2}\{\sinh^2(\delta/2)+[\cosh(\delta/2)\\&\qquad-2\delta^{-1}\sinh(\delta/2)]^2\}-1\end{aligned} \quad (20)$$

$$Q_{\mathrm{HP}}=\frac{\pi R^4\Delta p}{8L\eta_0\Re T}\ ,$$

where $\Re$ is the gas constant and we have used the equation of state $p=n\Re T$. Note that Q_{HP} is the Hagen-Poiseuille formula[4] for the volume flow rate apart from the factor $\Re T$ that appears since Q is the number of particles flowing through the tube of radius R per unit time, but not the volume flow per unit time. [Note that Q_{hp} results from (6) with $N_1=N_2=0$, (7) and (9) with $q_e=1$, i.e., the Navier-Stokes equation.] To understand the behavior of Q as p is varied, we now return to the definitions of parameters τ and δ. Since τ is inversely proportional to $p=n\Re T$ [see (12) above], so is δ. Therefore ΔQ vanishes as $\delta\to 0$ and thus Q becomes a linear function of p in the limit. This is the prediction by the Hagen-Poiseuille velocity profile, which also predicts that

Q vanishes at $p=0$. Experiments by Knudsen[2,3] and Gaede[6] show that Q does not vanish at $p=0$, as pointed out in Sec. I. Since δ is inversely proportional to p, the former diverges as $p \to 0$, and so does Q:

$$Q \sim p \exp(\delta_0/p) \quad \text{as} \quad p \to 0 \tag{21}$$

where δ_0 is such that

$$\delta_0 = p\delta = \tau_0 R \Delta p / 2L\eta_0 ,$$
$$\tau_0 = [2\eta_0(m_r k_B T/2)^{1/2}]^{1/2}/\sqrt{2}\sigma .$$

Therefore it is easy to infer that there exists a minimum in Q at some value of p. The location of the minimum is given by

$$\frac{d}{d\delta}\{\delta^{-3}[\tfrac{1}{2}\cosh\delta + \delta^{-2}(\cosh\delta - 1) - \delta^{-1}\sinh\delta]\} = 0 ,$$

which yields the transcendental equation for δ_m at the minimum

$$\delta_m(10+\delta_m^2)\sinh\delta_m - 5(2+\delta_m^2)\cosh\delta_m + 10 = 0 . \tag{22}$$

Thus the pressure at the minimum is given by

$$p_m = \delta_0/\delta_m , \tag{23}$$

where δ_m is the solution of (22). There are two solutions: $\delta_m=0$ and approximately 2.30. The latter is the value of interest to us. The parameter δ depends on Δp, temperature, viscosity, and other molecular parameters as well as the aspect ratio R/L. Especially the Δp dependence is significant since the flow rate depends sensitively on the applied pressure difference (or gradient) as well as the mean pressure. Knudsen reduced the flow rate with Δp and presented the data without stating the Δp values in his paper[3] and monograph.[2] This absence of Δp values is understandable since he, when judged from his interpolation formula (1) which does not include Δp in it, appeared to have thought that the flow rate is linear with respect to Δp and thus felt that Q_k is the same for all values of Δp. Contrary to his formula (1), the present flow-rate formula (19) depends on Δp. Although Gaede's data[6] do not cover the post-minimum region of pressure, being confined to the low-pressure region, he gives the pressure value at the entrance of the tube in the case of hydrogen. Since his formula contains a logarithmic singularity we, however, find it difficult to match it with the exponential singularity of the present theory and get a meaningful value of the desired pressure value. For this reason a quantitative comparison with experiment is found to be impossible with the kind of data available. Under these circumstances we are compelled to just verify the existence of a minimum in terms of dimensionless parameter δ, unless we arbitrarily guess the parameter δ_0. Guessing the value of δ_0 is equivalent to assuming a value of Δp. Since there seems to be some merit in this manner of comparison, by taking $\delta_0=0.07$ and 0.1, in Figs. 2 and 3 we have compared for CO_2 the flow rates Q_k and $\bar{Q} \equiv Q\Re T/\Delta p$, where Q is given by (19), calculated as a function of p in units of cm Hg. The parameters a, b, c_1 and c_2 have the following values in units commensurate to the cm Hg units for pressure:

$$a = 0.048\,80/\text{cm Hg}, \quad b = 0.034\,89 ,$$
$$c_1 = 43.13/\text{cm Hg}, \quad c_2 = 53.10/\text{cm Hg} .$$

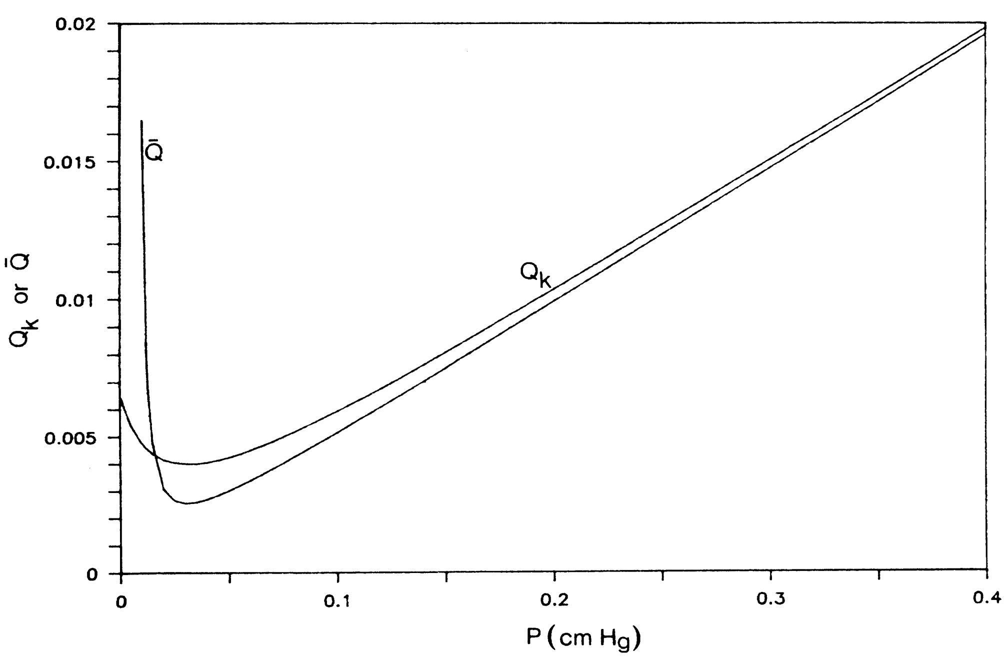

FIG. 2. Reduced flow rate (Q_k and $\bar{Q}$) vs pressure (in units of cm Hg). $\delta_0=0.07$.

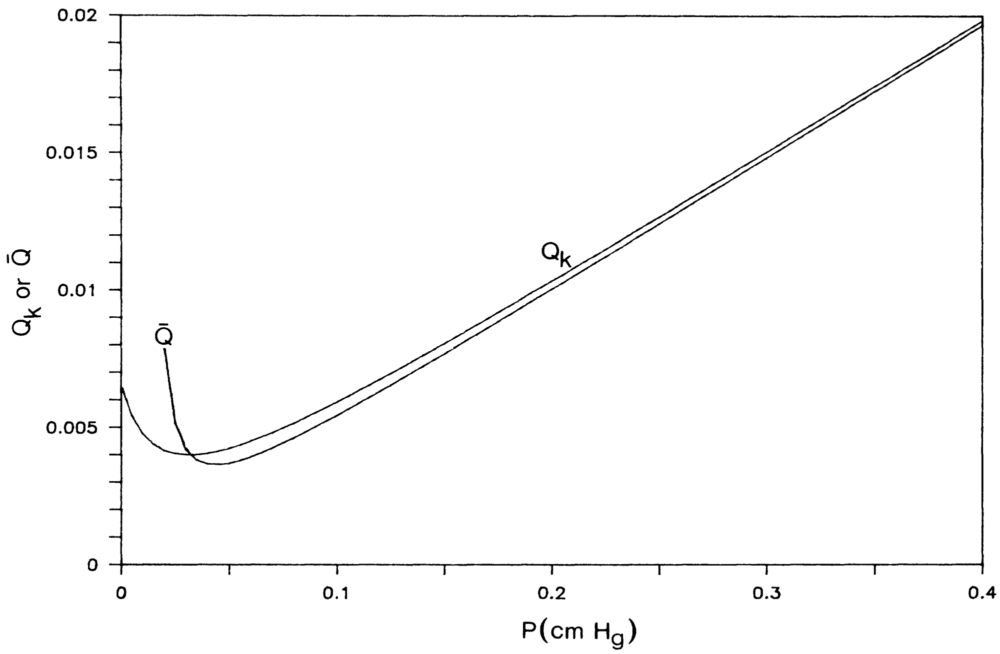

FIG. 3. Reduced flow rate (Q_k and $\bar{Q}$) vs pressure (in units of cm Hg). $\delta_0=0.1$.

See Ref. 3 for the source of these parameters which are here converted into the units indicated. The prediction by the formula (19) clearly shows a minimum at a small value of p. When the factor bc_1/c_2 is subtracted from (1), the two formulas (1) and (19) in the large-p limit not only give a linear p dependence in agreement with the Hagen-Poiseuille prediction, but also numerically coincide with each other. Therefore (19) is an interpolation formula covering the whole pressure range. It must be noted that Q_k in Figs. 2 and 3 are, in fact, the experimental data since Q_k is calculated from the interpolation formula (1) proposed by Knudsen[3] who fitted his data to it. Note also that a better comparison could have been achieved in Figs. 2 and 3 if an optimum value of Δp were chosen for fitting, but the present author does not consider such a fitting meaningful because the limitations arising from the lack of experimental information on the Δp values may make such a nice looking fitting misleading. Both theory and experiment may need improvement after all.

The basic reason for the appearance of the minimum is in the fact that the effective viscosity diminishes as the parameter δ increases, and thereby the gas becomes increasingly less frictional, as the gas density (or the mean pressure) decreases. The Navier-Stokes equation does not have a mechanism for a diminishing viscosity, and the absence of such a mechanism is the reason for the Hagen-Poiseuille result being in variance with experiment at low pressure. The mathematical mechanism for a diminishing effective viscosity is provided by the nonlinear factor q_e appearing in the constitutive equation (9) for Π, which gives rise to an effective viscosity vanishing[11] in the boundary layer that gets thinner as δ increases or the density decreases. We have already noted how this nonlinear factor enters the present generalized hydrodynamic theory. We find this factor plays many subtle and important roles in fluid dynamics when the system is far from equilibrium. Some of its effects are reported in references already cited.[14,16,17]

The entropy production associated with the flow can be calculated with the formula[12,14]

$$\sigma_{\rm ent}=k_B g^{-1}\kappa\sinh\kappa \ , \tag{24}$$

where

$$g=(m_r/2k_BT)^{1/2}/(n\sigma)^2 \ .$$

Since $\Pi=r\Delta p/2L$, there follows from (11)

$$\kappa=\delta\xi \ ,$$

where $\xi=r/R$, and therefore we obtain

$$\sigma_{\rm ent}=S_0\delta^{-1}\xi\sinh(\delta\xi) \ , \tag{25}$$

where

$$S_0=k_B\frac{\Delta p^2R^2}{4L^2\eta_0k_BT} \ .$$

The entropy production in (25) is at position ξ. A more interesting quantity is the global entropy production which is obtained by integrating (25) over the volume of a unit height of the cylinder. We define reduced global entropy production by the formula

$$\Sigma=2\pi S_0^{-1}\int_0^R dr\, r\sigma_{\rm ent}(r) \ .$$

This is easily calculated to be

$$\Sigma=4\pi R^2\delta^{-2}[\tfrac{1}{2}\cosh\delta+\delta^{-2}(\cosh\delta-1)-\delta^{-1}\sinh\delta] \ , \tag{26}$$

which is seen proportional to the flow rate Q; see (19). Thus we see that the energy is dissipated in direct proportion to the number of gas molecules drawn out of the tube.

Another quantity of interest is the drag coefficient at the tube wall. It gives a measure of the mean kinetic energy dissipated into frictional heat at the surface of a body over which the flow occurs. It is defined by the formula

$$C_d = \frac{2\pi R L \Pi(R)}{2\pi R L \frac{1}{2}\rho \langle u_z \rangle^2} , \quad (27)$$

where $\langle u_z \rangle$ is a mean velocity. In the present case of flow it may be defined by

$$\langle u_z \rangle = 2\pi \int_0^R dr\, r u_z(r) / \pi R^2 .$$

This is the usual definition of the mean velocity used in connection with the drag coefficient for tube flow. Note that this mean velocity is proportional to the flow rate Q already computed.

The formula (27) can be easily computed by using the shear stress and velocity profiles obtained earlier. It may be given in the following form:

$$C_d = \mathcal{R}^{-1} / F(\delta) , \quad (28)$$

for which we have used the maximum velocity at $r=0$

$$u_m = u_z(0) = (R/\tau\delta)(\cosh\delta - 1) ,$$

and have defined the Reynolds number $\mathcal{R}$ therewith

$$\mathcal{R} = \rho u_m R / \eta_0 .$$

The function $F(\delta)$ is then given by

$$F(\delta) = 2[\tfrac{1}{2}\cosh\delta + \delta^{-2}(\cosh\delta - 1) - \delta^{-1}\sinh\delta]^2 / \delta^2(\cosh\delta - 1) , \quad (29)$$

which is equal to $\frac{1}{16}$ at $\delta = 0$ and positive for $\delta > 0$. It increases exponentially as p decreases, and thus the drag coefficient decays like $\exp(-\delta)$ as $p \to 0$. We thus see that C_d agrees with the drag coefficient for the Hagen-Poiseuille flow in the small δ limit $C_{dHP} = 16/\mathcal{R}$. A detailed analysis[14] of the parameter δ shows that it is proportional to the Reynolds number $\mathcal{R}$ and the Knudsen number. Therefore the drag coefficient is a nonlinear function of $\mathcal{R}$. In Fig. 4, $C_d^* \equiv \mathcal{R} C_d$ is plotted against the mean pressure: it vanishes exponentially as p decreases and thus indicates that the kinetic energy of flow is dissipated less and less into viscous friction energy or, put in another term, heat, as the mean pressure diminishes. This vanishing of the drag coefficient at low pressure is helpful for understanding the singular behavior of Q at $p=0$ since the kinetic energy of the flow is, on the average, increasingly less dissipated into frictional heat at the wall in the low-pressure limit where the molecules tend to move in parallel to the tube wall under a pressure gradient without a significant number of collisions with either the wall or other molecules. Thus the rarefied gas acts under a pressure gradient as if it is a collimated beam moving along the tube. This seems to account for the singular flow rate, although the latter appears to be in variance with the experimental results of Knudsen.

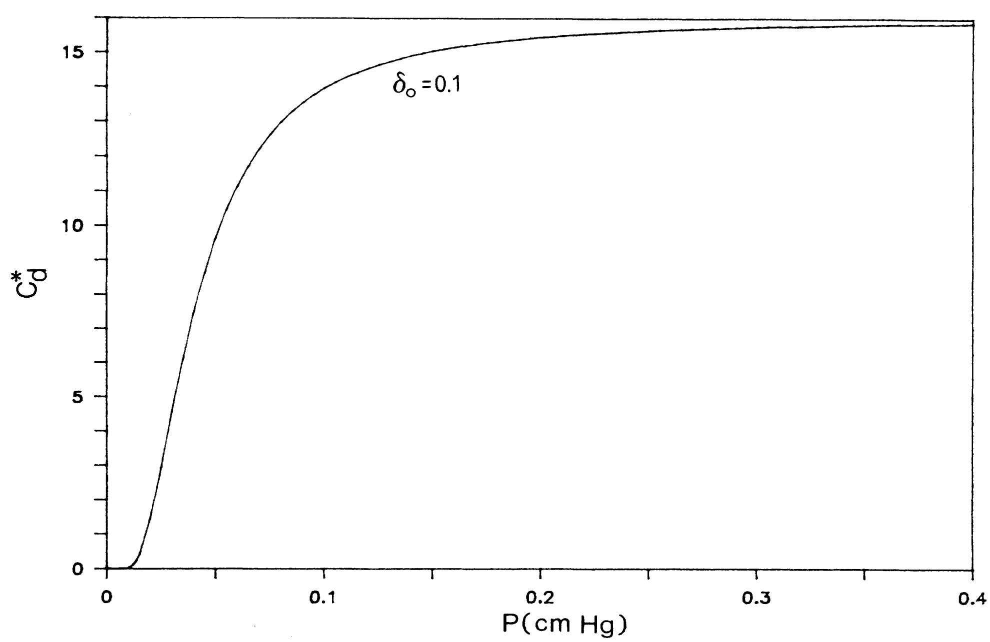

FIG. 4. Reduced drag coefficient $C_d^* = \mathcal{R} C_d$ vs pressure (in units of cm Hg) for $\delta_0 = 0.1$. The reduced drag coefficient for the case of $\delta_0 = 0.07$ is almost identical with the one presented in this figure.

III. DISCUSSION AND CONCLUSION

The resolution of the Knudsen problem presented in this work is achieved by a completely hydrodynamical method since the generalized hydrodynamic equations are used for calculating the flow rate of a gas in a circular tube. In generalized hydrodynamics we approach to flow properties of rarefied gases with the viewpoint that the unusual behavior exhibited by rarefied gases is a manifestation, at the hydrodynamic and thus macroscopic level, of nonlinear transport processes inherent to rarefied gases and the stick boundary conditions are adequate, provided that the transport processes are taken into consideration to a sufficiently nonlinear degree. This viewpoint is in contrast to the one taken in the conventional approach[5,7,8] in rarefied gas dynamics that attributes the unusual behavior of rarefied gases to slip phenomena arising from the specular and reflective scattering of molecules off the walls which become more manifest in the low-density regime. We have shown in the previous papers[12,14] on applications of generalized hydrodynamics to rarefied gas dynamics problems that there can occur thin boundary layers because of nonlinear transport processes and, as a consequence, the flow profiles may appear to be exhibiting a slip or jump behavior, especially if experimental resolutions are not of high quality. The generalized hydrodynamics calculation presented in this paper is done with the same viewpoint and spirit as for the previous calculations mentioned earlier.[10-12,14] Moreover, it shows that the present approach provides a flow-rate formula that can account for the existence of a minimum observed experimentally.

By solving the BGK (linearized) kinetic equation by a set of approximations Takao[7] and Cercignani[8] showed that in the case of plane Poiseuille flow there appears a minimum in the flow rate versus pressure curve, but there has not been a theoretical result for Poiseuille flow in a circular tube that exhibits such a minimum. Takao's method uses a special kind of moment method which yields linear differential equations for velocities and stress tensors to first order in nonuniformity defined by $\epsilon \sim \Pi / p$, and solves the differential equations to obtain the volume flow rate in terms of quadratures. On the other hand, Cercignani[8] reduces the BGK equation to an integral equation for the velocity and thus for the volume flow rate, which he solves by iteration. The approximate results obtained by both of them are quite complicated and does not reveal the Knudsen number dependence of the volume flow rate as simply as (19). More importantly, the leading approximations for their volume flow-rate formulas are found to yield the zero Knudsen number limit expressions that are significantly different from the volume flow-rate formula in the Hagen-Poiseuille theory, namely, Q_{HP} presented earlier, although the Hagen-Poiseuille formula must be recovered as the limiting formula in the limit of vanishing Knudsen number or δ. The inability of the Takao and Cercignani theories, at least in their leading approximations as presented in their papers,[7,8] to yield the Hagen-Poiseuille limit formula probably indicates the quality of the approximations and perhaps the methods used in their analysis of the problem. The present theory is rather simple and free from such a weakness.

There is, however, a common feature shared by the Takao, Cercignani, and present theory: that is, the divergent flow rate as $p \rightarrow 0$. Knudsen's data show that Q is finite at $p = 0$, but Cercignani's result[5,8] shows a logarithmic singularity while Takao's and the present results give an exponential singularity. Note also that Gaede fitted his low-pressure data to a logarithmic function of p. The theoretical variances with experiment (especially Knudsen's) are not possible to account for at present, but the $p = 0$ limit of Q appears to require removal of some of assumptions made for the present calculation, which is expected to result in more complicated equations not possible to solve analytically.

The present result not only exhibits a minimum in flow rate characteristic of the Knudsen problem for the experimental flow geometry, namely, the circular-tube flow within the framework of hydrodynamics, but also thereby demonstrates once more the utility of generalized hydrodynamic equations obtained by the modified moment method for the Boltzmann equation and the generalized Boltzmann equation.[18] The latter aspect is potentially more interesting than the resolution of the Knudsen problem itself since generalized hydrodynamics extends the scope of classical (Navier-Stokes-Fourier) hydrodynamics into highly nonlinear regimes of processes for which the density regime of rarefied gases is an important example. We remark that the generalized hydrodynamic equations are completely consistent with the thermodynamic laws and within the framework of the theory of extended irreversible thermodynamics reported previously.[18,19]

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