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Unbinding transition of semiflexible membranes in (1+1) dimensions

G. Gompper* and T. W. Burkhardt†
Department of Physics, FM-15, University of Washington, Seattle, Washington 98195
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The universality classes and critical exponents of the unbinding transition in a model of semiflexible membranes (or polymers) in (1+1) dimensions are determined for wall potentials that fall off with distance z as z^{-p}. For $p > \frac{2}{3}$ the unbinding transition is first order, and for attractive potentials with $p < \frac{2}{3}$ the membrane is always bound. The marginal case $p = \frac{2}{3}$ is quite similar to the intermediate-fluctuation regime in (1+1)-dimensional wetting.

The fluctuations of membranes are governed by their bending energy, in contrast to the fluctuations of interfaces, which are controlled by surface tension.[1] Continuum models for both fluid[2] and crystalline (or polymerized)[3] membranes have recently been proposed. The analysis of these models is rather difficult, even for free membranes. In studying unbinding transitions, it is useful to consider simpler solid-on-solid (SOS) models, which neglect configurations with overhangs.[4] Membranes with dimension $d-1$ are believed to be crumpled[5] for all temperatures $T > 0$ in spatial dimension $d < 3$, and to have a crumpling transition at finite temperatures for $d > 3$. In the crumpled phase the SOS approximation breaks down at large length scales. It is still useful to study SOS models, because the persistence length,[6] which defines the average size of regions with a well-defined orientation, may be very large, in particular for membranes near an attractive wall.[7]

The adsorption of semiflexible polymers or membranes in (1+1) dimensions has recently been studied by Maggs, Huse, and Leibler[8] for *short-range* wall potentials. They find that the unbinding transition is first-order. In this paper we determine the universality classes and critical exponents as a function of the range of the interaction potential and recover their results as a special case. Some strong similarities with critical wetting in (1+1) dimensions[9-11] are found.

A natural generalization for membranes of the SOS interface Hamiltonian in (1+1) dimensions is given by

$$H = \sum_i [\kappa(h_{i+1} - 2h_i + h_{i-1})^2 + V(h_i)] , \tag{1}$$

where the h_i are integer height variables. The membrane interacts with the boundary via the potential $V(h)$. We consider potentials of the form

$$V(h) = \begin{cases} -U\delta_{1,h} - Wh^{-p}, & h > 0 , \\ \infty, & h \le 0 . \end{cases} \tag{2}$$

Continuum models are often easier to analyze than lattice models. In a continuum version of the above model the partition function is defined by the path integral

$$Z_l(z,v \mid z_0,v_0) = \int Dz \exp\left\{ -\int_0^l dx \left[\frac{\kappa}{2} \left(\frac{d^2z}{dx^2} \right)^2 + V(z) \right] \right\} . \tag{3}$$

At $x = 0$ and $x = l$ the position and slope of the membrane are fixed at the values z,v and z_0,v_0, respectively. The path integral implies the Schrödinger-type equation[12]

$$\left(\frac{\partial}{\partial l} + v\frac{\partial}{\partial z} - \frac{1}{2\kappa}\frac{\partial^2}{\partial v^2} + V(z) \right) Z_l(z,v \mid z_0,v_0) = 0 . \tag{4}$$

The prefactor $(2\kappa)^{-1}$ of the $\partial^2/\partial v^2$ term can be eliminated by rescaling z, v, and V and will be omitted below. In analogy with Eq. (2) potentials of the form

$$V(z) = \begin{cases} -wz^{-p}, & z > a_0 , \\ -(u + wa_0^{-p}), & 0 \le z < a_0 , \\ \infty, & z < 0 , \end{cases} \tag{5}$$

will be considered. We will need the Markov property

$$Z_{l_1+l_2}(z_2,v_2 \mid z_0,v_0) = \int_0^\infty dz_1 \int_{-\infty}^\infty dv_1 \, Z_{l_2}(z_2,v_2 \mid z_1,v_1) Z_{l_1}(z_1,v_1 \mid z_0,v_0) . \tag{6}$$

Three different scaling regimes[4,13] can be defined by comparing the asymptotic decay z^{-p} of the potential $V(z)$ and the fluctuation-induced repulsion[4,14] $V_{\rm fl} \sim z^{-\tau}$, with $\tau = -2(d-1)/(d-5)$. The conditions $p > \tau$, $p = \tau$, and $p < \tau$, with $\tau = \frac{2}{3}$ for $d = 2$, correspond to the strong, intermediate, and weak-fluctuation regimes, respectively.

Equation (4) can be solved[15] for a free membrane using Fourier transforms, with the result

$$Z_l(z,v \mid z_0,v_0) = \frac{\sqrt{3}}{2\pi} l^{-2} \exp\{ -3l^{-3}[(z - z_0 - v_0 l)^2 - l(z - z_0 - v_0 l)(v - v_0) + \tfrac{1}{3} l^2 (v - v_0)^2] \} . \tag{7}$$

This implies $\langle v^2 \rangle = 2l$ and $\langle z^2 \rangle = \frac{2}{3} l^3$ for $z_0 = v_0 = 0$. Below we only consider systems with one end of the membrane pinned close to the wall and suppress the z_0 and the v_0 dependence of Z_l. The results for the free membrane motivate the

scaling ansatz[8]

$$Z_l(z,v) = z^{\alpha} l^{-\psi} g(z l^{-3/2}, v l^{-1/2}) \tag{8}$$

for wall potentials V that decay as z^{-r} or faster and are *not* strong enough to *bind* the membrane. In Eq. (8), the exponents α and ψ are defined by the condition $g(0,0) = \text{const}$.

The Markov property (6) can be used to relate the exponents α and ψ. Expressing the partition function of a membrane of length $2l$ with both ends near the wall in terms of the partition function of a membrane of length l with one end close to the wall and making use of Eq. (8), we obtain

$$l^{-2\psi} \int_0^{\infty} dz \int_{-\infty}^{\infty} dv\, z^{2\alpha} g(z l^{-3/2}, v l^{-1/2}) \times g(z l^{-3/2}, -v l^{-1/2}) \sim l^{-\psi},$$

which implies

$$\psi = 2 + 3\alpha . \tag{9}$$

This holds for $\alpha > -\frac{1}{2}$, so that the integral converges for $z \to 0$.

The exponent α can be determined from the differential equation (4). Being interested in unbound solutions, we set $a_0 = 0$ in (5). In the large-l limit with z,v fixed, we look for solutions

$$g(z l^{-3/2}, v l^{-1/2}) \to H(v z^{-1/3}) . \tag{10}$$

First we consider the intermediate-fluctuation regime. For potentials with $p = \frac{2}{3}$, Eqs. (4) and (10) lead to the differential equation

$$H'' + \tfrac{1}{3} y^2 H' - (\alpha y - w) H = 0 , \tag{11}$$

where $y = v z^{-1/3}$. For $v \ll -z^{1/3}$ or $y \to -\infty$, $H(y) \to \text{const} \times |y|^{3\alpha}$, and the z dependence in Eq. (8) cancels out. Due to the bending energy (1), configurations with steep *positive* slopes are energetically suppressed near the wall. This is compatible with an *exponential* decay of $H(y)$ for $y \to +\infty$, i.e., $v \gg z^{1/3}$. These boundary conditions determine the possible values of α. In the special case $w = 0$ considered in Ref. 8, H is a linear combination of Kummer's confluent hypergeometric functions,[16]

$$H(y) = \frac{M(-\alpha, \frac{2}{3}, -\frac{1}{9} y^3)}{\Gamma(\frac{1}{3} - \alpha)\Gamma(\frac{2}{3})} + (\tfrac{1}{9} y^3)^{1/3} \frac{M(\frac{1}{3} - \alpha, \frac{4}{3}, -\frac{1}{9} y^3)}{\Gamma(-\alpha)\Gamma(\frac{4}{3})} , \tag{12}$$

where $\alpha = \frac{1}{6} + n$, with $n = 0, \pm 1, \pm 2, \ldots$. It will be argued below that in fact $n = 0$. For $w \neq 0$ no special function seems available, and we proceed numerically as described below. The resulting function $\alpha(w)$, for $p = \frac{2}{3}$, is shown in Fig. 1. For all potentials with $p > \frac{2}{3}$, $\alpha = \frac{1}{6}$, and $\psi = \frac{5}{2}$, just as for $w = 0$.

We have also studied numerically the restricted solid-on-solid (RSOS) model considered by Maggs, Huse, and Leibler.[8] The partition function satisfies the recurrence

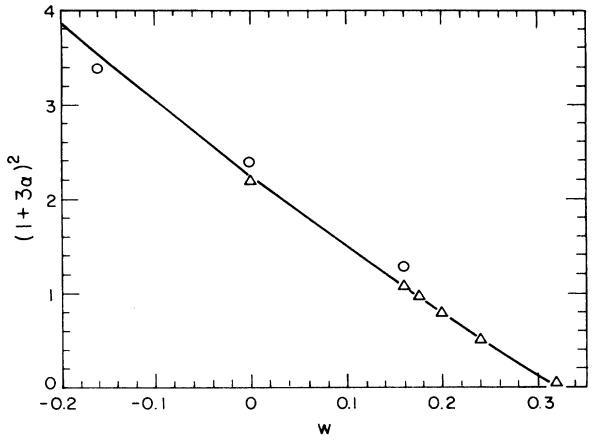

FIG. 1. Dependence of the exponents α and β, defined in Eqs. (8) and (16), with $1 + 3\alpha = -(1 + 3\beta)$, on the potential parameter w. The solid line shows the solution to the differential equations (11) and (16). The triangles and circles indicate values of α and of β, respectively, for the RSOS model with $w = 1.6W$. The uncertainty in α is of the size of the symbols. The uncertainty in β, about ± 0.025, is larger.

relation

$$Z_{l+1}(z,v) = [0.5 Z_l(z-v,v) + 0.25 Z_l(z-v+1,v-1) + 0.25 Z_l(z-v-1,v+1)] \exp[-V(z)] . \tag{13}$$

The RSOS model (13) allows calculations with much larger lattice sizes than the SOS model (1). Iterating Eq. (13) with the initial condition $Z_{l=1}(z,v) = 0$ except at $z = 1$, $v = 0$ for systems with $0 < z < L$ with L up to 2000, we have determined ψ from the l dependence of $Z_l(1,0)$, for $U \equiv 0$. The results for α, calculated using Eq. (9), are shown in Fig. 1. A direct determination of α by comparison with Eq. (8) gave consistent but less precise results. The value of α for $w = 0$ corresponds to the $n = 0$ branch of solutions (12). According to Fig. 1 the potential parameters of the continuous and discrete models are related by $w \approx 1.6W$.

The results in Fig. 1 have a simple physical interpretation. As the amplitude w of the asymptotic form $-w z^{-2/3}$ of the potential increases, the probability of finding the free end of the membrane near the wall also increases, and ψ and α decrease correspondingly.

The effect of the short-range part of the potential can be easily inferred from the necklace model for wetting.[17,18] As U is varied, there is an unbinding transition at a critical value $U = U_c(W)$. For $1 < \psi < 2$, the transition is continuous, with critical exponents determined by ψ. [For example, the exponent $\nu_{\parallel}$ of the longitudinal correlation length $\xi_{\parallel}$ is given by $\nu_{\parallel} = (\psi - 1)^{-1}$.] For $\psi > 2$ the transition is first order (with $\nu_{\parallel} = 1$). The results presented in Fig. 1 imply the three subregimes[10] A, B, and C of unbinding transitions indicated in Fig. 2. For $w > w_{mc} \approx 0.321\,14$ or $W > W_{mc} \approx 0.20$, the tail of the potential is strong enough to bind the membrane, and the short-range part of V is irrelevant. As $w - w_{mc} \to 0+$ with $-\infty < u < u_{mc}$ (subregime A), one expects an essential singularity in the correlation length, in analogy

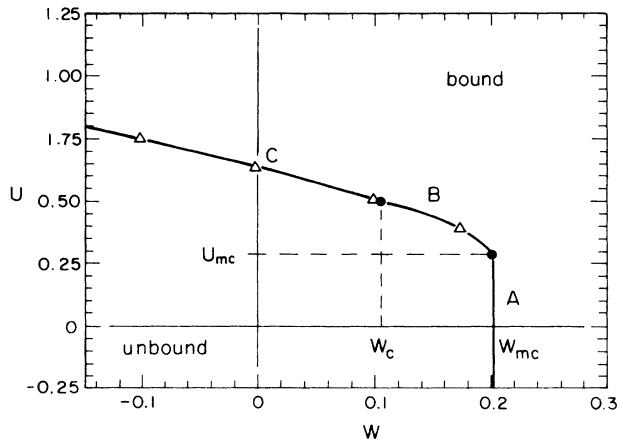

FIG. 2. Phase diagram for the RSOS model (13) with the potential (2). The phase boundary between the bound and the unbound states has three distinct subregimes *A*, *B*, and *C*. The value of U_{mc} is a rough extrapolation.

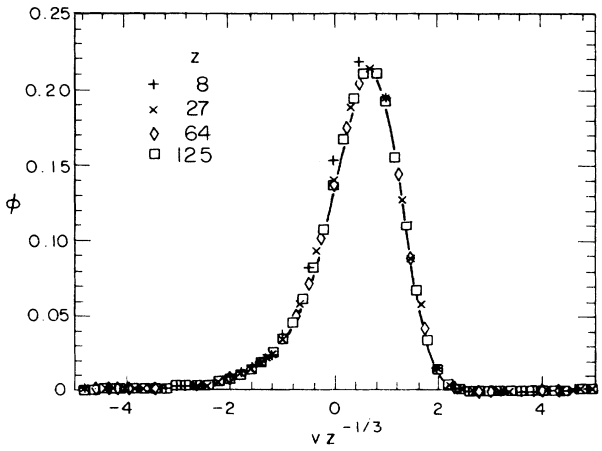

FIG. 3. Scaling function ϕ of Eq. (15) for $w=-0.16$. The solid line is the solution to the differential equation (16) with the horizontal axis rescaled by the factor 0.625. Predictions of the RSOS model for four different values of z are also shown.

with wetting.[9-11] In the subregime *B* of continuous transitions, corresponding to $w_c < w < w_{mc}$, with $w_c \approx 0.170245$ or $W_c \approx 0.105$,

$$\xi_{\parallel} \sim [u - u_c(w)]^{-\nu_{\parallel}} \text{ with } \nu_{\parallel} = (1+3a)^{-1}. \tag{14}$$

For $w \lesssim w_{mc}$, $\nu_{\parallel} \sim (w_{mc} - w)^{-1/2}$. Finally, in subregime *C*, corresponding to $w < w_c$, the transition is first order.

Let us take a closer look at subregime *C*. Since the transition is first order, there is a zero-energy bound state right at the transition,[11] i.e., $(\partial/\partial l) Z_l = 0$ for $l \to \infty$ and $u = u_c$. Inserting the scaling ansatz

$$Z_l(z,v) = z^{\beta} \phi(v z^{-1/3}) \tag{15}$$

for $l \to \infty$ into (4), we find

$$\phi'' + \tfrac{1}{3} y^2 \phi' - (\beta y - w)\phi = 0, \tag{16}$$

where $y = v z^{-1/3}$ for $z > a_0$. This is the same as the differential equation (11) for the scattering states. Since the boundary conditions as $y \to \pm\infty$ are also the same, $\beta(w)$ is determined in exactly the same way as $a(w)$. However, a different branch of solutions now applies. The integral in Eq. (6) only exists for $\beta(w) < -\frac{2}{3}$, which excludes the branches $n \geq 0$. Comparison with the RSOS results for $W = 0$ leads to the identification $n = -1$ or $\beta(0) = -\frac{5}{6}$. Numerically we find that

$$\beta(w) = -\tfrac{2}{3} - a(w) < -\tfrac{2}{3}, \tag{17}$$

which implies $1 + 3\beta(w) = -[1 + 3a(w)]$ (see Fig. 1).

Here we have only determined the large-z behavior of $Z_l(z,v)$. We have not used the short-range part of V, which, however, must be taken into account in calculating the phase boundary, i.e., $u_c(w)$.

We have also calculated the shape function ϕ for various values of w. The result for $w = -0.16$ is shown in Fig. 3. Predictions of the RSOS model with $W = -0.1$ are also indicated, and the agreement is excellent.

The power-law decay of the bound state (15) leads to an unusual behavior[10,11] of the moments

$$\langle z^n \rangle = \lim_{l \to \infty} \frac{\int dv \int dz\, z^n Z_l(z,v) Z_l(z,-v)}{\int dv \int dz\, Z_l(z,v) Z_l(z,-v)}. \tag{18}$$

The integrals in (18) contain a short-distance cutoff for z, corresponding to the underlying lattice model. Thus there are no divergences at the lower limit of the z integration. Substituting Eq. (15) into Eq. (18), one sees that $\langle z^n \rangle$ is finite at the transition for $\beta(w) < -\frac{1}{6}(4+3n)$ and diverges *continuously* for $-\frac{1}{6}(4+3n) < \beta(w) < -\frac{2}{3}$ as the transition is approached. This situation can occur when the long-range part of the potential is *attractive*, in contrast to the interface case with z^{-2} potentials,[10,11] where the long-range part has to be sufficiently repulsive.

Finally, we consider potentials that fall off more slowly than $z^{-2/3}$. The membrane is bound to the wall for all $w > 0$. We expect scaling of the form

$$Z_l(z,v) = z^{a} l^{-\psi} g(z l^{-3/2}, v l^{-1/2}, w l^{\Delta}) \exp(\sigma l), \tag{19}$$

with $a = \frac{1}{6}$ and $\psi = 2 + 3a = \frac{5}{2}$, which is identical with (8) for $w = 0$. For $w > 0$ in the limit $l \to \infty$, Eq. (16) implies the scaling form

$$Z_l(z,v) = z^{a} w^{\psi/\Delta} \tilde{g}(z w^{3/2\Delta}, v w^{1/2\Delta}) \exp(\sigma l), \tag{20}$$

for the bound state. The relations

$$\sigma = \text{const} \times w^{1/\Delta}, \quad \Delta = 1 - \tfrac{3}{2} p \tag{21}$$

follow from Eq. (4). One finds $\langle z \rangle \sim w^{-3/2\Delta}$ for $w \to 0+$.

The unbinding transition of semiflexible fluid membranes in (2+1) dimensions has been argued to be in the same universality class as wetting in (1+1) dimensions.[19] Together with our results, the following picture emerges: For all $d < 5$ there is an intermediate-fluctuation regime with subregimes *A*, *B*, and *C* for potentials that decay as $z^{-\tau}$, where $\tau = -2(d-1)/(d-5)$. In $d = (1+1)$ dimen-

sions the short-range potentials belong to subregime C, and the transition is first order. As d increases, the fluctuations become less violent, and at a critical dimension $d_c > 2$ the short-range potentials move to subregime B. Renormalization-group studies[19] and Monte Carlo simulations[20] both indicate that $d_c < 3$. For $d_c < d < 5$, the transition should stay in subregime B.

In closing we note that for a "true" polymer (self-avoiding walk without the SOS restriction) subject to a z^{-p} potential, the strong, intermediate, and weak-fluctuation regimes correspond to $p > p_c$, $p = p_c$, and $p < p_c$, where in $d = 2$, $p_c = \frac{4}{3}$ instead of $\frac{2}{3}$. This follows from an independent-blob picture[7] and also from the equivalence with the $O(n)$ model of magnetism[21] in the limit $n \to 0$ and the result $p_c = \nu^{-1}$ for magnetic systems with inhomogeneous coupling constants.[22] For attractive potentials with $p < p_c$ the polymer is always bound. For $p > p_c$ the polymer unbinding transition corresponds to the "special" transition[21,23] of the magnetic system. At $p = p_c$ the surface magnetic exponents are known to be nonuniversal,[22] but the magnetic analog of the polymer unbinding transition has not yet been studied in detail.

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*Permanent address: Sektion Physik der Ludwig-Maximilians-Universität München, 8000 München 2, West Germany.

†Permanent address: Department of Physics, Temple University, Philadelphia, PA 19122.

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