# Photon-number distributions for fields with Gaussian Wigner functions

S. Chaturvedi and V. Srinivasan
*School of Physics, University of Hyderabad, Hyderabad 500 134, India*

(Received 27 April 1989)

We compute the generating function for the photon-number distribution for single-mode fields described by a Gaussian Wigner distribution considered previously by Agarwal and Adam [Phys. Rev. A **38**, 750 (1988)] and obtain simpler formulas for the photon-number distribution than those given by them. We also give analytical results for the factorial cumulants of the photon-number distribution and consider some limiting cases.

## I. INTRODUCTION

In a series of papers Agarwal[1] and Agarwal and Adam[2,3] have shown that a very large class of systems in nonlinear optics can be described in terms of a density matrix that corresponds to a Gaussian Wigner function and have studied in detail, the properties thereof both analytically as well as numerically.

The purpose of this Brief Report is twofold.

(i) We show that the density matrix investigated by Agarwal and Adam[3] can be obtained by applying a unitary transformation on the density matrix for a harmonic oscillator, the unitary transformation being the product of a displacement and a squeeze operator.[4,5] This permits easy identification of the limiting cases.

(ii) We show that the analytical expressions for the photon-number distribution given by Agarwal and Adam[3] can be considerably simplified by considering the generating function for the photon-number distribution rather than the photon-number distribution itself. This not only yields simpler expressions for the photon-number distribution but also permits us to obtain analytical expressions for all the factorial cumulants as well.

## II. GAUSSIAN WIGNER FUNCTION AND CORRESPONDING DENSITY OPERATOR

The Gaussian Wigner function investigated by Agarwal and Adam[3] is given by

$$\Phi(z,z^*)=\frac{1}{\pi(\tau^2-4|\mu|^2)^{1/2}}\exp\left[-\frac{\mu(z-z_0)^2+\mu^*(z^*-z_0^*)^2+\tau|z-z_0|^2}{\tau^2-4|\mu|^2}\right] \tag{2.1}$$

where the parameters $z_0$, $\mu$, and $\tau$ are related to the lower-order moments of the annihilation and creation operators $a$ and $a^\dagger$:

$$\begin{aligned}&\langle a\rangle=z_0,\quad \langle a^2\rangle=-2\mu^*+|z_0^2\ ,\\ &\langle (a^\dagger)^2\rangle=-2\mu+(z_0^*)^2,\quad \langle a^\dagger a\rangle=\tau-\tfrac{1}{2}+|z_0|^2\ .\end{aligned} \tag{2.2}$$

The positive definiteness of the density matrix corresponding to (2.1) puts certain restrictions on $\mu$ and $\tau$. These restrictions can be taken into account through the following parameterization of $\mu$ and $\tau$:

$$\mu=\frac{Q}{4}(\sinh x)e^{-i\theta}\ , \tag{2.2a}$$

$$\tau=\frac{Q}{2}\cosh x\ , \tag{2.2b}$$

with

$$Q\geq 1\ . \tag{2.2c}$$

The density matrix corresponding to (2.1) is

$$\begin{aligned}\rho=&[\tfrac{1}{4}(e^{2\phi}-1)]^{-1/2}\\ &\times\exp\{-2e^{-\phi}\cosh^{-1}(\coth\phi)\\ &\times[\mu(a-z_0)^2+\mu^*(a^\dagger-z_0^*)^2\\ &+\tau(a^\dagger-z_0^*)(a-z_0)+\tau/2]\}\ ,\end{aligned} \tag{2.3}$$

where

$$e^{2\phi}=4(\tau^2-4|\mu|^2)=Q^2\ . \tag{2.4}$$

The form of $\rho$ in (2.3) suggests that it can be expressed in terms of the density matrix $\rho_0$ for the harmonic oscillator at finite temperature in the following manner:

$$\rho=D(z_0)S(-\alpha)\rho_0S^\dagger(-\alpha)D^\dagger(z_0)\ , \tag{2.5}$$

where[4,5]

$$D(z_0)=\exp(z_0a^\dagger-z_0^*a)\ , \tag{2.6a}$$

$$S(\alpha)=\exp[\tfrac{1}{2}\alpha(a^\dagger)^2-\tfrac{1}{2}\alpha^*a^2]\ , \tag{2.6b}$$

and

$$\rho_0=\left[2\sinh\frac{\beta}{2}\right]^{1/2}\exp[-\beta(a^\dagger a+\tfrac{1}{2})]\ . \tag{2.6c}$$

Indeed, if we write

$$\alpha=re^{i\theta}\ ,$$

then (2.3) and (2.5) may easily shown to be identical when the following identifications are made:

$$x=2r \tag{2.7a}$$

 

$$e^{\phi}=Q=\coth\frac{\beta}{2}\ . \tag{2.7b}$$

This completes the decomposition of the density matrix $\rho$ corresponding to the Wigner distribution of (2.1) in terms of the density matrix $\rho_0$ for a harmonic oscillator at finite temperature, the displacement operator $D(z_0)$ and the squeeze operator $S(\alpha)$. The form (2.5) renders transparent the various limiting cases of (2.3).

## III. GENERATING FUNCTION FOR THE PHOTON-NUMBER DISTRIBUTION FOR THE FIELD CHARACTERIZED BY (2.1)

We compute the generating function $G(\lambda)$

$$G(\lambda)=\sum_{n=0}^{\infty}\lambda^n P(n)\ , \tag{3.1}$$

for the photon-number distribution $P(n)$ corresponding to the field characterized by (2.1) or (2.3) in two different ways by choosing to work with (a) the Wigner function given by (2.1) and (b) the form of the density operator (2.3) as given in (2.5).

(a) Given the Wigner function $\Phi(z,z^*)$, the corresponding $P(n)$ can be calculated using the well-known formula

$$P(n)=\int d^2z\ \Phi(z,z^*)2(-1)^n L_n(4|z|^2)\exp(-2|z|^2)\ , \tag{3.2}$$

where $L_n$'s are the Laguerre polynomials.[6] For the generating function $G(\lambda)$ we then have

$$G(\lambda)=2\left[\frac{1}{1+\lambda}\right]\int d^2z\ \exp(-4\lambda_1|z|^2)\Phi(z,z^*)\ , \tag{3.3}$$

where

$$\lambda_1=\frac{1}{2}\frac{(1-\lambda)}{(1+\lambda)}\ . \tag{3.4}$$

Since the Wigner function (2.1) under consideration is a Gaussian, the integral in (3.3) may easily be carried out. The result is given in Agarwal and Adam.[3] Expressing their result for (3.3) in terms of $\lambda$ we get

$$G(\lambda)=\Phi_0(z_0)g\left[\frac{C_+}{\nu_+(\nu_+-1)},-\lambda\left[1-\frac{1}{\nu_+}\right]\right] g\left[\frac{C_-}{\nu_-(\nu_--1)},-\lambda\left[1-\frac{1}{\nu_-}\right]\right]\ , \tag{3.5}$$

where

$$\Phi_0(z_0)=\frac{1}{Q\sqrt{\nu_+\nu_-}}\times\exp\left[\frac{-2r_+^2(\nu_+-\frac{1}{2})\nu_- -2r_-^2(\nu_--\frac{1}{2})\nu_+}{\nu_+\nu_-}\right] \tag{3.6}$$

and

$$g(x,z)=\frac{1}{\sqrt{1-z}}\exp\left[\frac{xz}{z-1}\right]\ , \tag{3.7}$$

with

$$\nu_\pm=\frac{1}{2}+\frac{e^{\pm x}}{2Q}\ , \tag{3.8}$$

$$C_\pm=\left[\frac{e^{\pm x}r_\pm}{Q}\right]^2\ , \tag{3.9}$$

The quantities $r_\pm$ are given by

$$r_+=r_0\cos\left[\phi_0-\frac{\theta}{2}\right]\ , \tag{3.10a}$$

$$r_-=r_0\left[\sin\phi_0-\frac{\theta}{2}\right]\ , \tag{3.10b}$$

where $r_0$ and $\phi_0$ are, respectively, the amplitude and phase of $z_0$,

$$z_0=r_0e^{i\phi_0}\ . \tag{3.11}$$

The expression (3.6) for $\Phi_0(z_0)$ when reexpressed in terms of $\mu$, $\tau$, and $z_0$ becomes

$$\Phi_0(z_0)=\frac{1}{[(\tau+\frac{1}{2})^2-4|\mu|^2]^{1/2}}\times\exp\left[-\frac{\mu z_0^2+\mu^*(z_0^*)^2+(\tau+\frac{1}{2})|z_0|^2}{(\tau+\frac{1}{2})^2-4|\mu|^2}\right]\ . \tag{3.12}$$

The function $g(x,z)$ is readily recognized to be the generating function for the associated Laguerre polynomials

$$g(x,z)=\sum_{n=0}^{\infty}z^nL_n^{-1/2}(x)\ . \tag{3.13}$$

From (3.5) and (3.13) we readily obtain

$$\begin{aligned}P(n)&=\frac{1}{n!}\left[\frac{\partial^nG}{\partial\lambda^n}\right]_{\lambda=0}\\&=\frac{1}{n!}\phi_0(z_0)\left\{\frac{\partial^n}{\partial\lambda^n}\left[g\left[\frac{C_+}{\nu_+(\nu_+-1)},-\lambda\left[1-\frac{1}{\nu_+}\right]\right]g\left[\frac{C_-}{\nu_-(\nu_--1)},-\lambda\left[1-\frac{1}{\nu_-}\right]\right]\right]\right\}_{\lambda=0}\\&=\frac{1}{n!}\phi_0(z_0)\sum_{k=0}^{n}\binom{n}{k}\left[\frac{\partial^{n-k}}{\partial\lambda^{n-k}}g\left[\frac{C_+}{\nu_+(\nu_+-1)},-\lambda\left[1-\frac{1}{\nu_+}\right]\right]\right]_{\lambda=0}\left[\frac{\partial^k}{\partial\lambda^k}g\left[\frac{C_-}{\nu_-(\nu_--1)},-\lambda\left[1-\frac{1}{\nu_-}\right]\right]\right]_{\lambda=0}\\&=(-1)^n\sum_{k=0}^{n}\left[1-\frac{1}{\nu_+}\right]^{n-k}\left[1-\frac{1}{\nu_-}\right]^kL_{n-k}^{-1/2}\left[\frac{C_+}{\nu_+(\nu_+-1)}\right]L_k^{-1/2}\left[\frac{C_-}{\nu_-(\nu_--1)}\right]\ .\end{aligned} \tag{3.14}$$

The formula (3.14) is considerably simpler than that given in Ref. 3 in that it involves a single summation.

(b) We now briefly outline the derivation of the formula (3.5) starting from the expression (2.5) for the density matrix corresponding to the Wigner function (2.1).

The generating function

$$G(\lambda)=\sum_{n=0}^{\infty}\langle n|\rho|n\rangle\lambda^n\ , \tag{3.15}$$

on using the resolution of the identity in terms of coherent states may be rewritten as

$$\begin{aligned}G(\lambda)&=\frac{1}{\pi^2}\int d^2\chi\, d^2\delta(\sum_n\langle n/\chi\rangle\langle\delta|n\rangle\lambda^n)\langle\chi|\rho|\delta\rangle\\&=\frac{1}{\pi^2}\int d^2\chi\, d^2\delta\exp[\lambda\chi\delta^*-|\chi|^2-|\delta|^2]R(\chi^*,\delta)\ ,\end{aligned}\tag{3.16}$$

where

$$R(\chi^*,\delta)=\langle\chi|\rho|\delta\rangle\exp[\tfrac{1}{2}|\chi|^2+\tfrac{1}{2}|\delta|^2]\ . \tag{3.17}$$

Using[7]

$$\frac{1}{\pi}\int d^2\delta\exp[\mu\delta^*-|\delta|^2]R(\gamma^*,\delta)=R(\gamma^*,\mu)\ , \tag{3.18}$$

we get

$$G(\lambda)=\frac{1}{\pi}\int d^2\chi\exp[-\tfrac{1}{2}(1-\lambda^2)|\chi|^2]\langle\chi|\rho|\lambda\chi\rangle\ . \tag{3.19}$$

The next step is the calculate $\langle\chi|\rho|\lambda\chi\rangle$. Since $\rho$ may be decomposed as in (2.5) we have

$$\begin{aligned}\langle\chi|\rho|\lambda\chi\rangle&=\langle\chi|D(z_0)S(-\alpha)\rho_0S^\dagger(-\alpha)D^\dagger(z_0)|\lambda\chi\rangle\\&=\exp[-\tfrac{1}{2}(1-\lambda)(\chi z_0^*=\chi^*z_0)]\\&\quad\times\langle\chi-z_0|S(-\alpha)\rho_0S^\dagger(-\alpha)|\lambda\chi-z_0\rangle\ .\end{aligned}\tag{3.20}$$

Since $\rho_0$ is given by (2.6c) we have

$$\begin{aligned}\langle\gamma|S(-\alpha)\rho_0S^\dagger(-\alpha)|\delta\rangle&=(1-e^{-\beta})\sum_m[\langle m|S^\dagger(-\alpha)|\delta\rangle\langle\gamma|S(-\alpha)|m\rangle\\&\quad\times(e^{-\beta})^m]\ .\end{aligned}\tag{3.21}$$

Using the results of Ref. 4, we find that

$$\langle m|S^\dagger(-\alpha)|\delta\rangle=(m!a)^{-1/2}\left[\frac{b}{2a}\right]^{m/2}H_m\left[\frac{\delta}{\sqrt{2ab}}\right]\times\exp\left[-\tfrac{1}{2}|\delta|^2+\frac{b^*}{2a}\delta^2\right]\ , \tag{3.22}$$

where $H_m$ are the Hermite polynomials and

$$a=\cosh r\ , \tag{3.23a}$$

$$b=-e^{i\theta}\sinh r\ . \tag{3.23b}$$

Using (3.22) in (3.21) and carrying out the summation over $m$ using the formula

$$\begin{aligned}&\sum_{m=0}^{\infty}\frac{1}{2^m}\frac{1}{m!}t^mH_m(z_1)H_m(z_2)\exp(-\tfrac{1}{2}z_1^2-\tfrac{1}{2}z_2^2)\\&\quad=\frac{1}{(1-t^2)^{1/2}}\exp\left[\frac{2z_1z_2t}{(1-t^2)}-\frac{(z_1^2+z_2^2)}{2(1-t^2)}(1+t^2)\right]\ ,\end{aligned}\tag{3.24}$$

we obtain

$$\begin{aligned}\langle\gamma|S(-\alpha)\rho_0S^\dagger(-\alpha)|\delta\rangle&=\exp\left[-\tfrac{1}{2}|\delta|^2+\frac{b^*}{2a}\delta^2+\frac{1}{2}\frac{\delta^2}{2ab}+\frac{1}{2}|\gamma|^2+\frac{b}{2a}(\gamma^*)^2+\frac{1}{2}\frac{(\gamma^*)^2}{2ab^*}\right]\\&\quad\times\frac{1}{(1-t^2)^{1/2}}\exp\left\{\left[\frac{2}{1-t^2}\right]\left[\frac{2\delta\gamma^*t}{2a(bb^*)^{1/2}}-\frac{1}{2}\left[\frac{\delta^2}{2ab}\right](1+t^2)-\frac{1}{2}\left[\frac{(\gamma^*)^2}{2ab^*}\right](1+t^2)\right]\right\}\ ,\end{aligned}\tag{3.25}$$

where

$$t=\left[\frac{bb^*}{a}\right]^{1/2}e^{-\beta}\ . \tag{3.26}$$

Putting $\gamma=\chi-z_0$ and $\delta=\lambda\chi-z_0$ in (3.25) and using (3.20) we obtain $\langle\chi|\rho|\lambda\chi\rangle$. The expression for $\langle\chi|\rho|\lambda\chi\rangle$ thus obtained has a Gaussian form in $\chi$. Substituting it in (3.19) and carrying out the Gaussian integral we obtain (3.5).

## IV. FACTORIAL CUMULANTS OF $P(n)$

$G(\lambda)$ defined in (3.1) or (3.15) generates factorial moments of $P(n)$

$$\begin{aligned}\langle n^m\rangle_f&\equiv\sum_{n=0}^{\infty}n(n-1)\cdots(n-m)P(n)\\&=\left.\frac{\partial^mG(\lambda)}{\partial\lambda^m}\right|_{\lambda=1}\ .\end{aligned}\tag{4.1}$$

Since $G(\lambda)$ is given explicitly by (3.5)–(3.7), we may compute these analytically if desired.

Of greater interest are the factorial cumulants[7] (FC)

$$\langle n^m\rangle_{\rm FC}=\left.\frac{\partial^m\ln G(\lambda)}{\partial\lambda^m}\right|_{\lambda=1}\ , \tag{4.2}$$

for which we obtain

$$\langle n^m\rangle_{FC}=(m-1)!\left[\frac{1-\nu_+}{2\nu_+-1}\right]^m\left[\frac{1}{2}+\frac{mC_+}{(1-\nu_+)(2\nu_+-1)}\right]+(m-1)!\left[\frac{1-\nu_-}{2\nu_--1}\right]^m\times\left[\frac{1}{2}+\frac{mC_-}{(1-\nu_-)(2\nu_--1)}\right]. \quad (4.3)$$

These are of interest in discussing sub-Poissonian statistics.

## V. LIMITING CASES

We discuss two limiting cases.

(i) $x=0$. In this case the density operator (2.5) becomes

$$\rho=D(z_0)\rho_0D^\dagger(z_0) \quad (5.1)$$

and describes a mixture of a coherent and an incoherent field. Further, in this case, we have

$$\nu_+=\nu_-=\nu;\quad C_++C_-=\left[\frac{r_0}{Q}\right]^2=C,\quad Q=2\tau, \quad (5.2)$$

and the expression for $G(\lambda)$ becomes

$$G(\lambda)=\left[\frac{1}{(\tau+\frac{1}{2})}\exp\left[\frac{-|z_0|^2}{(\tau+\frac{1}{2})}\right]\right]\frac{1}{1+\lambda(1-1/\nu)}\times\exp\left[\frac{C}{\nu(\nu-1)}\frac{\lambda(1-1/\nu)}{1+\lambda(1-1/\nu)}\right], \quad (5.3)$$

which is easily seen to be the generating function for Laguerre polynomials,

$$G(\lambda)=\Phi_0(z_0)\sum_{n=0}^{\infty}\lambda^n\left[\frac{1}{\nu}-1\right]^nL_n\left[\frac{C}{\nu(\nu-1)}\right], \quad (5.4)$$

from which we readily obtain, after reexpressing $\nu$ in terms of

$$P(n)=\frac{(\tau-\frac{1}{2})^n}{(\tau+\frac{1}{2})^{n+1}}\exp\left[-\frac{|z_0|^2}{\tau+\frac{1}{2}}\right]L_n\left[-\frac{|z_0|^2}{\tau^2-\frac{1}{4}}\right]. \quad (5.5)$$

The corresponding factorial cumulants are given by

$$\langle n^m\rangle_{FC}=(m-1)!(\tau-\tfrac{1}{2})^m\left[1+\frac{m|z_0|^2}{\tau-\frac{1}{2}}\right]. \quad (5.6)$$

(ii) $z_0=0$. In this case the density matrix reduces to

$$\rho=S(-\alpha)\rho_0S^\dagger(-\alpha). \quad (5.7)$$

The corresponding $G(\lambda)$ becomes

$$G(\lambda)=\frac{1}{[(\tau+\frac{1}{2})^2-4|\mu|^2]^{1/2}}\frac{1}{[1+\lambda(1-1/\nu_+)]^{1/2}}\times\frac{1}{[1+\lambda(1-1/\nu_-)]^{1/2}} \quad (5.8)$$

which is easily seen to be related to the generating function for Legendre polynomials,

$$G(\lambda)=\frac{1}{[(\tau+\frac{1}{2})^2-4|\mu|^2]^{1/2}}\frac{1}{(1-2tz+t^2)^{1/2}}, \quad (5.9)$$

where

$$t=\lambda\left[\frac{Q^2+1-2Q\cosh x}{Q^2+1+2Q\cosh x}\right]^{1/2}. \quad (5.10a)$$

$$z=\frac{(Q^2-1)}{[(Q^2+1)^2-4Q^2\cosh^2x]^{1/2}}, \quad (5.10b)$$

and hence gives

$$P(n)=\frac{2}{(Q^2+1+2Q\cosh x)^{1/2}}\left[\frac{Q^2+1-2Q\cosh x}{Q^2+1+2Q\cosh x}\right]^{n/2}\times P_n\left[\frac{Q^2-1}{[(Q^2+1)^2-4Q^2\cosh^2x]^{1/2}}\right]. \quad (5.11)$$

For $Q=1$, this gives the known result[8]

$$P(2n+1)=0, \quad (5.12a)$$

$$P(2n)=\frac{(2n-1)!!}{2^nn!}\frac{1}{\left[\cosh\frac{x}{2}\right]}\left[\tanh\frac{x}{2}\right]^{2n}. \quad (5.12b)$$

The factorial cumulants in this case are given by

$$\langle n^m\rangle_{FC}=(m-1)!\frac{1}{2}\left[\left[\frac{Q}{2}e^{-x}-\frac{1}{2}\right]^m+\left[\frac{Q}{2}e^{x}-\frac{1}{2}\right]^m\right]. \quad (5.13)$$

For $Q=1$, this expression becomes

$$\langle n^m\rangle_{FC}=\begin{cases}(m-1)!\left[\sinh\frac{x}{2}\right]^m\cosh\left[\frac{mx}{2}\right], & m \text{ even} \quad (5.14a)\\ (m-1)!\left[\sinh\frac{x}{2}\right]^m\sinh\left[\frac{mx}{2}\right], & m \text{ odd}. \quad (5.14b)\end{cases}$$

## CONCLUSIONS

We have computed the generating function for the photon-number distribution corresponding to a density matrix with a Gaussian Wigner function of a fairly general structure, considered by Agarwal and Adam.[3] We have shown that the density operator can be decomposed into a form which makes the various limiting cases rather transparent and have presented the formulas of Agarwal and Adam for the photon-number distribution in a simpler form. We also present analytical expressions for the factorial cumulants which are easily obtained from the knowledge of the generating function.

## ACKNOWLEDGMENTS

We are extremely grateful to Professor G. S. Agarwal for numerous discussions.


[1] G. S. Agarwal, J. Mod. Opt. **34**, 909 (1987).

[2] G. S. Agarwal and G. Adam, Phys. Rev. A **38**, 750 (1988).

[3] G. S. Agarwal and G. Adam, Phys. Rev. A **39**, 6259 (1989).

[4] H. P. Yuen, Phys. Rev. A **13**, 2226 (1976).

[5] J. Hollenhorst, Phys. Rev. D **19**, 1669 (1979).

[6] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1965).

[7] See, for instance, C. W. Gardiner, *Handbook of Stochastic Methods* (Springer, Berlin, 1983).

[8] W. Schlech and J. A. Wheeler, J. Opt. Soc. Am. B **4**, 1715 (1987).