Electromagnetic-wave propagation in anisotropic stratified media

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The electromagnetic field generated in a nonabsorbing anisotropic multilayer by a plane incident wave with given wave vector is considered. By taking into account the fact that the field within a given layer is a superposition of four proper waves, namely, four waves that propagate without changing their polarization state, the field is associated with a four-dimensional complex vector space. In this space a new algebra is defined such that the norm of a vector gives the energy flux density of the corresponding field, and the scalar product of two proper waves is zero. This allows one (1) to simply derive the amplitudes of the four proper waves for any layer, (2) to explicitly write the reflection and transmission coefficients of a structure whose layers are uniaxial media with arbitrary direction of the optical axis, and (3) to deduce the propagation equations for the four waves in the limiting case of a plane stratified medium with continuous variation of the dielectric tensor. Similar equations are found in the literature only for some very particular cases and are generally used to obtain approximate solutions for the wave equation. Here the general case is considered. The given equations contain as specific cases most of the already known approximations and give a unifying method for their discussion and generalization. In particular, they contain the geometrical-optics approximation (GOA) for uniaxial stratified media with arbitrary directions of the optical axis and of the incidence angle, as well as a generalization of the GOA, where the coupling between the ordinary and extraordinary waves is taken into account. The theory developed here has a wide range of applications in many fields of physics, as for instance the propagation of electromagnetic waves in magnetoactive plasmas, the optical properties of liquid crystals, and more generally the optics of anisotropic media.

I. INTRODUCTION

The first systematic studies on the propagation of the electromagnetic waves in anisotropic stratified media are related to the transmission of radio waves in the ionosphere. The main results obtained up to 1969 and a reference list of more than a hundred papers are found in the textbooks of Ginsburg[1] and of Budden.[2] A new interest in such studies is related to the development of electrooptic devices, and, in particular, of liquid-crystal devices.

In order to clearly expound on the aim of this paper, let us first consider the simple case of a linear homogeneous anisotropic layer between two parallel planes orthogonal to the z axis of a Cartesian-coordinate system. A plane monochromatic wave with given wave vector $\vec{K}_i$ generates in the layer two proper waves with different wave vectors, because the medium is doubly refracting. Since the second boundary plane gives rise to two reflected waves, the electromagnetic field within the layer is a linear combination of four proper waves (namely four waves propagating without changing their polarization states). The coefficients of the linear combination can be considered the components of a four-dimensional complex vector $\boldsymbol{\phi}$, which defines the "state" of the field. In a multilayer the state vector $\boldsymbol{\phi}$ changes from layer to layer. In the limiting case where the thickness of the layers goes to zero, giving rise to a stratified medium in which the dielectric tensor continuously changes by increasing z, the state vector becomes a function of z.

The aim of the work is to compute the $\boldsymbol{\phi}$ vector, defining the field generated in planar stratified media by a plane incident wave. The starting point will be the propagation equation for a different four-dimensional vector function $\boldsymbol{\psi}(z)$, representing the same field and defined by making use of four components of the complex electric and magnetic vectors $\vec{E}(z)$ and $\vec{H}(z)$. This propagation equation is well known,[2,3] because it is directly and simply derived from the Maxwell equations.

The vectors $\boldsymbol{\psi}(z)$ and $\boldsymbol{\phi}(z)$ can be considered different representations of the same vector space. The transformation between the two representations is defined by the relation $\boldsymbol{\psi}(z)=\underline{T}(z)\boldsymbol{\phi}(z)$. The interest in the $\boldsymbol{\phi}$ representation is due to the fact that the components of $\boldsymbol{\phi}$ are directly related the measured physical quantities. Knowledge of these components gives a deeper insight into the physics of the problem, a fact which is particularly useful if one makes use of approximated methods. To this purpose, we observe that simple analytic solutions of the propagation equations for inhomogeneous media are known in few very particular cases. Generally, numerical analysis[4] or approximate methods are required.[5-8] The most important approximations make use—or can be reformulated by making use—of the $\boldsymbol{\phi}$ representation. This representation gives a sound and unifying basis which allows one to generate, discuss, and compare the different approximations, such as, for instance, the geometrical optics approximation and various types of perturbation expansions.

Until now, this technique has been used only for particularly simple cases,[9] since the general expressions of the matrices $\underline{T}$ and $\underline{T}^{-1}$ are very complicated. The preliminary—and in a sense the central point—of our analysis will be the definition of a matrix algebra which allows one to simplify these expressions and their deduction. This goal is achieved by the use of a suitably defined scalar product between two vectors of the ψ space. With such a definition, the scalar product between the vectors corresponding to the four proper waves of a nonabsorbing layer is zero: these waves can therefore be considered orthogonal. For normally incident light, the wave vectors of the four proper waves are parallel and their polarization states are indeed mutually orthogonal. Our definition of a scalar product extends the concept of orthogonality to waves which propagate in different directions.

The advantages obtained by the use of an orthogonal basis for a vector space are evident. As an example, a simple relation between the matrix elements of $\underline{T}$ and $\underline{T}^{-1}$ is expected, and is indeed found. The starting point of our analysis is the theory developed in Ref. 3. In Secs. II and III we define the scalar product, on the basis of energy considerations, and state the orthogonality relations. These relations are used in Secs. IV and V to derive the reflectances and transmittances of multilayers and the propagation equation for $\phi(z)$ in nonabsorbing continuous media.

II. ENERGY FLUX AND DEFINITON OF THE SCALAR PRODUCT

We consider a linear medium whose optical properties depend on the single Cartesian coordinate z. The full translational invariance of the Maxwell equations with respect to x, y, and t suggests that one look for solutions having the structure

$$\begin{bmatrix} \vec{\mathcal{E}} \\ \vec{\mathcal{H}} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \vec{\mathrm{E}}(z) \\ \vec{\mathrm{H}}(z) \end{bmatrix} \exp[i(K_x x + K_y y - \omega t)] + \text{c.c.} \ , \tag{2.1}$$

which will be used to solve the physical problem where a plane monochromatic wave is incident on the medium through the plane boundary surface $z=0$. In the following, we assume (x,z) is the incidence plane. This gives

$$K_y = 0 \ . \tag{2.1'}$$

Only four of the six components of the fields $\vec{\mathrm{E}}, \vec{\mathrm{H}}$ are independent. Generally, the x and y components are considered.[2,3] In Berreman's formalism the Maxwell equations are cast in the form[3]

$$\frac{d\psi(z)}{dz} = iK\underline{D}(z)\psi(z) \ , \tag{2.2}$$

where $K=\omega/c$, $\underline{D}$ is a 4×4 complex matrix, and

$$\psi = \begin{bmatrix} E_x \\ H_y \\ E_y \\ -H_x \end{bmatrix} . \tag{2.3}$$

The time average of the energy flux density through the planes $z=$const is given by the quantity

$$\frac{c}{16\pi}(H_y^* E_x + E_x^* H_y - H_x^* E_y - E_y^* H_x) = \frac{c}{16\pi}\psi^\dagger \underline{M}\psi \ , \tag{2.4}$$

which is the time average of the z component of the Poynting vector. Here, $\dagger$ is the symbol of Hermitian conjugation, namely

$$\psi^\dagger = (E_x^*, H_y^*, E_y^*, -H_x^*) \ , \tag{2.5}$$

and $\underline{M}$ is the matrix,

$$\underline{M} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} . \tag{2.6}$$

For nonabsorbing media we expect that the flux will be z independent, namely $d(\psi^\dagger \underline{M}\psi)/dz = 0$, and that this property will be related to some property of the matrix $\underline{D}$. Now from Eq. (2.2) and its Hermitian conjugate,

$$\frac{d}{dz}\psi^\dagger = -iK\psi^\dagger \underline{D}^\dagger \ , \tag{2.7}$$

it immediately follows that

$$\frac{d}{dz}(\psi^\dagger \underline{M}\psi) = iK\psi^\dagger(\underline{M}\,\underline{D} - \underline{D}^\dagger \underline{M})\psi \ , \tag{2.8}$$

and the condition for energy conservation is therefore

$$\underline{M}\,\underline{D} = \underline{D}^\dagger \underline{M} \ , \tag{2.9}$$

or, equivalently,

$$\underline{M}\,\underline{D}^\dagger \underline{M} = \underline{D} \ , \tag{2.9'}$$

where use has been made of the properties $(\psi^\dagger)^\dagger = \psi$, $(\underline{M}\psi)^\dagger = \psi^\dagger \underline{M}^\dagger$, and $\underline{M} = \underline{M}^\dagger = \underline{M}^{-1}$.

For a medium with unit permeability, the matrix $\underline{D}$ is given by[3]

$$\underline{D} = \begin{bmatrix} -m\dfrac{\epsilon_{zx}}{\epsilon_{zz}} & 1 - \dfrac{m^2}{\epsilon_{zz}} & -m\dfrac{\epsilon_{zy}}{\epsilon_{zz}} & 0 \\ \epsilon_{xx} - \dfrac{\epsilon_{xz}\epsilon_{zx}}{\epsilon_{zz}} & -m\dfrac{\epsilon_{xz}}{\epsilon_{zz}} & \epsilon_{xy} - \dfrac{\epsilon_{xz}\epsilon_{zy}}{\epsilon_{zz}} & 0 \\ 0 & 0 & 0 & 1 \\ \epsilon_{yx} - \dfrac{\epsilon_{yz}\epsilon_{zx}}{\epsilon_{zz}} & -m\dfrac{\epsilon_{yz}}{\epsilon_{zz}} & \epsilon_{yy} - \dfrac{\epsilon_{yz}\epsilon_{zy}}{\epsilon_{zz}} - m^2 & 0 \end{bmatrix} , \tag{2.10}$$

where $\underline{\epsilon}$ is the dielectric tensor, and

$$m = K_x/K = n_i \sin\vartheta_i \ . \tag{2.11}$$

Here, ϑ_i is the incidence angle and n_i is the refractive index of the incident medium.

For a nonabsorbing medium the matrix $\underline{\epsilon}$ is Hermitian (namely $\epsilon_{ij} = \epsilon_{ji}^*$), and $\underline{M}\,\underline{D} = \underline{D}^\dagger \underline{M}$, as is easily verified: the energy flux is indeed conserved.

The scalar quantity $\psi_b^\dagger \underline{M} \psi_a$ will appear repeatedly in the following. Owing to its evident analogy with a scalar product (with the matrix $\underline{M}$ playing the role of the metric tensor of the theory of relativity), it will be named the "scalar product (ψ_a, ψ_b) of ψ_a and ψ_b" in the ψ-vector space. Consequently, two vectors whose scalar product is zero will be considered orthogonal, and $\psi^\dagger \underline{M} \psi$ will be considered the "norm" of the vector ψ. This norm is real but not necessarily positive. With the above definition of the scalar product, the property of the $\underline{D}$ matrix given by Eq. (2.9) implies:

$$(\underline{D}\psi_a, \psi_b) = (\psi_a, \underline{D}\psi_b) . \tag{2.12}$$

This means that the operator of the ψ defined by the matrix $\underline{D}$ is self-adjoint. If we consider that it is the operator that determines the spatial evolution of the electromagnetic field [through Eq. (2.2)], we realize that the property (2.12) has extremely important consequences for the whole theory (similar to those deriving in quantum mechanics from the fact that the Hamiltonian operator is self-adjoint).

In the following sections we will consider nonabsorbing media whose $\underline{D}$ matrix has the structure (2.10) and therefore satisfies Eqs. (2.9) and (2.12). The theory which follows has a wide range of applications, such as, for instance, the propagation of radio waves in horizontal ionosphere and the optics of crystals and liquid crystals.

III. ORTHOGONALITY RELATIONS AND ϕ REPRESENTATION FOR NONABSORBING MEDIA

For homogeneous media the matrix $\underline{D}$ is z independent and the propagation equation (2.2) admits solutions of the type

$$\psi^j \exp(iKd_j z) , \tag{3.1}$$

which are the four proper electromagnetic waves (two forward- and two backward-propagating) with the same values of ω and K_x, and with $K_y = 0$.

By inserting Eq. (3.1) into (2.2), we obtain the characteristic equation for $\underline{D}$:

$$\underline{D}\psi^j = d_j \psi^j . \tag{3.2}$$

Let us now consider two proper vectors ψ^i and ψ^j. Equation (3.2) gives the relations $d_j \underline{M}\psi^j = \underline{M}\,\underline{D}\psi^j$ and $d_i^* \psi_i^\dagger \underline{D}^\dagger \underline{M} = \psi_i^\dagger \underline{D}^\dagger \underline{M}$, where $\psi_i^\dagger = (\psi^i)^\dagger$. From these relations and from Eq. (2.9), it follows that

$$(d_j - d_i^*)\psi_i^\dagger \underline{M}\psi^j = \psi_i^\dagger(\underline{M}\,\underline{D} - \underline{D}^\dagger \underline{M})\psi^j = 0 . \tag{3.3}$$

By taking $i = j$, Eq. (3.3) shows that the proper vector of a complex proper value has zero norm. Such vectors represent evanescent waves, a case requiring separate analysis which will not be considered here.[10] The same equation shows that two proper vectors corresponding to different real proper values are orthogonal. For degenerated proper values ($d_i = d_j$) the scalar product $\psi_i^\dagger \underline{M}\psi^j$ can be different from zero. However, in this case any linear combination of ψ^i, ψ^j is a proper vector of d_i, and in this two-dimensional vector space we can always choose two orthogonal vectors. We can therefore write

$$\psi_i^\dagger \underline{M}\psi^j = N_i \delta_{ij} , \tag{3.4}$$

or, equivalently,

$$N_i^{-1}\psi_i^\dagger \underline{M}\psi^j = \delta_{ij} , \tag{3.4'}$$

where N_i is the norm of the vector ψ^i. Let us now choose the indexes 1,2 for the forward-propagating waves and 3,4 for the other two waves. With such an assumption, N_1 and N_2 are positive and N_3 and N_4 are negative. It is natural to assume as normalization conditions

$$N_1 = N_2 = 1, \quad N_3 = N_4 = -1 . \tag{3.5}$$

The ϕ representation is obtained by expressing ψ as a superposition of the four proper vectors, namely

$$\psi = \sum_{j=1}^{4} f_j \psi^j = \underline{T}\phi , \tag{3.6}$$

where

$$\phi = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} , \tag{3.7}$$

and $\underline{T}$ is the matrix whose element t_{ij} is the i component of the vector ψ^j. This means that the columns of $\underline{T}$ are the column vectors $\psi^1, \psi^2, \psi^3, \psi^4$ and that the rows of $\underline{T}^\dagger$ are the vectors $\psi_1^\dagger, \psi_2^\dagger, \psi_3^\dagger, \psi_4^\dagger$. The orthogonality relations (3.4') can therefore be written as

$$\underline{N}^{-1}\underline{T}^\dagger \underline{M}\,\underline{T} = \underline{1} , \tag{3.8}$$

where $\underline{N}$ is the diagonal matrix whose nonzero elements are the norms N_1, N_2, N_3, N_4 of the proper vectors of $\underline{D}$, and $\underline{1}$ is the unity 4×4 matrix.

Equation (3.8) shows that $\underline{N}^{-1}\underline{T}^\dagger \underline{M}$ is the inverse of the matrix $\underline{T}$, namely

$$\underline{T}^{-1} = \underline{N}^{-1}\underline{T}\,\underline{M} , \tag{3.9}$$

and, with the normalization condition (3.5), is expressed as

$$\underline{T}^{-1} = \begin{bmatrix} t_{21}^* & t_{11}^* & t_{41}^* & t_{31}^* \\ t_{22}^* & t_{12}^* & t_{42}^* & t_{32}^* \\ -t_{23}^* & -t_{13}^* & -t_{43}^* & -t_{33}^* \\ -t_{24}^* & -t_{14}^* & -t_{44}^* & -t_{34}^* \end{bmatrix} . \tag{3.10}$$

IV. LAYERED MEDIA WITH DISCONTINUITY PLANES

Here we consider a plane wave incident on a set of N homogeneous layers separated by $N+1$ planes at $z = z_n$ ($n = 0, 1, \ldots, N$), with the aim of finding out analytic expressions for the reflected and transmitted waves.

It is convenient to consider the case $N = 0$ first, corresponding to a single planar interface at $z = 0$. With the symbols of Sec. III, the incident and transmitted waves are represented by ϕ vectors with $f_3=f_4=0$, and the reflected wave by a vector with $f_1=f_2=0$. The ϕ vectors at the two sides of the boundary plane are, therefore, with obvious symbols,

$$\phi(0^-)=\begin{bmatrix} i_1 \\ i_2 \\ r_1 \\ r_2 \end{bmatrix}, \quad \phi(0^+)=\begin{bmatrix} t_1 \\ t_2 \\ 0 \\ 0 \end{bmatrix} . \tag{4.1}$$

Our problem consists of expressing the amplitudes r_1,r_2,t_1,t_2 of the reflected and transmitted waves as a function of the amplitudes i_1,i_2 of the incident proper waves.

The continuity of the tangential components of the vectors $\vec{E}$ and $\vec{H}$ is expressed by the relation

$$\psi(0^-)=\psi(0^+) . \tag{4.2}$$

These simple preliminary considerations clearly show the relative roles played by the ψ and ϕ vectors. The first enters directly into the boundary conditions and in the propagation equations. The second is more directly related to the physical quantities of interest in experiments, which in our case are the reflectances and transmittances, namely the ratios r_i/i_j and t_i/i_j $(i,j=1,2)$. The transformation matrices $\underline{T}$ and $\underline{T}^{-1}$ play the role of interfaces between the two aspects of the problem.

From Eqs. (4.1) and (4.2), and the relation $\psi=\underline{T}\phi$, we immediately obtain

$$\phi(0^-)=T_i^{-1}T_t\phi(0^+) , \tag{4.3}$$

where $\underline{T}_i$ and $\underline{T}_t$ are the $\underline{T}$ matrices of the first and second medium, respectively. Equation (4.3) gives four linear equations in the four unknowns r_1,r_2,t_1,t_2, and can therefore be considered the solution of our problem.

Now we consider the case $N=1$, corresponding to a single homogeneous layer between the planes $z=0$ and $z=z_1$. The continuity of the ψ vectors at the layer boundaries gives two relations of the type (4.3), namely

$$\begin{aligned} &\phi(0^-)=\underline{T}_i^{-1}\underline{T}\phi(0^+) , \\ &\phi(z_1^-)=\underline{T}^{-1}\underline{T}_t\phi(z_1^+) , \end{aligned} \tag{4.4}$$

where the quantity $\underline{T}$ without indices refers to the inner medium. The relation between $\phi(0^+)$ and $\phi(z_1^-)$ is, according to Eq. (3.1),

$$\phi(z_1^-)=\underline{K}(z_1)\phi(0^+) , \tag{4.5}$$

where $\underline{K}(z_1)$ is the diagonal matrix whose nonzero elements are

$$K_{jj}(z_1)=\exp(iKd_jz_1) . \tag{4.6}$$

The quantities d_j $(j=1,2,3,4)$ are the proper values of the $\underline{D}$ matrix for the inner medium. From Eqs. (4.4)–(4.6) we immediately obtain

$$\phi(0^-)=\underline{T}_i^{-1}\underline{U}\,\underline{T}_t\phi(z^+) , \tag{4.7}$$

where $\underline{U}$ is the transfer matrix for the ψ vectors within the layer, implicitly defined by $\psi(z_1)=\underline{U}\psi(0)$. It is given by

$$\underline{U}=\underline{T}\,\underline{K}^{-1}(z_1)\underline{T}^{-1} , \tag{4.8}$$

or, equivalently,

$$\underline{U}=\sum_{j=1}^{4}\psi^j\exp(-iKd_jz_1)\tilde{\psi}_j , \tag{4.8'}$$

where $\tilde{\psi}_1,\tilde{\psi}_2,\tilde{\psi}_3,\tilde{\psi}_4$ are the rows of the matrix $\underline{T}^{-1}$.

For the general case of N layers, the solution of our problem is given by Eq. (4.7), where $\underline{U}$ is the transfer matrix of the layer set, and is given by the product of the N transfer matrices of each layer.

All of the above results are very simply stated and largely known.[3] The new important fact is that for any homogeneous medium the matrix $\underline{T}^{-1}$ is known if we know $\underline{T}$. In order to obtain an explicit expression for $\underline{T}$, one must find the proper vectors ψ^j. By taking into account the structure of the matrix $\underline{D}$, given by Eq. (2.10), the first three equations of the set (3.2) give

$$\psi^j=C_j\begin{bmatrix} d_{12}d_{23}-d_{13}(d_{22}-d_j) \\ d_{13}d_{21}-d_{23}(d_{11}-d_j) \\ (d_{11}-d_j)(d_{22}-d_j)-d_{12}d_{21} \\ d_j[(d_{11}-d_j)(d_{22}-d_j)-d_{12}d_{21}] \end{bmatrix} , \tag{4.9}$$

where d_{ij} are the elements of $\underline{D}$, and C_j is the normalization constant. The proper values d_j appearing in Eq. (4.9) are the solutions of the characteristic equation of $\underline{D}$. Since $\underline{D}$ is a 4×4 matrix, the characteristic equation is a quartic. It admits analytic solutions, which are generally given by rather complicated expressions, as is well known. For our problem we therefore expect reasonably simple expressions only if the quartic has simple solution. This is the case of uniaxial, nonoptically active media, where the two roots of the quartic corresponding to the ordinary waves are well known. Taking into account this fact, the quartic can be reduced to a quadratic equation for the extraordinary proper values. The explicit expressions for the matrix $\underline{D}$ and its proper values and vectors are

$$\underline{D}=\begin{bmatrix} m_t\cos\varphi & 1-m^2\epsilon_t/\epsilon_o\epsilon_e & m_t\sin\varphi & 0 \\ \epsilon_f & m_t\cos\varphi & (\epsilon_t-\epsilon_o)\sin\varphi\cos\varphi & 0 \\ 0 & 0 & 0 & 1 \\ (\epsilon_t-\epsilon_o)\sin\varphi\cos\varphi & m_t\sin\varphi & \epsilon_o+\epsilon_t-\epsilon_f-m^2 & 0 \end{bmatrix} , \tag{4.10}$$

$$d_e^{\pm}=m_t\cos\varphi\pm\left[\epsilon_t-m^2\frac{\epsilon_t\epsilon_f}{\epsilon_e\epsilon_o}\right]^{1/2},\quad d_o^{\pm}=\pm(\epsilon_0-m^2)^{1/2}, \tag{4.11}$$

$$\psi_e^{\pm}=C_e^{\pm}\begin{bmatrix}(1-m^2/\epsilon_o)\cos\varphi-m(\cot\vartheta)d_e^{\pm}/\epsilon_o\\ d_e^{\pm}\cos\varphi-m\cot\vartheta\\ \sin\varphi\\ d_e^{\pm}\sin\varphi\end{bmatrix}, \tag{4.12a}$$

$$\psi_o^{\pm}=C_o^{\pm}\begin{bmatrix}-\sin\varphi\\ -\epsilon_o(\sin\varphi)/d_o^{\pm}\\ \cos\varphi-m(\cot\vartheta)/d_o^{\pm}\\ d_o^{\pm}\cos\varphi-m\cot\vartheta\end{bmatrix}, \tag{4.12b}$$

where

$$\begin{aligned}&\epsilon_0=n_o^2,\quad \epsilon_e=n_e^2,\\ &\frac{1}{\epsilon_t}=\frac{\cos^2\vartheta}{\epsilon_o}+\frac{\sin^2\vartheta}{\epsilon_e},\\ &\epsilon_f=\epsilon_o\sin^2\varphi+\epsilon_t\cos^2\varphi,\\ &m=K_x/K=n_i\sin\vartheta_i,\\ &m_t=m(1-\epsilon_t/\epsilon_o)\cot\vartheta.\end{aligned} \tag{4.13}$$

n_o, n_e are the ordinary and extraordinary refractive indices of the medium, and ϑ, φ are the polar angles of the optical axis.

Generally, the multilayer is sandwiched between homogeneous media and the incident, reflected and transmitted waves are analyzed in terms of p- (or TM) and s- (or TE) polarizations. The ψ-vectors for these waves are obtained from Eq. (4.12) by assuming $n_e=n_o$ and $\varphi=0$. This gives

$$\underline{T}_i=\frac{1}{\sqrt{2}}\begin{bmatrix}1/p_i & 0 & 1/p_i & 0\\ p_i & 0 & -p_i & 0\\ 0 & 1/s_i & 0 & 1/s_i\\ 0 & s_i & 0 & -s_i\end{bmatrix},$$

$$\underline{T}_i^{-1}=\frac{1}{\sqrt{2}}\begin{bmatrix}p_i & 1/p_i & 0 & 0\\ 0 & 0 & s_i & 1/s_i\\ p_i & -1/p_i & 0 & 0\\ 0 & 0 & s_i & -1/s_i\end{bmatrix}, \tag{4.14}$$

where

$$p_i=(n_i/\cos\vartheta_i)^{1/2},\quad s_i=(n_i\cos\vartheta_i)^{1/2}, \tag{4.15}$$

and ϑ_i is the incidence angle and n_i the refractive index of the first medium. Analogous relations hold for T_t.

Equations (4.1), (4.7), and (4.14) give

$$t_1=\frac{b_2^+i_1-a_2^+i_2}{a_1^+b_2^+-a_2^+b_1^+},\quad t_2=\frac{-b_1^+i_1+a_1^+i_2}{a_1^+b_2^+-a_2^+b_1^+},$$

$$r_1=a_1^-t_1+a_2^-t_2,\quad r_2=b_1^-t_1+b_2^-t_2,$$

where

$$\begin{aligned}2a_1^{\pm}&=u_{11}\frac{p_i}{p_t}+u_{12}p_ip_t\pm u_{21}\frac{1}{p_ip_t}\pm u_{22}\frac{p_t}{p_i},\\ 2a_2^{\pm}&=u_{13}\frac{p_i}{s_t}+u_{14}p_is_t\pm u_{23}\frac{1}{p_is_t}\pm u_{24}\frac{s_t}{p_i},\\ 2b_1^{\pm}&=u_{31}\frac{s_i}{p_t}+u_{32}s_ip_t\pm u_{41}\frac{1}{s_ip_t}\pm u_{42}\frac{p_t}{s_i},\\ 2b_2^{\pm}&=u_{33}\frac{s_i}{s_t}+u_{34}s_is_t\pm u_{43}\frac{1}{s_is_t}\pm u_{44}\frac{s_t}{s_i},\end{aligned}$$

and where u_{ij} are the elements of the matrix $\underline{U}$, which is given by Eqs. (4.5) and (4.8). For a multilayer of uniaxial media with arbitrary directions of the optical axis, the matrix $\underline{U}$ is straightforwardly computed by making use of Eqs. (4.11)–(4.13) and (3.10). This long-standing problem receives a satisfactory solution here.

V. PROPAGATION EQUATION FOR THE VECTOR ϕ IN CONTINUOUS MEDIA

A medium with continuous variation of the tensor $\underline{\epsilon}(z)$ can be considered the limit for $\Delta z\to 0$ of a stratified medium constituted of homogeneous layers of thickness Δz. The analysis given in Sec. III is valid for any layer, and in the limit $\Delta z\to 0$ we obtain

$$\psi(z)=\underline{T}(z)\phi(z), \tag{5.1}$$

where $\underline{T}(z)$ is the 4×4 matrix whose four columns are the four proper vectors of the matrix $\underline{D}(z)$. This property can be expressed by the equation

$$\underline{D}(z)\underline{T}(z)=\underline{T}(z)\underline{D}_0(z), \tag{5.2}$$

where $\underline{D}_0(z)$ is the diagonal matrix,

$$\underline{D}_0=\begin{bmatrix}d_1 & 0 & 0 & 0\\ 0 & d_2 & 0 & 0\\ 0 & 0 & d_3 & 0\\ 0 & 0 & 0 & d_4\end{bmatrix}. \tag{5.3}$$

By inserting Eq. (5.1) into Eq. (2.2) and taking into account Eq. (5.2), we immediately obtain

$$\frac{d\phi}{dz}=iK(\underline{D}_0+\underline{V}), \tag{5.4}$$

where

$$\underline{V}=\frac{i}{K}\underline{T}^{-1}\frac{d\underline{T}}{dz}. \tag{5.4'}$$

An important property of the matrix $\underline{V}$ is given by

$$\underline{V}=\underline{N}^{-1}\underline{V}^{\dagger}\underline{N}. \tag{5.5}$$

This follows from the relation $d(\underline{T}^{-1}\underline{T})/dz$

$=d(\underline{1})/dz=0$, which gives

$$ik\left[\underline{T}^{-1}\frac{d\underline{T}}{dz}+\frac{d\underline{T}^{-1}}{dz}\underline{T}\right]=0 .$$

In fact, $ik\underline{T}^{-1}d\underline{T}/dz=\underline{V}$ and $iK(d\underline{T}^{-1}/dz)\underline{T}=-\underline{N}^{-1}\underline{V}^{\dagger}\underline{N}$, as is easily found by considering Eq. (3.9) and its Hermitian conjugate.

The properties of $\underline{T}$ and $\underline{V}$ given by Eqs. (3.9) and (5.5) are the most important consequences of the orthogonality relations (3.4).

For media whose dielectric tensor $\underline{\epsilon}$ has real components, the elements of $\underline{D}$ are also real, and its proper vectors corresponding to real proper values can be chosen as real. In this case, and with the normalization conditions (3.5), the property (5.5) gives to the matrix $\underline{V}$ the structure

$$\underline{V}=\begin{bmatrix} 0 & V_{12} & V_{13} & V_{14} \\ V_{12}^{*} & 0 & V_{23} & V_{24} \\ -V_{13}^{*} & -V_{23}^{*} & 0 & V_{34} \\ -V_{14}^{*} & -V_{24}^{*} & V_{34}^{*} & 0 \end{bmatrix} , \tag{5.6}$$

where the elements V_{ij} are purely imaginary.

For uniaxial media, the elements of $\underline{V}$ are easily computed from Eqs. (4.12). The computations are greatly simplified by considering that the derivatives dC_j/dz of the normalization constants C_j are unnecessary. In fact, it is easily found that each derivative dC_j/dz is multiplied by a nondiagonal element of the matrix $\underline{T}^{-1}\underline{T}$, which is identically zero.

We explicitly observe that Eqs. (5.5) and (5.6) are valid only if the norms of the proper vectors are z independent. Another choice of the norms can easily be taken into account by inserting $\phi(z)=\underline{R}^{-1/2}\tilde{\phi}(z)$ in Eq. (5.4), where $\underline{R}$ is a diagonal matrix whose elements R_{ii} are the ratios between the new z-dependent norms and the previous ones. This gives a $\underline{V}$ matrix whose diagonal elements are different from zero.

We finally recall that Eqs. (5.5) and (5.6) are valid only for nonabsorbing media. The energy conservation is expressed by the conservation of the quantity $\phi^{\dagger}\underline{N}\phi$. In fact, from the relations $\psi=\underline{T}\phi$, $\psi^{\dagger}=\psi^{\dagger}\underline{T}^{\dagger}$, and Eq. (3.9), we obtain

$$\psi^{\dagger}\underline{M}\psi=\phi^{\dagger}\underline{N}\phi , \tag{5.7}$$

which shows that $\underline{N}$ plays the role of the metric matrix in the ϕ representation. With the normalization conditions (3.5), the norm is simply given by

$$\phi^{\dagger}\underline{N}\phi=f_1^{*}f_1+f_2^{*}f_2-f_3^{*}f_3-f_4^{*}f_4 . \tag{5.8}$$

By using the propagation equation (5.4) and the property given by (5.5), the conservation of the norm $\phi^{\dagger}\underline{N}\phi$ can be easily stated.

VI. CONCLUSIONS

In anisotropic linear media, the radiation field is generally analyzed in terms of proper waves, which propagate without changing their polarization state. The orthogonality of the polarization states of the proper wave propagating in a given direction through a nonabsorbing uniaxial medium has been known since the early works of Fresnel. A more complicated problem arises when a plane wave of given wave vector $\vec{K}_i$ is obliquely incident on the surface of an anisotropic medium, because of the $\vec{K}$ change due to refraction. In a layer between parallel planes four proper waves with generally different wave vectors are generated. The central point of this paper has been the analysis of the polarizations of these waves, namely of the amplitude and phase relations between the Cartesian components of the electric and magnetic vectors $\vec{E}$ and $\vec{H}$. The most important results are summarized by the orthogonality relations (3.4) and by the property (3.9) of the transformation matrix $\underline{T}$ between the set of the vector components E_i,H_i and the set of the complex amplitudes f_j of the proper waves. This matrix is similar to a unitary matrix, since the elements of the inverse matrix $\underline{T}^{-1}$ are the complex conjugates of the elements of $\underline{T}$, taken in a different order.

The interest in the above results is due to the fact that the matrix $\underline{T}$ is the interface between two fundamental aspects of our problem: the solution of the Maxwell equation, where the vector components E_i,H_i appear, and the application of the obtained results to physical problems, which generally requires the knowledge of the amplitudes f_j.

Explicit expressions for the matrices $\underline{T}$ and $\underline{T}^{-1}$ are found in the literature only in a few particular cases, because of the complicated calculations needed. Our analysis greatly simplifies the calculation and, in particular, avoids the problem of matrix inversion.

With the help of the formalism developed, two important problems are solved: the evaluation of the reflection and transmission coefficients of a multilayer, given in Sec. IV, and the propagation equations (5.4) for the quantities f_j in stratified media with continuous variation of the dielectric tensor. Explicit expressions are given only for the particularly simple and important case of nonabsorbing uniaxial media.

Let us finally add some comments about the possible applications of the results obtained.

The interest in stratified media (considered in Sec. IV) comes from the widespread use of such structures and from the fact that they can approximate continuous media, a fact which allows one to develop methods for the numerical integration of the wave equation for such media.[4] Explicit expressions for the wave propagation through stratified media are found in the literature, but they refer to particular directions of the dielectric tensor,[3] or make use of rather complicated expressions. The analysis given here allows one to treat the general case on the basis of very simple expressions. From the numerical point of view, the most complicated problem is now the computation of the proper values of the matrix $\underline{D}$, since a numerical inversion of the matrix $\underline{T}$ is no longer required.

The propagation equations (obtained in Sec. V) for the amplitudes f_i of the proper waves are a good starting point for other methods of numerical analysis.[11] However, the main interest in such equations is due to the fact that they are the basis of most approximate methods, at least concerning aperiodic media. All the approximations quoted in this paper make use of similar equations, which for slowly varying media have the structure of coupled equations with small coupling terms. They are therefore the starting point for the approximations based on perturbation expansions or coupled-mode analysis. The most important of these approximations is the geometrical optics approximation (GOA), which consists of simply dropping the coupling terms. Only a few very particular cases have been considered until now, such as the cases of normally incident light[5,7,9] and of media with the optical axis lying in the plane of incidence.[6,8] The GOA in the most general case of oblique incidence and arbitrary direction of the optical axis is contained in the Eq. (5.4), by simply neglecting the matrix $\underline{V}$.

In the optics of anisotropic layers a particularly important role is played by media having a periodic helicoidal structure, such as the cholesteric and ferroelectric smectic-*C* liquid crystals. The wave equation for such media is a Mathieu equation, whose solutions are Bloch waves. For this problem specific analytical methods are available,[12–15] and simple solutions are known for the particular case of waves propagating along the helix axis. Also, for such periodic media the theory developed here gives (1) new and interesting solutions in the case where the helical pitch is greater than the light wavelength, and (2) the exact solution for a wave propagating along the helix axis, since Eq. (5.4) becomes z independent.

We finally observe that the amplitudes f_i have a very simple physical meaning, being closely related to measured quantities. This gives a clear meaning to all the terms which appear in the propagation equations (5.4), a fact which is of great help when we are looking for the most useful approximations. As an example, it is evident (and indeed well known) that for slowly varying media the reflection is generally negligible, whereas the interference between the two forward-propagating proper waves may play an essential role and must be taken into account.[9,15] This is very easily achieved by simply assuming $f_3=f_4=0$. The set of four differential equations given by Eq. (5.4) reduces to a set of two equations, which is formally identical to the time-dependent Schrödinger equation for a two-level system (such as, for instance, a spin-$\frac{1}{2}$ particle), with the coordinate z playing the role of the time. In fact, Eq. (5.6) shows that the reduced matrix $\underline{V}$ becomes Hermitian, since $V_{21}=V_{12}^*$. This gives a very good approximation for many problems in the physics of liquid crystals, already used for the particular cases of normally incident light and of a medium with the optical axis orthogonal to the z axis.[9] Once again, the theory developed here extends this approximation to the most general case, and the formal analogy with the Schrödinger equation allows one to make use of the well-known methods developed in quantum mechanics for this problem.

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[10]From the property $\underline{D}=\underline{M}\,\underline{D}^{\dagger}\underline{M}$ given by Eq. (2.9) we can further deduce that the complex proper values of $\underline{D}$ occur in pairs d_i and $d_j=d_i^*$. In fact, if d_1, d_2, d_3, and d_4 comprise the set of proper values of $\underline{D}$, the set of proper values of $\underline{D}^{\dagger}$ and of $\underline{M}\,\underline{D}^{\dagger}\underline{M}$ is $d_1^*, d_2^*, d_3^*, d_4^*$, and the two sets are coincident since $\underline{M}\,\underline{D}^{\dagger}\underline{M}$ is equal to $\underline{D}$.

[11]If the matrix $\underline{V}$ in Eq. (5.4) is small with respect to $\underline{D}_0$ a perturbation expansion (similar to the one discussed in Ref. 4 for the particular case of small anisotropy) gives fast convergence of numerical computations.

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