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Sojourn time, sojourn time operators, and perturbation theory for one-dimensional scattering by a potential barrier

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We show that a useful concept of the time spent by a quantum-mechanical particle in a given spatial region can be expressed in terms of a Hermitian sojourn time operator. Mean values of this operator give the mean sojourn time (dwell time) of the particle. We show that the sojourn time operators occur in a natural way in first-order perturbation theory for the barrier perturbed by any finite-range potential. The perturbed S operator, and also the effect of the perturbation on the change of observables due to the scattering, are, to a first-order approximation, fully expressible in terms of sojourn time operators. We anticipate that this result can be generalized to the case when the translational motion of the tunneling particle is weakly coupled to additional degrees of freedom. As a specific example of such a system, we explicitly reconsider the Larmor clock, originally proposed as a device measuring interaction times. We show that the change of the spin components, to a first-order approximation, is indeed fully expressible in terms of a sojourn time operator. In particular, when set as a clock, the system measures mean values of the sojourn time operator or real parts of some of its matrix elements.

I. INTRODUCTION

Let Ψ be a wave function describing an initial state of a quantum-mechanical particle with Hamiltonian H. If Ω is an arbitrary spatial region and (t_1,t_2) an arbitrary time interval (with $-\infty \le t_1 < t_2 \le \infty$) then the quantity

$$\tau(\Omega,t_1,t_2;\Psi)=\int_{t_1}^{t_2}dt\int_{\Omega}dx\,|[\exp(-itH)\Psi](x)|^2 \tag{1.1}$$

is usually interpreted as the mean sojourn time of the particle in Ω during the interval (t_1,t_2) (we put $\hbar=1$). Similarly, the quantity

$$\tau'(x,t_1,t_2;\Psi)=\int_{t_1}^{t_2}dt|[\exp(-itH)\Psi](x)|^2 \tag{1.2}$$

can be interpreted as the mean local sojourn time at x, i.e., the mean sojourn time per unit volume. Clearly,

$$\tau(\Omega,t_1,t_2;\Psi)=\int_{\Omega}dx\,\tau'(x,t_1,t_2;\Psi)\ . \tag{1.3}$$

If Ψ is an eigenstate of the Hamiltonian H, then

$$\tau(\Omega,t_1,t_2;\Psi)=(t_2-t_1)\int_{\Omega}|\Psi(x)|^2dx\ .$$

A more interesting situation arises when Ψ is a scattering state and Ω a bounded region containing the scattering center. Then $\tau(\Omega,-\infty,\infty;\Psi)$ can be finite and gives a measure of the duration of the collision. The time delay can be rigorously defined as, essentially, the difference between two sojourn times—one for the interacting particle and one for the reference free particle. The standard sojourn time based definition refers, in the case of two-body scattering,[1] to the delay averaged over all scattering angles, and in the case of many-body scattering to the delay averaged over all angles and all channels.[2] Recently, we have shown how the sojourn time definition can be extended to include separate time delays for transmission and reflection in one-dimensional scattering by a potential barrier.[3] In the context of temporal aspects of one-dimensional scattering the sojourn time (1.1), usually referred to as dwell time in the literature on tunneling times,[4–6] has also been recognized as relevant.

In this paper we want to further elucidate the role played by the sojourn times (1.1) and (1.2) in one-dimensional scattering. It seems that considering insight into the nature of these quantities can be gained by introducing Hermitian sojourn time operators $T(\Omega,t_1,t_2)$ and $T'(x,t_1,t_2)$ defined in the subspace of scattering states by the requirements

$$\tau(\Omega,t_1,t_2;\Psi)=\langle\Psi|T(\Omega,t_1,t_2)\Psi\rangle \tag{1.4}$$

and

$$\tau'(x,t_1,t_2;\Psi)=\langle\Psi|T'(x,t_1,t_2)\Psi\rangle\ , \tag{1.5}$$

respectively. We will show that the operators $T(\Omega,-\infty,\infty)$ and $T'(x,-\infty,\infty)$ occur in a natural way in the first-order perturbation theory for the scattering problem with the Hamiltonian

$$H(\lambda)=-\tfrac{1}{2}\frac{d^2}{dx^2}+V(x)+\lambda M(x)\ ,$$

where $\lambda M(x)$ is the perturbation potential. If $S(\lambda)$ is the corresponding scattering operator, then in the first-order approximation we have $S(\lambda)\approx S(0)+\lambda\partial S/\partial\lambda(0)$, and our result is

$$iS^\dagger\frac{\partial S}{\partial\lambda}=\int_{-\infty}^{\infty}M(x)\Omega_+^\dagger T'(x,-\infty,\infty)\Omega_+dx\ , \tag{1.6}$$

where Ω_+ is the Moller operator. Moreover, if A is an observable which is a constant of motion with respect to free (asymptotic) motion, then the change of its expectation value due to the scattering reads

$$\langle \Delta A \rangle_{\Psi^{\rm in}}(\lambda) = \langle S(\lambda)\Psi^{\rm in} | AS(\lambda)\Psi^{\rm in} \rangle - \langle \Psi^{\rm in} | A\Psi^{\rm in} \rangle$$
$$= \langle \Delta A \rangle_{\Psi^{\rm in}}(0) - i\lambda \left\langle \Psi^{\rm in} \left| \left[S^\dagger(0)AS(0), iS^\dagger(0)\frac{\partial S}{\partial \lambda}(0) \right] \right| \Psi^{\rm in} \right\rangle + O(\lambda^2) . \quad (1.7)$$

We believe Eqs. (1.6) and (1.7) can be generalized to more complicated scattering systems, in particular to the case when the translational degree of freedom of the particle is weakly coupled to some additional degrees of freedom, e.g., to the internal degrees of freedom of the particle, and the coupling is treated as a perturbation. An easily tractable example of such a system is the Larmor clock, or spin clock, proposed as a device measuring tunneling times.[4,6,7] We reconsider this example from a more general perspective. Here the additional degree of freedom is the spin $\frac{1}{2}$ of the particle, which is coupled to the translational motion via a homogeneous magnetic field extending over a bounded spatial region. In this case an appropriate generalization of (1.6) and (1.7) is straightforward and allows one to compute changes of the spin components. In particular, when set as a clock the system actually measures mean values of the sojourn operator (1.4) or real parts of some of its matrix elements. In general, the example shows that a generally useful concept of interaction times is probably not expressible directly in terms of numbers, real or complex,[8] but rather in the language of linear operators.

It should be realized that the origin and interpretation of formula (1.1) in quantum mechanics is not as self-evident as it may appear at first sight. The difficulty is that formula (1.1) cannot be derived in the way one can derive its analog in classical mechanics or, more generally, in the theory of stochastic processes. These derivations explicitly rely on the existence of a trajectory of the particle. The concept of the sojourn time operator was originally introduced by Ekstein and Siegert[9] in connection with the theory of decay of unstable states. They constructed the sojourn time operator $T(\Omega,t_1,t_2)$ as a quantum-mechanical image of the corresponding classical trajectory-dependent quantity [this construction is equivalent to (1.4)]. Both the sojourn time and the sojourn time operator have rather unexpected properties. For example, when the particle moves freely in $n \geq 2$ dimensions (i.e., $H = -\frac{1}{2}\sum_{i=1}^{n} \partial^2/\partial x_i^2$), and when Ω is a bounded region then the sojourn time $\tau(\Omega,-\infty,\infty;\Psi)$ is bounded with a bound independent of the initial state Ψ of the particle (provided $\|\Psi\|=1$). Correspondingly, the sojourn time operator $T(\Omega,-\infty,\infty)$ is bounded. One can also generally show that the usual Born probability interpretation of the spectral measure associated with the sojourn time operator yields rather strange, if not paradoxical results resembling the well-known quantum-mechanical Zeno paradox.[10] These questions as well as some of the mathematical aspects of sojourn time operators are discussed in Ref. 11. In the present context, however, they are not of immediate concern, since here we consider the sojourn time and the sojourn time operator rather from the point of view of their potential usefulness in scattering theory.

II. SCATTERING THEORY; UNPERTURBED SYSTEM

We consider a one-dimensional quantum-mechanical system with Hilbert space $\mathcal{H}=L^2(\mathbb{R})$ and the Hamiltonian $H=-\frac{1}{2}d^2/dx^2+V(x)$. The potential $V(x)$ is assumed to be constant outside a bounded interval (a,b), i.e., $V(x)=V_1$ for $x \leq a$ and $V(x)=V_2$ for $x \geq b$. By $\mathcal{H}_b \subseteq L^2(\mathbb{R})$ we denote the subspace of all bound states of H. We put $\hbar=1$ throughout.

We introduce two "free" Hamiltonians

$$H_0^{\rm in} = -\frac{1}{2}\frac{d^2}{dx^2} + V_1F_+ + V_2F_- ,$$
$$H_0^{\rm out} = -\frac{1}{2}\frac{d^2}{dx^2} + V_1F_- + V_2F_+ , \quad (2.1)$$

where F_+ and F_- are projections onto positive and negative momenta, respectively. The Hamiltonian $H_0^{\rm in}$ describes the asymptotic time evolution of our system for $t \to -\infty$, while $H_0^{\rm out}$ describes the asymptotic time evolution for $t \to \infty$. More precisely, $H_0^{\rm in}$ and $H_0^{\rm out}$ define the Moller operators

$$\Omega_+ = s - \lim_{t \to -\infty} \exp(itH)\exp(-itH_0^{\rm in}) ,$$
$$\Omega_- = s - \lim_{t \to -\infty} \exp(itH)\exp(-itH_0^{\rm out}) . \quad (2.2)$$

These operators map $\mathcal{H}=L^2(\mathbb{R})$ isometrically onto the orthogonal complement $\mathcal{H}_b^\perp$ of $\mathcal{H}_b$. Every state vector $\Psi \in \mathcal{H}_b^\perp$ represents a scattering state, i.e., possesses in and out asymptotes $\Psi^{\rm in}$ and $\Psi^{\rm out}$, $\Omega_+\Psi^{\rm in}=\Psi=\Omega_-\Psi^{\rm out}$,

$$\lim_{t \to -\infty} \|\exp(-itH)\Psi - \exp(-itH_0^{\rm in})\Psi^{\rm in}\| = 0$$
$$= \lim_{t \to +\infty} \|\exp(-itH)\Psi - \exp(-itH_0^{\rm out})\Psi^{\rm out}\| . \quad (2.3)$$

The asymptotes $\Psi^{\rm in}$ and $\Psi^{\rm out}$ are related by the scattering operator $S=\Omega_-^\dagger\Omega_+$, $\Psi^{\rm out}=S\Psi^{\rm in}$, which is a unitary operator from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.

It is convenient to introduce a (two-valued) energy representation employing the Hilbert space $L^2((0,\infty),\mathbb{C}^2)$ which consists of pairs $F=(f_1,f_1)$ of square integrable functions defined on the half line $(0,\infty)$. The scalar product of $F=(f_1,f_2)$ and $G=(g_1,g_2)$ is given by

$$\langle F|G\rangle = \int_0^\infty [f_1^*(E)g_1(E)+f_2^*(E)g_2(E)]dE , \quad (2.4)$$

and the unitary correspondence $U:L^2(\mathbb{R}) \to L^2((0,\infty),\mathbb{C}^2)$ between the usual position representation and our energy representation reads

$$(U\Psi)(E) = (2E)^{-1/4}(\hat{\Psi}(\sqrt{2E}),\hat{\Psi}(-\sqrt{2E})), \quad E>0 \quad (2.5)$$

where $\hat{\Psi}$ denotes the Fourier transform (momentum representation) of Ψ. Correspondingly,

$$[U^{-1}(\Phi_1,\Phi_2)](x)=(2\pi)^{-1/2}\left[\int_0^\infty dk\sqrt{k}\exp(ikx)\Phi_1(k^2/2)+\int_{-\infty}^0 dk\sqrt{-k}\exp(ikx)\Phi_2(k^2/2)\right]$$
$$=\int_0^\infty dE\,\varepsilon_E^+(x)\Phi_1(E)+\int_0^\infty dE\,\varepsilon_E^-(x)\Phi_2(E)\ , \tag{2.6}$$

where

$$\varepsilon_E^\pm(x)=(2\pi)^{-1/2}(2E)^{-1/4}\exp(\pm ix\sqrt{2E}\,) \tag{2.7}$$

are the two linearly independent continuous spectrum eigenfunctions of the free Hamiltonian $-\frac{1}{2}d^2/dx^2$, normalized so that

$$\int_{-\infty}^\infty dx\,\varepsilon_E^i(x)^*\varepsilon_{E'}^j(x)=\delta_{ij}\delta(E-E')\quad (i,j=+,-)\ . \tag{2.8}$$

In the above energy representation, the free Hamiltonian $-\frac{1}{2}d^2/dx^2$ acts simply as multiplication by E. The momentum operator p and the projections $F_\pm$ take the form

$$p(\Phi_1,\Phi_2)(E)=\sqrt{2E}\,(\Phi_1(E),-\Phi_2(E))\ ,$$
$$F_+(\Phi_1,\Phi_2)=(\Phi_1,0)\ , \tag{2.9}$$
$$F_-(\Phi_1,\Phi_2)=(0,\Phi_2)\ .$$

Hence for the asymptotic Hamiltonians H_0^{in} and H_0^{out} we have

$$H_0^{\text{in}}(\Phi_1,\Phi_2)(E)=((E+V_1)\Phi_1(E),(E+V_2)\Phi_2(E))\ ,$$
$$H_0^{\text{out}}(\Phi_1,\Phi_2)(E)=((E+V_2)\Phi_1(E),(E+V_1)\Phi_2(E))\ . \tag{2.10}$$

The state vectors $(\Phi_1,0),(0,\Phi_2)\in L^2(\mathbb{R},\mathbb{C}^2)$ describe states (wave packets) with positive and negative momentum, respectively, cf. (2.6) and (2.7). If the actual state of the system at $t=0$ has an in asymptote of the form (in the energy representation) $\Psi^{\text{in}}=(\Phi_1^{\text{in}},0)$, then this means that the particle (wave packet) approaches the potential barrier from the left before colliding with it. Long before the collision the time evolution is essentially the free time evolution

$$(\exp[-it(E+V_1)]\Phi_1^{\text{in}}(E),\,0)$$

determined by H^{in}. Long after the collision the time evolution is essentially the free time evolution

$$(\exp[-it(E+V_2)]\Phi_1^{\text{out}}(E),\exp[-it(E+V_1)]\Phi_2^{\text{out}}(E))$$

determined by H^{out} and the out asymptote $\Psi^{\text{out}}=(\Phi_1^{\text{out}},\Phi_2^{\text{out}})=S\Psi^{\text{in}}$. Ψ^{out} is the superposition $\Psi^{\text{out}}=F_+\Psi^{\text{out}}+F_-\Psi^{\text{out}}=(\Phi_1^{\text{out}},0)+(0,\Phi_2^{\text{out}})$ of states corresponding to the particle moving to the right (transmitted) and to the left (reflected). The probabilities of transmission and reflection are

$$\langle\Psi^{\text{out}}|F_+\Psi^{\text{out}}\rangle=\int_0^\infty dE|\Phi_1^{\text{out}}(E)|^2\ ,$$
$$\langle\Psi^{\text{out}}|F_-\Psi^{\text{out}}\rangle=\int_0^\infty dE|\Phi_2^{\text{out}}(E)|^2\ ,$$

respectively.

The Moller operators $\Omega_\pm$ and the scattering operator S can be expressed in terms of the continuous spectrum eigenfunctions of the Hamiltonian $H=-\frac{1}{2}d^2/dx^2+V$. These are solutions of the differential equation

$$\frac{d^2}{dx^2}h(x)=2[V(x)-\mu]h(x)\ , \tag{2.11}$$

where $\mu>\min(V_1,V_2)$, $\mu\neq\max(V_1,V_2)$. We define solutions $g_\mu^\pm$ and $f_\mu^\pm$ by the requirements

$$g_\mu^\pm(x)=\varepsilon_{\mu-V_1}^\pm(x)=(2\pi)^{-1/2}[2(\mu-V_1)]^{-1/4}\times\exp[\pm ix\sqrt{2(\mu-V_1)}]\quad\text{for } x\le a\ ,$$
$$f_\mu^\pm(x)=\varepsilon_{\mu-V_2}^\pm(x)=(2\pi)^{-1/2}[2(\mu-V_2)]^{-1/4}\times\exp[\pm ix\sqrt{2(\mu-V_2)}]\quad\text{for } x\ge b\ . \tag{2.12}$$

It is to be understood that $\sqrt{-|E|}=i\sqrt{|E|}$, $(-|E|)^{1/4}=|E|^{1/4}\exp(i\pi/4)$.

The functions $g_\mu^\pm$ constitute a pair of linearly independent solutions of (2.11) and so do the functions $f_\mu^\pm$. With $W(h_1,h_2)$ denoting the Wronskian, one has $W(f_\mu^+,f_\mu^-)=W(g_\mu^+,g_\mu^-)=-i/\pi$, and $W(f_\mu^+,g_\mu^-)\neq 0$. One can thus write

$$\sigma_{11}(\mu)f_\mu^+=g_\mu^++\sigma_{21}(\mu)g_\mu^-\ , \tag{2.13}$$

$$\sigma_{22}(\mu)g_\mu^-=f_\mu^-+\sigma_{12}(\mu)f_\mu^+\ , \tag{2.14}$$

where

$$\sigma_{11}(\mu)=\sigma_{22}(\mu)=\sigma(\mu)=[\pi iW(f_\mu^+,g_\mu^-)]^{-1}\ ,$$
$$\sigma_{21}(\mu)=W(f_\mu^+,g_\mu^-)^{-1}W(g_\mu^+,f_\mu^+)\ , \tag{2.15}$$
$$\sigma_{12}(\mu)=W(f_\mu^+,g_\mu^-)^{-1}W(g_\mu^-,f_\mu^-)\ .$$

We note that the matrix $\sigma_{ij}(\mu)$ is unitary for $\mu>\max(V_1,V_2)$, while for $\min(V_1,V_2)<\mu<\max(V_1,V_2)$ we have $|\sigma_{12}(\mu)|=|\sigma_{21}(\mu)|=1$.

Using the solutions $f_\mu^\pm$ and $g_\mu^\pm$, the Moller operators can be expressed as follows:

$$[\Omega_+ U(\Phi_1,\Phi_2)](x)=\int_0^\infty dE\{[g^+_{V_1+E}(x)+\sigma_{21}(V_1+E)g^-_{V_1+E}(x)]\Phi_1(E)+[f^-_{V_2+E}(x)+\sigma_{12}(V_2+E)f^+_{V_2+E}(x)]\Phi_2(E)\}\ , \tag{2.16}$$

$$[\Omega_- U(\Phi_1,\Phi_2)](x)=\int_0^\infty dE\{[f^-_{V_2+E}(x)+\sigma_{12}(V_2+E)f^+_{V_2+E}(x)]^*\Phi_1(E)+[g^+_{V_1+E}(x)+\sigma_{21}(V_1+E)g^-_{V_1+E}(x)]^*\Phi_2(E)\}\ . \tag{2.17}$$

The scattering operator satisfies $\Omega_- S=\Omega_+$, and it can now be directly checked that in our energy representation S acts as follows:

$$[S(\Phi_1,\Phi_2)](E)=(S_{11}(E)\Phi_1(E+V_2-V_1)+S_{12}(E)\Phi_2(E),S_{21}(E)\Phi_1(E)+S_{22}(E)\Phi_2(E+V_1-V_2))\ , \tag{2.18}$$

where the $S_{ij}(E)$ matrix is given by

$$\begin{aligned} S_{11}(E)&=\begin{cases}\sigma(E+V_2) & \text{for } E+V_2>V_1\ ,\quad S_{12}(E)=\sigma_{12}(E+V_2)\\ 0 & \text{otherwise,}\end{cases}\\ S_{22}(E)&=\begin{cases}\sigma(E+V_1) & \text{for } E+V_1>V_2\ ,\quad S_{21}(E)=\sigma_{21}(E+V_1)\\ 0 & \text{otherwise}\cdot\end{cases}\end{aligned} \tag{2.19}$$

Similarly, for the adjoint $S^\dagger$ we have

$$[S^\dagger(\Phi_1,\Phi_2)](E)=(S_{11}(E)^\dagger\Phi_1(E+V_1-V_2)+S^\dagger_{12}(E)\Phi_2(E),S^\dagger_{21}(E)\Phi_1(E)+S_{22}(E)\Phi_2(E+V_2-V_1))\ , \tag{2.20}$$

with

$$\begin{aligned} S^\dagger_{11}(E)&=\begin{cases}\sigma^*(E+V_1) & \text{for } E+V_1>V_2\ ,\quad S^\dagger_{12}(E)=\sigma^*_{21}(E+V_1)\\ 0 & \text{otherwise,}\end{cases}\\ S^\dagger_{22}(E)&=\begin{cases}\sigma^*(E+V_2) & \text{for } E+V_2>V_1\ ,\quad S^\dagger_{21}(E)=\sigma^*_{12}(E+V_2)\\ 0 & \text{otherwise,}\end{cases}\end{aligned} \tag{2.21}$$

III. SCATTERING THEORY: PERTURBED SYSTEM

Let $M(x)$ be a potential vanishing outside a bounded interval (α,β). We want to study first-order perturbation theory for scattering by the perturbed Hamiltonian $H(\lambda)=H+\lambda M$, where λ is a small real parameter. Without loss of generality we can assume that $(\alpha,\beta)\subseteq(a,b)$.

Clearly, the mathematical apparatus outlined in Sec. I is applicable to the Hamiltonian $H(\lambda)$. Now the Moller operators, the scattering operator, and the continuous spectrum eigenfunctions are dependent on λ. First-order perturbation theory means that we want to find the derivative of the scattering operator $\partial S(\lambda)/\partial\lambda$ evaluated at $\lambda=0$. This derivative will be abbreviated $\partial S/\partial\lambda$. We will also write S for $S(0)$, H for $H(0)$, $f^\pm_\mu, g^\pm_\mu$ for the eigenfunctions of $H(\lambda)$ at $\lambda=0$, etc., and $\partial/\partial\lambda$ will always denote the partial derivative evaluated at $\lambda=0$.

Explicitly, the operator $\partial S/\partial\lambda$ is described in the energy representation as follows:

$$\left[\frac{\partial S}{\partial\lambda}(\Phi_1,\Phi_2)\right](E)=\left[\frac{\partial S_{11}(E)}{\partial\lambda}\Phi_1(E+V_2-V_1)+\frac{\partial S_{12}(E)}{\partial\lambda}\Phi_2(E),\frac{\partial S_{21}(E)}{\partial\lambda}\Phi_1(E)+\frac{\partial S_{22}(E)}{\partial\lambda}\Phi_2(E+V_1-V_2)\right]\ , \tag{3.1}$$

cf. (2.18).

Since the S matrix is fully expressible in terms of the Wronskians $W(f^i_\mu,g^j_\mu)$ (with $i,j=+,-$), the problem amounts to computation of the derivatives $(\partial/\partial\lambda)W(f^i_\mu,g^j_\mu)$. This can be accomplished using standard methods of the theory of second-order ordinary differential equations (see the Appendix). We obtain

$$\frac{\partial}{\partial\lambda}W(f^i_\mu,g^j_\mu)=2\int_{-\infty}^{\infty}d\xi\, M(\xi)f^i_\mu(\xi)g^j_\mu(\xi)=2\int_\alpha^\beta d\xi\, M(\xi)f^i_\mu(\xi)g^j_\mu(\xi)\ . \tag{3.2}$$

The above, via (2.15) and (2.19), directly yield

$$\frac{\partial S_{11}(E)}{\partial\lambda}=\begin{cases}-2\pi i\sigma^2(E+V_2)\int_\alpha^\beta d\xi\, M(\xi)f^+_{E+V_2}(\xi)g^-_{E+V_2}(\xi) & \text{when } E+V_2>V_1\ ,\\ 0 & \text{otherwise}\ ,\end{cases} \tag{3.3}$$

$$\frac{\partial S_{22}(E)}{\partial\lambda}=\begin{cases}-2\pi i\sigma^2(E+V_1)\int_\alpha^\beta d\xi\, M(\xi)f^+_{E+V_1}(\xi)g^-_{E+V_1}(\xi) & \text{when } E+V_1>V_2\ ,\\ 0 & \text{otherwise}\ ,\end{cases} \tag{3.4}$$

$$\frac{\partial S_{12}(E)}{\partial\lambda}=-2\pi i\sigma^2(E+V_2)\int_\alpha^\beta d\xi\, M(\xi)[g_{E+V_2}^-(\xi)]^2 , \tag{3.5}$$

$$\frac{\partial S_{21}(E)}{\partial\lambda}=-2\pi i\sigma^2(E+V_1)\int_\alpha^\beta d\xi\, M(\xi)[f_{E+V_1}^+(\xi)]^2 . \tag{3.6}$$

The significance of the operator $\partial S/\partial\lambda$ lies in the following. Suppose A is an observable which is a constant of motion with respect to the free asymptotic Hamiltonians $H_0^{\rm in}$ and $H_0^{\rm out}$. The change of the expectation value of this observable due to scattering of a fixed in-asymptote $\Psi^{\rm in}$ reads

$$\begin{aligned}\langle \Delta A\rangle_{\Psi^{\rm in}}(\lambda)&=\lim_{t\to\infty}\langle e^{-itH(\lambda)}\Omega_+\Psi^{\rm in}|\,Ae^{-itH(\lambda)}\Omega_+\Psi^{\rm in}\rangle-\lim_{t\to-\infty}\langle e^{-itH(\lambda)}\Omega_+\Psi^{\rm in}|\,Ae^{-itH(\lambda)}\Omega_+\Psi^{\rm in}\rangle\\ &=\lim_{t\to\infty}\langle \exp(-itH_0^{\rm out})S(\lambda)\Psi^{\rm in}|\,A\exp(-itH_0^{\rm out})S(\lambda)\Psi^{\rm in}\rangle\\ &\quad-\lim_{t\to-\infty}\langle\exp(-itH_0^{\rm in})\Psi^{\rm in}|\,A\exp(-itH_0^{\rm in})\Psi^{\rm in}\rangle=\langle S(\lambda)\Psi^{\rm in}|\,AS(\lambda)\Psi^{\rm in}\rangle-\langle\Psi^{\rm in}|\,A\Psi^{\rm in}\rangle .\end{aligned}\tag{3.7}$$

As first-order approximation we have

$$\begin{aligned}\langle \Delta A\rangle_{\Psi^{\rm in}}(\lambda)&\approx\langle \Delta A\rangle_{\Psi^{\rm in}}(0)+\lambda\left[\left\langle\frac{\partial S}{\partial\lambda}\Psi^{\rm in}\,\middle|\,AS\Psi^{\rm in}\right\rangle+\left\langle S\Psi^{\rm in}\,\middle|\,A\frac{\partial S}{\partial\lambda}\Psi^{\rm in}\right\rangle\right]\\ &=\langle \Delta A\rangle_{\Psi^{\rm in}}(0)-i\lambda\left\langle\Psi^{\rm in}\,\middle|\left[S^\dagger AS, iS^\dagger\frac{\partial S}{\partial\lambda}\right]\middle|\,\Psi^{\rm in}\right\rangle ,\end{aligned}\tag{3.8}$$

where to obtain the last line we used the identity $S^\dagger(\partial S/\partial\lambda)+(\partial S^\dagger/\partial\lambda)S=0$, which is a direct consequence of the unitarity of S. Note that this identity also implies that the operator $iS^\dagger(\partial S/\partial\lambda)$ is Hermitian.

In our energy representation we have

$$\begin{aligned}&\left[\left[iS^\dagger\frac{\partial S}{\partial\lambda}\right](\Phi_1,\Phi_2)\right](E)\\ &=\left[\left[iS^\dagger\frac{\partial S}{\partial\lambda}\right]_{11}(E)\Phi_1(E)\right.\\ &\quad\left.+\left[iS^\dagger\frac{\partial S}{\partial\lambda}\right]_{12}(E)\Phi_2(E-V_2+V_1),\left[iS^\dagger\frac{\partial S}{\partial\lambda}\right]_{21}(E)\Phi_1(E-V_1+V_2)+\left[iS^\dagger\frac{\partial S}{\partial\lambda}\right]_{22}(E)\Phi_2(E)\right] ,\end{aligned}\tag{3.9}$$

with

$$\left[iS^\dagger\frac{\partial S}{\partial\lambda}\right]_{11}(E)=2\pi|\sigma(E+V_1)|^2\int_\alpha^\beta d\xi\, M(\xi)|f_{E+V_1}^+(\xi)|^2 , \tag{3.10}$$

$$\left[iS^\dagger\frac{\partial S}{\partial\lambda}\right]_{22}(E)=2\pi|\sigma(E+V_2)|^2\int_\alpha^\beta d\xi\, M(\xi)|g_{E+V_2}^-(\xi)|^2 , \tag{3.11}$$

$$\left[iS^\dagger\frac{\partial S}{\partial\lambda}\right]_{12}(E)=\begin{cases}2\pi|\sigma(E+V_1)|^2\int_\alpha^\beta d\xi\, M(\xi)f_{E+V_1}^-(\xi)g_{E+V_1}^-(\xi) & \text{when } E+V_1>V_2 ,\\ 0 \text{ otherwise} ,\end{cases}\tag{3.12}$$

$$\left[iS^\dagger\frac{\partial S}{\partial\lambda}\right]_{21}(E)=\begin{cases}2\pi|\sigma(E+V_2)|^2\int_\alpha^\beta d\xi\, M(\xi)f_{E+V_2}^+(\xi)g_{E+V_2}^+(\xi) & \text{when } E+V_2>V_1 ,\\ 0 \text{ otherwise} .\end{cases}\tag{3.13}$$

IV. TIME OF SOJOURN AND THE OPERATOR $iS^\dagger(\partial S/\partial\lambda)$

In Sec. I we defined the sojourn time $\tau(\Omega,t_1,t_2;\Psi)$ generally for arbitrary spatial region Ω, and arbitrary time interval (t_1,t_2). Here we will only need the case when Ω is a bounded interval, $\Omega=(x_0,x_1)$, and $t_1=-\infty$, $t_2=\infty$, and Ψ is a scattering state, i.e., $\Psi\in\mathcal{H}_b^\perp=\Omega_+\mathcal{H}$. We will write $\tau(x_0,x_1;\Psi)$ for $\tau((x_0,x_1),-\infty,\infty;\Omega_+\Psi)$, i.e.,

$$\tau(x_0,x_1;\Psi)=\int_{-\infty}^{\infty}dt\int_{x_0}^{x_1}d\xi\,|(e^{-itH}\Omega_+\Psi)(\xi)|^2 . \tag{4.1}$$

Similarly, the sojourn time per unit interval $\tau'(x,-\infty,\infty;\Omega_+\Psi)$ will be denoted by $\tau'(x;\Psi)$:

$$\begin{aligned}\tau'(x;\Psi)&=\int_{-\infty}^{\infty}dt\,|(e^{-itH}\Omega_+\Psi)(x)|^2 ,\\ \tau(x_0,x_1;\Psi)&=\int_{x_0}^{x_1}d\xi\,\tau'(\xi;\Psi) .\end{aligned}\tag{4.2}$$

The sojourn time operators $T((x_0,x_1),-\infty,\infty)$, corresponding to $\tau((x_0,x_1),-\infty,\infty;\Psi)$ and $\tau'(x,-\infty,\infty;\Psi)$, respectively, are defined in the subspace $\mathcal{H}_b^{\perp}$ of scattering states. It will be more convenient to introduce operators

$$T(x_0,x_1)=\Omega_+^{\dagger}T((x_0,x_1),-\infty,\infty)\Omega_+$$

and

$$T'(x)=\Omega_+^{\dagger}T'(x,-\infty,\infty)\Omega_+$$

defined in $\mathcal{H}$. Clearly, $T(x_0,x_1)$ and $T'(x)$ satisfy

$$\tau(x_0,x_1;\Psi)=\langle\Psi|T(x_0,x_1)\Psi\rangle\ , \tag{4.3}$$

$$\tau'(x;\Psi)=\langle\Psi|T'(x)\Psi\rangle\ . \tag{4.4}$$

The sojourn times and the sojourn time operators can be effectively dealt with using the energy representation and the Moller operator Ω_+ in the form (2.16) with Eqs. (2.13) and (2.14) applied to the integrand. Writing $\Psi=U(\Phi_1,\Phi_2)$, we evaluate $\tau'(x;\Psi)$ of Eq. (4.2) by applying the intertwining relation $\exp(-itH)\Omega_+=\Omega_+\exp(-itH_0^{\text{in}})$, then the energy representation (2.10) of H_0^{in}, and finally the formula $\int_{-\infty}^{\infty}dt\,e^{-it(E-E')}=2\pi\delta(E-E')$:

$$\begin{aligned}\tau'(x;\Psi)=&\int_{-\infty}^{\infty}dt\left|\int_{-\infty}^{\infty}dE\,e^{-itE}[\sigma(E)f_E^{+}(x)\Phi_1(E-V_1)\Theta(E-V_1)+\sigma(E)g_E^{-}(x)\Phi_2(E-V_2)\Theta(E-V_2)]\right|^2\\ =&\,2\pi\int_0^{\infty}dE\,|\sigma(E+V_1)f_{E+V_1}^{+}(x)\Phi_1(E)|^2+2\pi\int_0^{\infty}dE\,|\sigma(E+V_2)g_{E+V_2}^{-}(x)\Phi_2(E)|^2\\ &+2\pi\int_{\max(0,V_1-V_2)}^{\infty}dE\,|\sigma(E+V_2)|^2f_{E+V_2}^{+}(x)g_{E+V_2}^{+}\Phi_1(E+V_2-V_1)\Phi_2^{*}(E)\\ &+2\pi\int_{\max(0,V_2-V_1)}^{\infty}dE\,|\sigma(E+V_1)|^2f_{E+V_1}^{-}(x)g_{E+V_1}^{-}\Phi_1^{*}(E)\Phi_2(E+V_1-V_2)\ ,\end{aligned} \tag{4.5}$$

where $\theta(\mu)=0$ for $\mu\le 0$ and 1 for $\mu>0$.

From the above relation it is inferred that in our energy representation the operator $T'(x)$ is given by

$$[T'(x)(\Phi_1,\Phi_2)](E)=(T'_{11}(x,E)\Phi_1(E)+T'_{12}(x,E)\Phi_2(E+V_1-V_2),T'_{21}(x,E)\Phi_1(E+V_2-V_1)+T'_{22}(x,E)\Phi_2(E))\ , \tag{4.6}$$

$$T'_{11}(x,E)=2\pi|\sigma(E+V_1)|^2|f_{E+V_1}^{+}(x)|^2\ , \tag{4.7}$$

$$T'_{22}(x,E)=2\pi|\sigma(E+V_2)|^2|g_{E+V_2}^{-}(x)|^2\ , \tag{4.8}$$

$$T'_{12}(x,E)=\begin{cases}2\pi|\sigma(E+V_1)|^2f_{E+V_1}^{-}(x)g_{E+V_1}^{-}(x) & \text{when } E+V_1>V_2\ ,\\ 0 \text{ otherwise}\ ,\end{cases} \tag{4.9}$$

$$T'_{21}(x,E)=\begin{cases}2\pi|\sigma(E+V_2)|^2f_{E+V_2}^{+}(x)g_{E+V_2}^{+}(x) & \text{when } E+V_2>V_1\ ,\\ 0 \text{ otherwise}\ .\end{cases} \tag{4.10}$$

The sojourn time operator $T'(x)$ is related to the sojourn time operator $T(x_0,x)$ by the formula

$$T(x_0,x)=\int_{x_0}^{x}d\xi\,T'(\xi)\ , \tag{4.11}$$

and the significance of $T'(x)$ for the perturbation theory of Sec. III lies in the fact that

$$iS^{\dagger}\frac{\partial S}{\partial\lambda}=\int_{\alpha}^{\beta}M(\xi)T'(\xi)d\xi=\int_{-\infty}^{\infty}M(\xi)T'(\xi)d\xi\ . \tag{4.12}$$

Note that in the particular case when $M(\xi)=M_0=\text{const}$ on (α,β) and zero outside, then the operator $iS^{\dagger}(\partial S/\partial\lambda)$ is simply proportional to the time of sojourn operator $T(\alpha,\beta)$: $iS^{\dagger}(\partial S/\partial\lambda)=M_0T(\alpha,\beta)$, and thus this operator is directly relevant for the calculation of $\langle\Delta A\rangle_{\Psi}$ in the first-order perturbation theory, cf. (3.7) and (3.8). In general,

$$\left[S^{\dagger}AS,iS^{\dagger}\frac{\partial S}{\partial\lambda}\right]=\int_{\alpha}^{\beta}M(\xi)[S^{\dagger}AS,T'(\xi)]d\xi\ , \tag{4.13}$$

or

$$\langle\Delta A\rangle_{\Psi^{\text{in}}}(\lambda)\approx\langle\Delta A\rangle_{\Psi^{\text{in}}}(0)-i\lambda\int_{\alpha}^{\beta}M(\xi)\langle\Psi^{\text{in}}|[S^{\dagger}AS,T'(\xi)]\Psi^{\text{in}}\rangle d\xi\ . \tag{4.14}$$

Thus formulas (4.12)–(4.14) indicate that the effect of the perturbation M is closely related to the sojourn time of the unperturbed particle in the region where the perturbation is localized.

V. THE SPIN CLOCK REVISITED

The scattering theory of Sec. I describes translational motion along the x axis of a particle interacting with the potential V. We now assume that the particle has spin $\frac{1}{2}$ as its internal degree of freedom, and that a weak homogeneous magnetic field pointing in the z direction extends over a bounded interval (α,β) of the x axis. As in Sec. I, the potential V is constant outside a bounded interval (a,b), and without loss of generality we may take $(\alpha,\beta)\subseteq(a,b)$.

The Hilbert space is now $\tilde{\mathcal{H}}=L^2(\mathbb{R})\otimes\mathbb{C}^2=L^2(\mathbb{R},\mathbb{C}^2)$, i.e., consists of two-component spinors,

$$\begin{bmatrix}\Psi_1\\ \Psi_2\end{bmatrix}(x)=\begin{bmatrix}\Psi_1(x)\\ \Psi_2(x)\end{bmatrix} .$$

Denote by $\sigma_x,\sigma_y,\sigma_z$ the Pauli matrices, by ω the Larmor frequency inside the field, and let $M(x)$ be the function

$$M(x)=\begin{cases}\frac{1}{2} & \text{for } \alpha<x<\beta ,\\ 0 & \text{otherwise} .\end{cases}$$

The Hamiltonian reads $\tilde{H}(\omega)=-\frac{1}{2}d^2/dx^2+V(x)+\omega M(x)\sigma_z$,

$$\left[\tilde{H}(\omega)\begin{bmatrix}\Psi_1\\ \Psi_2\end{bmatrix}\right](x)=\begin{bmatrix}\left[-\frac{1}{2}\frac{d^2}{dx^2}+V(x)+\omega M(x)\right]\Psi_1(x)\\ \left[-\frac{1}{2}\frac{d^2}{dx^2}+V(x)-\omega M(x)\right]\Psi_2(x)\end{bmatrix} . \tag{5.1}$$

The scattering theory can be constructed as in Secs. I and II. The asymptotic Hamiltonians have still the form (2.1) but now they act on the two-component spinors. The structure of the Hamiltonian $\tilde{H}(\omega)$ and the asymptotic Hamiltonians immediately imply that the scattering operator $\tilde{S}(\omega)$ reads $\tilde{S}(\omega)=S(\omega)\oplus S(-\omega)$, i.e.,

$$\tilde{S}(\omega)\begin{bmatrix}\Psi_1\\ \Psi_2\end{bmatrix}=\begin{bmatrix}S(\omega)\Psi_1\\ S(-\omega)\Psi_2\end{bmatrix} , \tag{5.2}$$

where $S(\omega)$ is the scattering operator of Sec. II acting in $L^2(\mathbb{R})$ and corresponding to the perturbed Hamiltonian $H(\omega)=-\frac{1}{2}d^2/dx^2+V(x)+\omega M(x)$.

When $\omega=0$, then $\tilde{S}=\tilde{S}(0)=S(0)\oplus S(0)=S\oplus S$. For small ω, in the first-order approximation, we have $\tilde{S}(\omega)\approx\tilde{S}(0)+\omega(\partial\tilde{S}/\partial\omega)(0)=S\oplus S+\omega[(\partial S/\partial\omega)\oplus(-\partial S/\partial\omega)]$. The analog of Eq. (3.8) can be derived as before: When A is an observable commuting with the asymptotic Hamiltonians, and

$$\Psi^{\text{in}}=\begin{bmatrix}\Psi_1^{\text{in}}\\ \Psi_2^{\text{in}}\end{bmatrix}\in L^2(\mathbb{R},\mathbb{C}^2)$$

is an in-asymptote, then

$$\langle\Delta\tilde{A}\rangle_{\Psi^{\text{in}}}(\lambda)\approx\langle\Delta\tilde{A}\rangle_{\Psi^{\text{in}}}(0)-i\omega\left\langle\Psi^{\text{in}}\middle|\left[\tilde{S}^\dagger A\tilde{S},i\tilde{S}^\dagger\frac{\partial\tilde{S}}{\partial\lambda}\right]\Psi^{\text{in}}\right\rangle , \tag{5.3}$$

with

$$i\tilde{S}^\dagger\frac{\partial\tilde{S}}{\partial\lambda}=\left[iS^\dagger\frac{\partial S}{\partial\lambda}\right]\oplus\left[-iS^\dagger\frac{\partial S}{\partial\lambda}\right]=\tfrac{1}{2}\{T(\alpha,\beta)\oplus[-T(\alpha,\beta)]\} , \tag{5.4}$$

where $T(\alpha,\beta)$ is the sojourn time operator (4.3) for the spinless particle with the Hamiltonian $H=-\frac{1}{2}d^2/dx^2+V(x)$.

The sojourn times and the sojourn time operators for the spin-$\frac{1}{2}$ particle can be constructed in the obvious way. It is easy to find that the sojourn time operator $\tilde{T}(\alpha,\beta)$ for the particle with the Hamiltonian $\tilde{H}(0)$ in (α,β) reads $\tilde{T}(\alpha,\beta)=T(\alpha,\beta)\oplus T(\alpha,\beta)$ and thus (5.4) can be written as

$$i\tilde{S}^\dagger\frac{\partial\tilde{S}}{\partial\lambda}=\tfrac{1}{2}\sigma_z\tilde{T}(\alpha,\beta)=\tfrac{1}{2}\tilde{T}(\alpha,\beta)\sigma_z . \tag{5.5}$$

Formula (5.3) applies, in particular, to the spin operators $\frac{1}{2}\sigma_i$. Note that in this case $\tilde{S}^\dagger\sigma_i\tilde{S}=\sigma_i$, because $\tilde{S}=\tilde{S}(0)$ and σ_i's are constants of motion with respect to $\tilde{H}(0)$ [σ_z is a constant of motion with respect to $\tilde{H}(\omega)$ for any ω]. Applying formula (5.3) to the operator $\sigma=\frac{1}{2}(\sigma_x+i\sigma_y)$ one obtains

$$\begin{aligned}\langle\Delta\sigma\rangle_{\Psi^{\text{in}}}(\omega)&=\langle\tilde{S}(\omega)\Psi^{\text{in}}|\sigma\tilde{S}(\omega)\Psi^{\text{in}}\rangle-\langle\Psi^{\text{in}}|\sigma\Psi^{\text{in}}\rangle\\ &\approx i\omega\langle\Psi^{\text{in}}|\sigma\tilde{T}(\alpha,\beta)\Psi^{\text{in}}\rangle\\ &=i\omega\langle\Psi_1^{\text{in}}|T(\alpha,\beta)\Psi_2^{\text{in}}\rangle ,\end{aligned} \tag{5.6}$$

where

$$\Psi^{\text{in}}=\begin{bmatrix}\Psi_1^{\text{in}}\\ \Psi_2^{\text{in}}\end{bmatrix}$$

and the scalar products in the third line of (5.6) are scalar products in $L^2(\mathbb{R})$.

Once in the field region, i.e., in (α,β), the spin undergoes precession around the z direction with the Larmor frequency ω. The equation

$$\langle\tilde{S}\Psi^{\text{in}}|\sigma\tilde{S}\Psi^{\text{in}}\rangle=\langle\Psi^{\text{in}}|\sigma\Psi^{\text{in}}\rangle e^{i\omega\bar{t}} \tag{5.7}$$

defines a time $\bar{t}$ which for infinitesimal ω can be obtained from Eq. (5.6):

$$\bar{t}=\frac{\langle\Psi^{\text{in}}|\sigma\tilde{T}(\alpha,\beta)\Psi^{\text{in}}\rangle}{\langle\Psi^{\text{in}}|\sigma\Psi^{\text{in}}\rangle} . \tag{5.8}$$

In particular, when Ψ^{in} is an eigenstate of σ_x (or of any spin component in the xy plane), then one finds

$$\bar{t}=\langle\Psi^{\text{in}}|\tilde{T}(\alpha,\beta)\Psi^{\text{in}}\rangle , \tag{5.9}$$

and noting that $\sigma_x\Psi^{\text{in}}=\pm\Psi^{\text{in}}$ implies $\Psi^{\text{in}}=(1/\sqrt{2})[{}^1_1]\Psi$, one also has

$$\bar{t}=\langle\Psi|T(\alpha,\beta)\Psi\rangle . \tag{5.10}$$

Thus when Ψ^{in} is an eigenstate of σ_x then the system works as a clock measuring the mean sojourn time $\langle\Psi^{\text{in}}|\tilde{T}(\alpha,\beta)\Psi^{\text{in}}\rangle$.

It has been also proposed to use the clock to define separate transmission and reflection times.[4,6] The quantities

$$\langle\sigma\rangle^{\text{tr}}_{\Psi^{\text{in}}}(\omega)=\frac{\langle F_+\tilde{S}(\omega)\Psi^{\text{in}}|\sigma F_+\tilde{S}(\omega)\Psi^{\text{in}}\rangle}{\langle\tilde{S}(\omega)\Psi^{\text{in}}|F_+\tilde{S}(\omega)\Psi^{\text{in}}\rangle} , \tag{5.11}$$

and

$$\langle\sigma\rangle^{r}_{\Psi^{\text{in}}}(\omega)=\frac{\langle F_-\tilde{S}(\omega)\Psi^{\text{in}}|\sigma F_-\tilde{S}(\omega)\Psi^{\text{in}}\rangle}{\langle\tilde{S}(\omega)\Psi^{\text{in}}|F_-\tilde{S}(\omega)\Psi^{\text{in}}\rangle} \tag{5.12}$$

are mean values of the operator σ corresponding to transmitted and reflected particles, respectively. Let $\Psi^{\rm in}$ be an eigenstate of σ_x with the eigenvalue (e.g.) 1. The transmission time $\bar{t}_{\rm tr}$ and the reflection time $\bar{t}_r$ are defined by the equations

$$\langle \sigma \rangle^{\rm tr}_{\Psi^{\rm in}}(\omega)=\tfrac{1}{2}\exp(i\omega\bar{t}_{\rm tr}), \quad \langle \sigma \rangle^{r}_{\Psi^{\rm in}}(\omega)=\tfrac{1}{2}\exp(i\omega\bar{t}_{\rm r}) . \tag{5.13}$$

Formulas (5.3) and (5.5) yield

$$\begin{aligned}\langle \tilde{S}(\omega)\Psi^{\rm in}|F_\pm\tilde{S}(\omega)\Psi^{\rm in}\rangle-\langle \Psi^{\rm in}|F_\pm\Psi^{\rm in}\rangle &\approx \langle \tilde{S}\Psi^{\rm in}|F_\pm\tilde{S}\Psi^{\rm in}\rangle-\langle \Psi^{\rm in}|F_\pm\Psi^{\rm in}\rangle-\frac{i\omega}{2}\langle \Psi^{\rm in}|[\tilde{S}^{\,\dagger}F_\pm\tilde{S},\sigma_z\tilde{T}(\alpha,\beta)]\Psi^{\rm in}\rangle\\ &=\langle \tilde{S}\Psi^{\rm in}|F_\pm\tilde{S}\Psi^{\rm in}\rangle-\langle \Psi^{\rm in}|F_\pm\Psi^{\rm in}\rangle ,\end{aligned} \tag{5.14}$$

$$\begin{aligned}\langle F_\pm\tilde{S}(\omega)\Psi^{\rm in}|\sigma F_\pm\tilde{S}(\omega)\Psi^{\rm in}\rangle-\langle F_\pm\Psi^{\rm in}|\sigma F_\pm\Psi^{\rm in}\rangle &=\langle \tilde{S}(\omega)\Psi^{\rm in}|\sigma F_\pm\tilde{S}(\omega)\Psi^{\rm in}\rangle-\langle \Psi^{\rm in}|\sigma F_\pm\Psi^{\rm in}\rangle\\ &=\langle \tilde{S}\Psi^{\rm in}|\sigma F_\pm\tilde{S}\Psi^{\rm in}\rangle-\langle \Psi^{\rm in}|\sigma F_\pm\Psi^{\rm in}\rangle\\ &\quad-\frac{i\omega}{2}\langle \Psi^{\rm in}|[\tilde{S}^{\,\dagger}\sigma F_\pm\tilde{S},\sigma_z\tilde{T}(\alpha,\beta)]\Psi^{\rm in}\rangle\\ &=\langle \tilde{S}\Psi^{\rm in}|\sigma F_\pm\tilde{S}\Psi^{\rm in}\rangle-\langle \Psi^{\rm in}|\sigma F_\pm\Psi^{\rm in}\rangle\\ &\quad+\frac{i\omega}{2}{\rm Re}\langle \Psi^{\rm in}|\tilde{S}^{\,\dagger}F_\pm\tilde{S}\tilde{T}(\alpha,\beta)\Psi^{\rm in}\rangle .\end{aligned} \tag{5.15}$$

To obtain the final equalities in (5.14) and (5.15) we used the fact that the spin operators commute with $F_\pm,\tilde{T}(\alpha,\beta)$ and $\tilde{S}=\tilde{S}(0)$, and that $\Psi^{\rm in}=(1/\sqrt{2})[^1_1]\Psi$. For infinitesimal ω we can now easily find $\bar{t}_{\rm tr}$ and $\bar{t}_{\rm r}$:

$$\bar{t}_{\rm tr}=\frac{{\rm Re}\langle \Psi^{\rm in}|\tilde{S}^{\,\dagger}F_+\tilde{S}\tilde{T}(\alpha,\beta)\Psi^{\rm in}\rangle}{\langle \tilde{S}\Psi^{\rm in}|F_+\tilde{S}\Psi^{\rm in}\rangle} , \tag{5.16}$$

$$\bar{t}_{\rm r}=\frac{{\rm Re}\langle \Psi^{\rm in}|\tilde{S}^{\,\dagger}F_-\tilde{S}\tilde{T}(\alpha,\beta)\Psi^{\rm in}\rangle}{\langle \tilde{S}\Psi^{\rm in}|F_-\tilde{S}\Psi^{\rm in}\rangle} . \tag{5.17}$$

Note that

$$\langle \tilde{S}\Psi^{\rm in}|F_+\tilde{S}\Psi^{\rm in}\rangle\bar{t}_{\rm tr}+\langle \tilde{S}\Psi^{\rm in}|F_-\tilde{S}\Psi^{\rm in}\rangle\bar{t}_{\rm r}=\bar{t} .$$

Formulas (5.16), (5.17), and also (5.9) can be further developed, and their asymptotics for $\alpha\to\infty$, $\beta\to-\infty$ can be found. But this in fact was done elsewhere.[4] In the present context we only wanted to show that what the spin-clock measures are matrix elements of the sojourn time operator $\tilde{T}(\alpha,\beta)$ or $T(\alpha,\beta)$.

In general, the example shows the usefulness of the concept of the sojourn time operator. The effect of scattering on the spin of the particle can be, in the first-order approximation, fully described in terms of it. Generalization to the case of an inhomogeneous magnetic field is straightforward by applying the local sojourn time operator. We anticipate that other interesting models in which the translational degree of freedom of a tunneling particle is coupled to additional degrees of freedom can be also treated using the concept of sojourn time and sojourn time operator.

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APPENDIX: DERIVATION OF EQ. (3.2)

If h_1 and h_2 are any functions of $x\in\mathbb{R}$, then their Wronskian $W(h_1,h_2)(x)$ is defined by

$$W(h_1,h_2)(x)=h_1(x)\frac{dh_2}{dx}(x)-h_2(x)\frac{dh_1}{dx}(x) . \tag{A1}$$

When h_1 and h_2 are solutions of the differential equation $(d^2/dx^2)h(x)=A(x)h(x)$, then their Wronskian is independent of x and we write $W(h_1,h_2)$.

To derive Eq. (3.2) note that for any $x\in\mathbb{R}$,

$$\frac{\partial}{\partial\lambda}W(f^i_\mu,g^j_\mu)=W\left[\frac{\partial f^i_\mu}{\partial\lambda},g^j_\mu\right](x)+W\left[f^i_\mu,\frac{\partial g^j_\mu}{\partial\lambda}\right](x) , \tag{A2}$$

and that the functions $(\partial f^i_\mu/\partial\lambda)(x)$ and $(\partial g^j_\mu/\partial\lambda)(x)$ satisfy the differential equations

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\frac{\partial f^i_\mu}{\partial\lambda}(x)&=2[V(x)-\mu]\frac{\partial f^i_\mu}{\partial\lambda}(x)+2M(x)f^i_\mu(x) ,\\ \frac{\partial^2}{\partial x^2}\frac{\partial g^i_\mu}{\partial\lambda}(x)&=2[V(x)-\mu]\frac{\partial g^i_\mu}{\partial\lambda}(x)+2M(x)g^i_\mu(x) ,\end{aligned} \tag{A3}$$

and the initial conditions

$$\frac{\partial f^i_\mu}{\partial\lambda}(b)=0=\frac{\partial}{\partial x}\frac{\partial f^i_\mu}{\partial\lambda}(b), \quad \frac{\partial g^j_\mu}{\partial\lambda}(a)=0=\frac{\partial}{\partial x}\frac{\partial g^j_\mu}{\partial\lambda}(a) .$$

Hence we can find that the Wronskians $W(\partial f^i_\mu/\partial\lambda,g^j_\mu)(x)$ and $W(f^i_\mu,\partial g^j_\mu/\partial\lambda)(x)$ satisfy the differential equations

$$\frac{\partial}{\partial x} W\left[\frac{\partial f^i_\mu}{\partial \lambda}, g^j_\mu\right](x) = -2M(x) f^i_\mu(x) g^j_\mu(x) ,$$
$$\frac{\partial}{\partial x} W\left[f^i_\mu, \frac{\partial g^j_\mu}{\partial \lambda}\right](x) = 2M(x) f^i_\mu(x) g^j_\mu(x) , \quad \text{(A4)}$$

and the initial conditions $W(\partial f^i_\mu/\partial\lambda, g^j_\mu)(b) = 0 = W(f^i_\mu, \partial g^j_\mu/\partial\lambda)(a)$. Integrating (A4) yields

$$W\left[\frac{\partial f^i_\mu}{\partial \lambda}, g^j_\mu\right](x) = 2\int_x^b d\xi\, M(\xi) f^i_\mu(\xi) g^j_\mu(\xi) ,$$
$$W\left[f^i_\mu, \frac{\partial g^j_\mu}{\partial \lambda}\right](x) = 2\int_a^x d\xi\, M(\xi) f^i_\mu(\xi) g^j_\mu(\xi) . \quad \text{(A5)}$$

Adding the above equations and taking into account the fact that $M(x)$ is zero outside $(\alpha,\beta) \subseteq (a,b)$, we arrive at (3.2).

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