Frequency shifts of spectral lines generated by scattering from space-time fluctuations

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The scattering of a class of partially coherent electromagnetic fields by a model medium whose dielectric susceptibility fluctuates in space and in time is considered within the accuracy of the first-order Born approximation. It is shown that if the incident field is a linearly polarized, polychromatic plane wave whose spectrum is a line of Gaussian profile, the spectrum of the scattered light is, in some cases, approximately Gaussian and is shifted towards the shorter or the longer wavelengths, depending upon the angle of scattering.

I. INTRODUCTION

It was predicted not long ago that, in general, correlations between source fluctuations produce changes in the spectrum of the emitted light.[1,2] For radiation from planar secondary sources this prediction was subsequently verified by experiment.[3] It was also shown theoretically that the changes may be such as to produce red shifts or blue shifts of spectral lines,[4–6] and this prediction has also been verified.[7–9] A similar effect can also be expected to arise with acoustical waves and was, in fact, observed recently.[10]

Because of the well-known analogy that exists between the processes of radiation and scattering, one might expect that similar phenomena will arise when a polychromatic wave is scattered by a medium whose dielectric susceptibility fluctuates in space and in time. In fact, it was shown recently[11] that, in some cases, when a polychromatic plane wave field is scattered from a medium whose dielectric susceptibility is a static, random function of position, the spectrum of the scattered light will have approximately the same profile as the incident light, but be blue shifted or red shifted, depending upon the angle of scattering.

In the present paper we investigate the scattering, within the accuracy of the first-order Born approximation, of a linearly polarized, electromagnetic polychromatic plane wave from a model medium whose dielectric susceptibility fluctuates both in space and in time. We utilize the theory developed in the accompanying paper,[12] which applies to a very wide class of problems of this kind. We will show, in particular, that when the spectrum of the incident field is a line of Gaussian profile, and the correlation function of the dielectric susceptibility fluctuations at any two space-time points is a Gaussian function of the spatial and temporal variables, the spectrum of the scattered light is approximately also Gaussian, but is blue shifted or red shifted, depending upon the angle of scattering. An approximate formula for the line shift, which emphasizes the roles of the physical parameters, is developed, and some numerical results are presented.

II. FORMULATION OF THE PROBLEM

Let us begin with a brief outline of the problem that is studied in this paper. We consider the scattering of a fluctuating, polychromatic, linearly polarized electromagnetic plane wave with the electric field $\mathbf{E}^{(i)}(\mathbf{r},t)$ and magnetic field $\mathbf{H}^{(i)}(\mathbf{r},t)$, propagating in free space in the direction specified by the real unit vector $\mathbf{u}_0$. The wave is incident upon a medium which occupies a finite volume V and whose dielectric susceptibility fluctuates both in space and in time (see Fig. 1). We denote by $\mathbf{E}^{(s)}(\mathbf{r},t)$ and $\mathbf{H}^{(s)}(\mathbf{r},t)$ the scattered (i.e., total minus incident) electric and magnetic fields, respectively. We wish to determine the spectrum of the scattered light in the far zone when the spectrum of the incident light consists of a single line of Gaussian profile and the correlation function of the

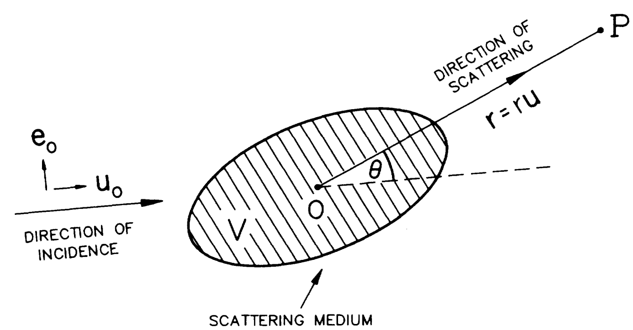

FIG. 1. Notation relating to the scattering geometry. $\mathbf{u}_0$ and $\mathbf{u}$ are real unit vectors in the direction of propagation of the incident field $\mathbf{E}^{(i)},\mathbf{H}^{(i)}$ and the scattered field $\mathbf{E}^{(s)},\mathbf{H}^{(s)}$, respectively. $\mathbf{e}_0$ is a real unit vector in the direction of polarization of the incident electric field. P is a typical observation point in the far zone.

dielectric susceptibility fluctuations is a Gaussian function both in the spatial and temporal variables.

A. The incident field

Let

$$\mathbf{E}^{(i)}(\mathbf{r},t)=\mathbf{e}_0\int_{-\infty}^{\infty} A(\omega)e^{i(k\mathbf{u}_0\cdot\mathbf{r}-\omega t)}d\omega , \tag{2.1a}$$

$$\mathbf{H}^{(i)}(\mathbf{r},t)=\mathbf{u}_0\times\mathbf{E}^{(i)}(\mathbf{r},t) \tag{2.1b}$$

be the (real[13]) electric and magnetic vectors of the incident wave. Here $\mathbf{u}_0$ is a real unit vector in the direction of propagation of the incident wave, $\mathbf{e}_0$ is a (real) unit vector in the direction of polarization of the incident electric field ($\mathbf{e}_0\cdot\mathbf{u}_0=0$), and

$$k=\omega/c , \tag{2.2}$$

c being the speed of light in a vacuum. We also assume that the field is stationary, at least in the wide sense,[14] as is well known; the representation (2.1a) must then be interpreted in terms of generalized functions. Under these circumstances the cross-spectral density tensor of the incident electric field is given by (cf. Sec. V A of Ref. 12)

$$W_{lm}^{(i)}(\mathbf{R},\omega)=S^{(i)}(\omega)e^{ik\mathbf{u}_0\cdot\mathbf{R}}e_{0l}e_{0m} , \tag{2.3}$$

where $S^{(i)}(\omega)$ is the spectrum of the incident field and the subscripts l and m label Cartesian components. In terms of the spectral amplitude $A(\omega)$ the spectrum $S^{(i)}(\omega)$ is given by

$$\langle A^*(\omega)A(\omega')\rangle=S^{(i)}(\omega)\delta(\omega'-\omega) , \tag{2.4}$$

where the angular brackets denote an ensemble average.

We will only consider the case where the spectrum is a line of Gaussian profile, centered on frequency ω_0 and of rms width Γ_0, i.e.,

$$S^{(i)}(\omega)=\frac{A}{\sqrt{2\pi}\Gamma_0}[g(\omega-\omega_0;\Gamma_0)+g(\omega+\omega_0;\Gamma_0)] , \tag{2.5}$$

where A, ω_0, and Γ_0 are positive constants and

$$g(\omega-\omega_0;\Gamma_0)=e^{-(\omega-\omega_0)^2/2\Gamma_0^2} . \tag{2.6}$$

For later purposes we note some coherence properties of the incident field. The space-time correlation tensor of the field is just the Fourier transform of the cross-spectral density tensor (2.3), i.e.,

$$\mathcal{E}_{lm}^{(i)}(\mathbf{R},T)=e_{0l}e_{0m}\int_{-\infty}^{\infty} S^{(i)}(\omega)e^{i(k\mathbf{u}_0\cdot\mathbf{R}-\omega T)}d\omega . \tag{2.7a}$$

If we choose the axes of our Cartesian coordinate system so that one of them coincides with the direction of the real unit vector $\mathbf{e}_0$, (i.e., along the direction of polarization of the incident electric field), then only one component of $\mathcal{E}_{lm}(\mathbf{R},T)$ will be nonzero and it will be given by

$$\mathcal{E}(\mathbf{R},T)=\int_{-\infty}^{\infty} S^{(i)}(\omega)e^{i(k\mathbf{u}_0\cdot\mathbf{R}-\omega T)}d\omega . \tag{2.7b}$$

The degree of coherence of the incident field is then given by

$$\gamma^{(i)}(\mathbf{R},T)\equiv\frac{\mathcal{E}^{(i)}(\mathbf{R},T)}{\mathcal{E}^{(i)}(0,0)} =\frac{\int_{-\infty}^{\infty} S^{(i)}(\omega)e^{i(k\mathbf{u}_0\cdot\mathbf{R}-\omega T)}d\omega}{\int_{-\infty}^{\infty} S^{(i)}(\omega)d\omega} . \tag{2.8}$$

In particular, with $T=0$,

$$\gamma^{(i)}(\mathbf{R},0)=\frac{\int_{-\infty}^{\infty} S^{(i)}(\omega)e^{ik\mathbf{u}_0\cdot\mathbf{R}}d\omega}{\int_{-\infty}^{\infty} S^{(i)}(\omega)d\omega} . \tag{2.9}$$

We see from Eq. (2.9) that when the two field points $\mathbf{r}_1$ and $\mathbf{r}_2$ ($\mathbf{R}=\mathbf{r}_2-\mathbf{r}_1$) are located in a plane perpendicular to the direction of propagation, i.e., when $\mathbf{u}_0\cdot\mathbf{R}=0$,

$$\gamma^{(i)}(\mathbf{R},0)=1 . \tag{2.10}$$

We may express this result by saying that the incident field has *complete transverse coherence.* On the other hand, when the two points are located along the direction of propagation, then $\mathbf{u}_0\cdot\mathbf{R}=R$, and if, in addition, the spectrum is given by Eq. (2.5) the effective width of $|\gamma^{(i)}(\mathbf{R},0)|$ is readily seen to be given by

$$L_0\sim\frac{c}{\Gamma_0} . \tag{2.11}$$

We will refer to this quantity as the *longitudinal coherence length* of the incident field.

B. The scattering medium

We will assume that the scattering medium is linear, isotropic, and nonmagnetic, and that the fluctuations of its generalized dielectric susceptibility $\hat{\eta}(\mathbf{r},t;\omega')$ (cf. Ref. 12, Sec. II) are statistically homogeneous and stationary, at least in the wide sense. As before, we will denote the space-time correlation function of the medium by $G(\mathbf{R},T;\omega')$:

$$\langle\hat{\eta}^*(\mathbf{r},t;\omega')\hat{\eta}(\mathbf{r}+\mathbf{R},t+T;\omega')\rangle=G(\mathbf{R},T;\omega') . \tag{2.12}$$

Here the asterisk denotes the complex conjugate. We will also assume that the scattering is so weak that, to a good accuracy, it may be described within the framework of the first-order Born approximation.

The resonance frequencies of the medium, i.e., the frequencies of its atomic or molecular transitions, will be assumed not to be in the immediate vicinity of the center frequencies $\pm\omega_0$ of the incident light. Under these circumstances the correlation function $G(\mathbf{R},T;\omega')$ may be approximated, over the effective frequency range (the spectral line) of the incident light, by $G(\mathbf{R},T;\omega_0)$ when $\omega'>0$ and by $G(\mathbf{R},T;-\omega_0)$ when $\omega'<0$.

We will take as our model scatterer one that satisfies the above requirements and has a space-time correlation function that is the product of a Gaussian function of $\mathbf{R}$ of rms width σ and a Gaussian function of T of rms width τ, i.e., that

$$G(\mathbf{R},T;\pm\omega_0)=\frac{B}{(2\pi\sigma^2)^{3/2}}e^{-R^2/2\sigma^2}e^{-T^2/2\tau^2} , \tag{2.13}$$

where B, σ, and τ are positive constants. Evidently, τ is the effective time duration over which the fluctuations of the dielectric susceptibility are correlated, i.e., it is the *correlation time of the medium.* The constant σ is clearly the effective distance over which the fluctuations of the dielectric constant are correlated, i.e., it is the *spatial correlation length* of the medium. We will assume that σ is much smaller than the linear dimensions of the scattering volume. Finally, we will assume that the fluctuations of the medium and the fluctuations of the incident field are statistically independent.

III. SPECTRUM OF THE SCATTERED LIGHT IN THE FAR ZONE

When the incident field is of the type described in Sec. II A and the medium fulfills the assumptions stated above, the spectrum of the scattered light at the position $\mathbf{r}=r\mathbf{u}$ ($\mathbf{u}\cdot\mathbf{u}=1$) in the far zone, $S^{(s)}(r\mathbf{u},\omega)$, is to a good approximation, given by Eq. (5.10) of Ref. 12, viz.,

$$S^{(s)}(r\mathbf{u},\omega)=\frac{(2\pi)^3\omega^4 V\sin^2\psi}{c^4r^2}\times\int_{-\infty}^{\infty}\mathscr{S}(\mathbf{k}-\mathbf{k}',\omega-\omega';\omega')S^{(i)}(\omega')d\omega' , \tag{3.1}$$

where ψ is the angle between $\mathbf{u}$ and $\mathbf{e}_0$, $\mathbf{k}$ is the wave vector of the scattered light,

$$\mathbf{k}=\frac{\omega}{c}\mathbf{u} , \tag{3.2}$$

$\mathbf{k}'$ is the wave vector of the ω' component of the incident field [cf. Eq. (2.1a)],

$$\mathbf{k}'=\frac{\omega'}{c}\mathbf{u}_0 , \tag{3.3}$$

and $\mathscr{S}(\mathbf{K},\Omega;\omega')$ is the *generalized structure function* of the medium [Ref. 12, Eq. (5.11a)]

$$\mathscr{S}(\mathbf{K},\Omega;\omega')=\frac{1}{(2\pi)^4}\int_V\int_{-\infty}^{\infty}G(\mathbf{R},T;\omega')e^{-i(\mathbf{K}\cdot\mathbf{R}-\Omega T)}d^3R\,dT . \tag{3.4}$$

The formula (3.1) shows that the spectrum of the scattered light in the far zone differs from the spectrum of the incident light $S^{(i)}(\omega)$ by the effect of two factors: namely, a factor proportional to ω^4 and a factor which is a linear transform of $S^{(i)}(\omega)$, the kernel of the transform being the generalized structure function of the medium. The ω^4 factor is a reflection of the fact that on the microscopic level the medium responds to the incident field as a set of dipole oscillators (cf. Ref. 15, Secs. 2.2.1 and 2.2.3). The generalized structure factor $\mathscr{S}$ depends upon the correlation between the oscillators. Consequently, the linear transform represents the effect of the interaction of the incident light with these correlated oscillators. Since $\mathscr{S}$ depends upon the "momentum transfer vector" $\mathbf{K}=\mathbf{k}-\mathbf{k}'$, it depends on the angle of scattering, and hence the spectrum of the scattered light in the far field will not only differ, in general, from the spectrum of the incident light, but will also be different in different directions of observation.

It is useful, for the purposes of calculation, to divide $S^{(s)}(r\mathbf{u},\omega)$ into two parts: the part denoted by $S_+^{(s)}(r\mathbf{u},\omega)$, generated by the positive frequency part of the spectrum of the incident light, and the part denoted by $S_-^{(s)}(r\mathbf{u},\omega)$, generated by the negative frequency part of the incident spectrum. Upon substituting Eq. (2.5) into Eq. (3.1) and assuming that $\Gamma_0/\omega_0\ll 1$ we find that, to a good approximation,

$$S^{(s)}(r\mathbf{u},\omega)=S_+^{(s)}(r\mathbf{u},\omega)+S_-^{(s)}(r\mathbf{u},\omega) , \tag{3.5}$$

where

$$S_\pm^{(s)}(r\mathbf{u},\omega)=\frac{(2\pi)^3\omega^4 V\sin^2\psi}{c^4r^2}\times\int_{-\infty}^{\infty}\mathscr{S}(\mathbf{k}-\mathbf{k}',\omega-\omega';\pm\omega_0)S_\pm^{(i)}(\omega')d\omega' , \tag{3.6}$$

with

$$S_\pm^{(i)}(\omega)=\frac{A}{(2\pi\Gamma_0^2)^{1/2}}g(\omega\mp\omega_0;\Gamma_0) . \tag{3.7}$$

We will now calculate the spectrum of the scattered light in the far zone for the case when the correlation function of the dielectric susceptibility is given by Eq. (2.13).

A. Evaluation of the generalized structure function $\mathscr{S}(\mathbf{k}-\mathbf{k}',\omega-\omega';\pm\omega_0)$

Upon substituting Eq. (2.13) into Eq. (3.4) and extending the spatial integration over all space (which is justified since σ was assumed to be small compared to the linear dimensions of V), we find that

$$\mathscr{S}(\mathbf{K},\Omega;\pm\omega_0)=\frac{B(2\pi\tau^2)^{1/2}}{(2\pi)^4}e^{-K^2\sigma^2/2}e^{-\Omega^2\tau^2/2} . \tag{3.8}$$

It will be useful to introduce the parameters

$$\Gamma_\sigma=c/\sigma , \tag{3.9}$$

$$\Gamma_\tau=1/\tau . \tag{3.10}$$

Each of these parameters has a clear physical significance. Γ_σ is the reciprocal of the time it takes light to cross a spatial correlation length σ. Γ_τ is the bandwidth of the temporal fluctuations of the medium.

Upon using Eqs. (3.9), (3.10), and (2.6), Eq. (3.8) can be rewritten as

$$\mathscr{S}(\mathbf{K},\Omega;\pm\omega_0)=\frac{B}{(2\pi)^4}\left[\frac{2\pi}{\Gamma_\tau^2}\right]^{1/2}g(Kc;\Gamma_\sigma)g(\Omega;\Gamma_\tau) . \tag{3.11}$$

Since $\mathbf{u}$ and $\mathbf{u}_0$ are unit vectors,

$$|\mathbf{k}-\mathbf{k}'|^2=\left|\frac{\omega}{c}\mathbf{u}-\frac{\omega'}{c}\mathbf{u}_0\right|^2 ,$$
$$=\frac{1}{c^2}(\omega^2-2\omega\omega'\cos\theta+\omega'^2) ,$$
$$=\frac{1}{c^2}\omega^2\sin^2\theta+\frac{1}{c^2}(\omega'-\omega\cos\theta)^2 , \quad (3.12)$$

where θ is the angle of scattering ($\mathbf{u}\cdot\mathbf{u}_0=\cos\theta$). It follows from Eqs. (3.11) and (3.12) that

$$\mathcal{S}(\mathbf{k}-\mathbf{k}',\omega-\omega';\pm\omega_0)=\frac{B}{(2\pi)^4}\left[\frac{2\pi}{\Gamma_\tau^2}\right]^{1/2} g(\omega\sin\theta;\Gamma_\sigma)\times g(\omega'-\omega\cos\theta;\Gamma_\sigma)g(\omega-\omega';\Gamma_\tau) . \quad (3.13)$$

B. Evaluation of the spectrum of the scattered field $S^{(s)}(r\mathbf{u},\omega)$

It is shown in Appendix A that upon substituting from Eqs. (3.13) and (3.7) into Eq. (3.6) we obtain, after a straightforward calculation, the following expression for the positive frequency part of the spectrum of the scattered field:

$$S_+^{(s)}(r\mathbf{u},\omega)=N(r)H(\theta,\psi)\omega^4\exp\left[-\frac{1}{2}\left[\frac{\omega-\hat{\omega}(\theta)}{\hat{\Gamma}(\theta)}\right]^2\right] . \quad (3.14)$$

Here

$$N(r)=\frac{VAB}{2\pi c^4 r^2}\left[\frac{2\pi}{\Gamma_\tau^2+\alpha_\tau^2\Gamma_0^2}\right]^{1/2} , \quad (3.15)$$

$$H(\theta,\psi)=\sin^2\psi\exp\left[-\frac{1}{2}\left[\frac{\omega_0}{\Gamma_\sigma}\right]^2\frac{\alpha_\tau^2}{\alpha^2(\theta)}\times\left[\sin^2\theta+\frac{(1-\cos\theta)^2}{\alpha_\tau^2}\right]\right] , \quad (3.16)$$

$$\hat{\omega}(\theta)=\frac{\omega_0}{\alpha^2(\theta)}\left[1+\frac{\Gamma_\tau^2}{\Gamma_\sigma^2}\cos\theta\right] , \quad (3.17)$$

$$\hat{\Gamma}(\theta)=\frac{(\Gamma_\tau^2+\alpha_\tau^2\Gamma_0^2)^{1/2}}{\alpha(\theta)} , \quad (3.18)$$

and

$$\alpha_\tau^2=1+\frac{\Gamma_\tau^2}{\Gamma_\sigma^2} , \quad (3.19)$$

$$\alpha^2(\theta)=1+4\left[\frac{\Gamma_0}{\Gamma_\sigma}\right]^2\sin^2(\theta/2)+\frac{\Gamma_\tau^2}{\Gamma_\sigma^2}\left[1+\frac{\Gamma_0^2}{\Gamma_\sigma^2}\sin^2\theta\right] . \quad (3.20)$$

In a similar way, one can show from Eqs. (3.13), (3.7), and (3.6) that the negative frequency part of the scattered spectrum is given by

$$S_-^{(s)}(r\mathbf{u},\omega)=N(r)H(\theta,\psi)\omega^4\exp\left[-\frac{1}{2}\left[\frac{\omega+\hat{\omega}(\theta)}{\hat{\Gamma}(\theta)}\right]^2\right] . \quad (3.21)$$

For physically reasonable values of the parameters ω_0, Γ_0, σ, and τ, the frequency $\hat{\omega}(\theta)$ is close to ω_0 and $\hat{\Gamma}(\theta)$ is much smaller than ω_0. Under these circumstances,

$$S_-^{(s)}(r\mathbf{u},\omega)\approx 0 \quad \text{when } \omega>0 \quad (3.22)$$

for all positions in the far zone $\mathbf{r}=r\mathbf{u}$. It then follows from Eqs. (3.22), (3.5), and (3.14) that for $\omega>0$ the spectrum of the scattered light in the far zone is given by, to a good approximation,

$$S^{(s)}(r\mathbf{u},\omega)=S_+^{(s)}(r\mathbf{u},\omega)=N(r)H(\theta,\psi)\omega^4\exp\left[-\frac{1}{2}\left[\frac{\omega-\hat{\omega}(\theta)}{\hat{\Gamma}(\theta)}\right]^2\right] , \quad (3.23)$$

where $N(r)$, $H(\theta,\psi)$, $\hat{\omega}(\theta)$, and $\hat{\Gamma}(\theta)$ are given by Eqs. (3.15)–(3.18).

It is evident from Eq. (3.23) that, in terms of its behavior as a function of frequency, the spectrum of the scattered light is a product of two factors: ω^4 and a Gaussian function of rms width $\hat{\Gamma}(\theta)$ and center frequency $\hat{\omega}(\theta)$. It is shown in Appendix C that for all values of the parameters Γ_0, ω_0, σ, and τ, and for all scattering angles θ ($0\le\theta\le\pi$),

$$\hat{\omega}(\theta)\le\omega_0 . \quad (3.24)$$

It is also shown there that the equality holds only in the following three special cases: (a) $\theta=0$, i.e., for forward scattering; (b) $\Gamma_\sigma\to\infty$ ($\sigma\to 0$), i.e., when the dielectric susceptibility fluctuations are spatially uncorrelated; and (c) $\Gamma_0\to 0$ and $\Gamma_\tau\to 0$ ($\tau\to\infty$), i.e., when the incident light is monochromatic and the dielectric susceptibility fluctuations are static.

The fact that, except in these special cases, $\hat{\omega}(\theta)<\omega_0$ implies that the Gaussian function in expression (3.23) is centered at a lower frequency (i.e., is red shifted) with respect to the Gaussian spectral line which produced it [$S_+^{(i)}(\omega)$], the magnitude of the shift depending on angle of scattering θ. On the other hand, the factor ω^4 in the expression (3.23) is an increasing function of frequency and hence will produce a shift towards the higher frequencies (i.e., a blue shift). Consequently, the spectrum of the scattered light will be either red shifted or blue shifted with respect to $S_+^{(i)}(\omega)$, depending on the relative magnitudes of these two contributions.

IV. FREQUENCY SHIFT OF THE SPECTRUM OF THE SCATTERED LIGHT

A. General form

By a straightforward calculation (cf. Ref. 11, Appendix C), one can show that the spectrum of the scattered light, given by Eq. (3.23), is maximum as a function of ω when $\omega=\omega_0'(\theta)$, where

$$\omega_0'(\theta)=\frac{\hat{\omega}(\theta)}{2}\left\{1+\left[1+\left[\frac{4\hat{\Gamma}(\theta)}{\hat{\omega}(\theta)}\right]^2\right]^{1/2}\right\} . \quad (4.1)$$

For the purposes of the present calculation, we will rewrite Eq. (3.17) as

$$\hat{\omega}(\theta)=\omega_0\frac{f(\theta)}{\alpha^2(\theta)} , \quad (4.2)$$

where

$$f(\theta)=1+\left[\frac{\Gamma_\tau}{\Gamma_\sigma}\right]^2\cos\theta . \quad (4.3)$$

It follows from Eqs. (3.18) and (4.2) that

$$\frac{\hat{\Gamma}(\theta)}{\hat{\omega}(\theta)}=\frac{\alpha(\theta)}{f(\theta)}\left[\frac{\Gamma_\tau^2+\alpha_\tau^2\Gamma_0^2}{\omega_0^2}\right]^{1/2} . \quad (4.4)$$

Upon substituting Eqs. (4.2) and (4.4) into Eq. (4.1), we find, after some algebra, that

$$\omega_0'(\theta)=\frac{\omega_0}{2\alpha^2(\theta)}\left\{f(\theta)+\left[f^2(\theta)+16\alpha^2(\theta)\left[\frac{\Gamma_\tau^2+\alpha_\tau^2\Gamma_0^2}{\omega_0^2}\right]\right]^{1/2}\right\} . \quad (4.5)$$

It is customary, especially in astronomy, to specify frequency shifts by the quantity

$$z=\frac{\lambda_0'-\lambda_0}{\lambda_0}=\frac{\omega_0-\omega_0'}{\omega_0'} , \quad (4.6)$$

where $\lambda_0=2\pi c/\omega_0$ is the original wavelength and $\lambda_0'=2\pi c/\omega_0'$ is the corresponding "shifted" wavelength. Evidently $z>0$ when the line is red shifted and $z<0$ when it is blue shifted. Upon substituting from Eq. (4.5) into Eq. (4.6), we find that in the present case

$$z(\theta)=\frac{2\alpha^2(\theta)-\left\{f(\theta)+\left[f^2(\theta)+16\alpha^2(\theta)\left[\frac{\Gamma_\tau^2+\alpha_\tau^2\Gamma_0^2}{\omega_0^2}\right]\right]^{1/2}\right\}}{f(\theta)+\left[f^2(\theta)+16\alpha^2(\theta)\left[\frac{\Gamma_\tau^2+\alpha_\tau^2\Gamma_0^2}{\omega_0^2}\right]\right]^{1/2}} . \quad (4.7)$$

B. Approximate forms for $z(\theta)$ and $H(\theta,\psi)$

The expression (4.7) for $z(\theta)$ is too complicated to make it possible to draw any conclusions about the roles that the various physical mechanisms play in generating the line shift. Similar remarks apply to the expression (3.16) for $H(\theta,\psi)$, which describes the strength of the scattered light. However, if

$$\frac{\Gamma_\tau^2}{\Gamma_\sigma^2}\ll 1 , \quad (4.8a)$$

$$\frac{\Gamma_0^2}{\Gamma_\sigma^2}\ll 1 , \quad (4.8b)$$

$$16\left[\frac{\Gamma_\tau^2+\Gamma_0^2}{\omega_0^2}\right]\ll 1 \quad (4.8c)$$

(conditions which are usually fulfilled in practice), $z(\theta)$ as given by Eq. (4.7) and $H(\theta,\psi)$ as given by Eq. (3.16) can be approximated by expressions which clearly indicate the roles of the various physical parameters, as we will now show.

It follows from Eq. (4.7) that $z(\theta)$ can be rewritten as

$$z(\theta)=\frac{2\alpha^2(\theta)-\{f(\theta)+[1+q(\theta)]^{1/2}\}}{f(\theta)+[1+q(\theta)]^{1/2}} , \quad (4.9)$$

where

$$\begin{aligned}q(\theta)&=f^2(\theta)-1+16\alpha^2(\theta)\left[\frac{\Gamma_\tau^2+\alpha_\tau^2\Gamma_0^2}{\omega_0^2}\right]\\&=2\left[\frac{\Gamma_\tau}{\Gamma_\sigma}\right]^2\cos\theta+\left[\frac{\Gamma_\tau}{\Gamma_\sigma}\right]^4\cos^2\theta\\&\quad+16\alpha^2(\theta)\left[\frac{\Gamma_\tau^2+\alpha_\tau^2\Gamma_0^2}{\omega_0^2}\right] .\end{aligned} \quad (4.10)$$

It follows from Eq. (3.20) that to first order in the small quantities, which appear in the inequalities (4.8),

$$\alpha^2(\theta)\approx 1+\left[\frac{\Gamma_\tau}{\Gamma_\sigma}\right]^2+4\frac{\Gamma_0^2}{\Gamma_\sigma^2}\sin^2(\theta/2) , \quad (4.11)$$

and if we use this approximation, we deduce from Eq. (4.10) that

$$\sqrt{1+q(\theta)}\approx 1+\left[\frac{\Gamma_\tau}{\Gamma_\sigma}\right]^2\cos\theta+8\left[\frac{\Gamma_\tau^2+\Gamma_0^2}{\omega_0^2}\right] . \quad (4.12)$$

Upon substituting from Eqs. (4.11) and (4.12) into Eq. (4.9), we find that, to first order in small quantities,

$$z(\theta)\approx 2\left[\frac{\Gamma_\tau^2}{\Gamma_\sigma^2}+2\frac{\Gamma_0^2}{\Gamma_\sigma^2}\right]\sin^2(\theta/2)-4\left[\frac{\Gamma_\tau^2+\Gamma_0^2}{\omega_0^2}\right] . \quad (4.13)$$

We note that in the static limit ($\Gamma_\tau\rightarrow 0$), Eq. (4.13) has the same form as the corresponding expression of scalar scattering theory [Ref. 11, Eq. (4.15)]. The two terms on the right-hand side of Eq. (4.13) have a simple physical interpretation as regards to the shift they provide: since $\sin^2(\theta/2) \geq 0$, *the first term represents a red shift* (as long as $\theta \neq 0$); on the other hand, *the second term represents a blue shift.*

We will now show that each of the terms can be related to a particular physical mechanism. Let us recall from Sec. III B, that the total shift of the line is comprised of two parts: the shift of the center frequency $\hat{\omega}(\theta)$ of the Gaussian term in Eq. (3.23) from the center frequency ω_0 of the incident line, and a shift due to the ω^4 factor. The shift of the Gaussian term can be described by the corresponding shift parameter

$$z_G(\theta) = \frac{\omega_0 - \hat{\omega}(\theta)}{\hat{\omega}(\theta)} , \quad = \frac{\alpha^2(\theta) - f(\theta)}{f(\theta)} , \tag{4.14}$$

where Eq. (4.2) was used above. Upon using Eq. (4.11), we find that to first order in small parameters,

$$z_G(\theta) \approx 2 \left[\frac{\Gamma_\tau^2}{\Gamma_\sigma^2} + 2 \frac{\Gamma_0^2}{\Gamma_\sigma^2} \right] \sin^2(\theta/2) . \tag{4.15}$$

Upon comparing Eqs. (4.15) and (4.13), we see the first term in Eq. (4.13) represents a *red shift* due to *correlations in the fluctuations of the physical properties of the medium and the finite bandwidth of the incident light* and the second term represents a *blue shift* due to the *dipole factor* ω^4.

In order to discuss the details of the dependences of the frequency shifts upon the physical parameters, we will rewrite Eq. (4.13) in a slightly different form. It follows from Eqs. (3.9) and (3.10) that the ratio $\Gamma_\tau/\Gamma_\sigma$ may be expressed as

$$\frac{\Gamma_\tau}{\Gamma_\sigma} = \frac{\sigma/c}{\tau} , \tag{4.16}$$

i.e., it is the ratio of the time light takes to cross the spatial correlation length of the medium to the correlation time of the medium. In view of Eqs. (2.11) and (3.9), the ratio Γ_0/Γ_σ may be expressed as

$$\frac{\Gamma_0}{\Gamma_\sigma} = \frac{\sigma}{L_0} , \tag{4.17}$$

i.e., it is the ratio of the spatial correlation length of the medium to the coherence length of the incident light. Upon substituting from Eqs. (4.16) and (4.17) into Eq. (4.13), we find that the frequency shift may be expressed as

$$z(\theta) \approx 2 \left[\left[\frac{\sigma/c}{\tau} \right]^2 + 2 \left[\frac{\sigma}{L_0} \right]^2 \right] \sin^2(\theta/2) - 4 \left[\left[\frac{\Gamma_\tau}{\omega_0} \right]^2 + \left[\frac{\Gamma_0}{\omega_0} \right]^2 \right] \tag{4.18}$$

It follows from Eq. (4.18) that the red-shift term is a monotonically increasing function of the two ratios mentioned above and the blue-shift term is a monotonically increasing function of the bandwidth of the temporal fluctuations, as measured in units of ω_0, and the bandwidth of the incident light, as measured in units of ω_0. Moreover, Eq. (4.18) shows that for appropriate values of the parameters it contains, the spectrum of the scattered light will be blue shifted $[z(\theta) < 0]$ for small angles of scattering θ and will become red shifted $[z(\theta) > 0]$ for larger values of θ.

We conclude this section with a brief discussion of the approximate form for the factor $H(\theta,\psi)$. Since $c = \omega_0 \lambda\!\!\!^{-}_0$ ($\lambda\!\!\!^{-}_0 = \lambda_0/2\pi$), it follows from Eq. (3.9) that $\omega_0/\Gamma_\sigma = \sigma/\lambda\!\!\!^{-}_0$. Therefore Eq. (3.16) may be rewritten as

$$H(\theta,\psi) = \sin^2\psi \exp\left[-\frac{1}{2} \left[\frac{\sigma}{\lambda\!\!\!^{-}_0} \right]^2 \frac{\alpha_\tau^2}{\alpha^2(\theta)} \times \left[\sin^2\theta + \frac{(1-\cos\theta)^2}{\alpha_\tau^2} \right] \right] . \tag{4.19}$$

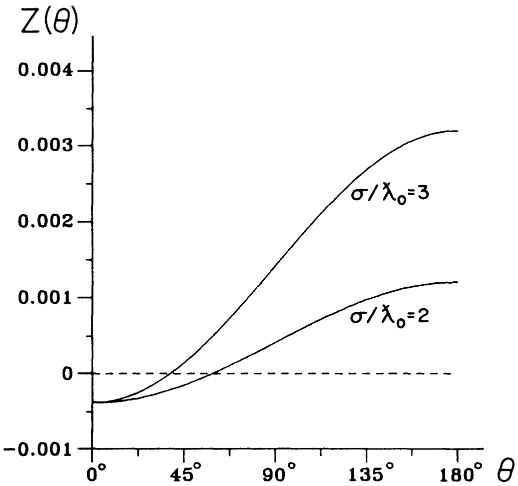

FIG. 2. Plots of the frequency shift parameter $z(\theta)$ with $\Gamma_0/\omega_0 = 0.01$ and $\Gamma_\tau/\Gamma_\sigma = 0$, for two selected values of $\sigma/\lambda\!\!\!^{-}_0$.

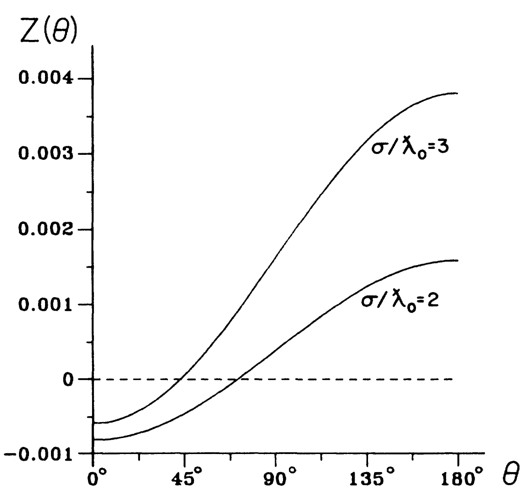

FIG. 3. Plots of the frequency shift parameter $z(\theta)$ with $\Gamma_0/\omega_0 = 0.01$ and $\Gamma_\tau/\Gamma_\sigma = 0.02$, for two selected values of $\sigma/\lambda\!\!\!^{-}_0$.

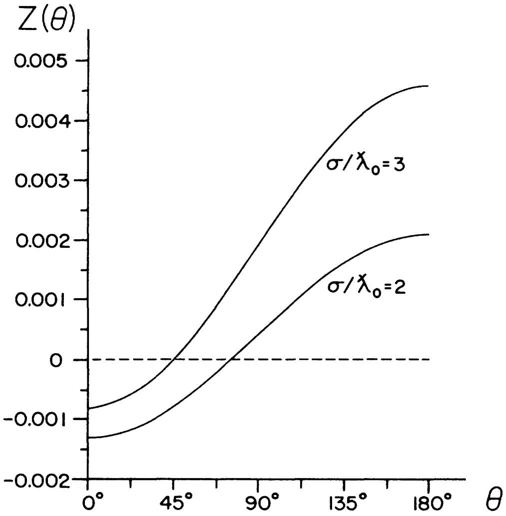

FIG. 4. Plots of the frequency shift parameter $z(\theta)$ with $\Gamma_0/\omega_0=0.01$ and $\Gamma_\tau/\Gamma_\sigma=0.03$, for two selected values of $\sigma/\lambda\!\!\!^{-}_0$.

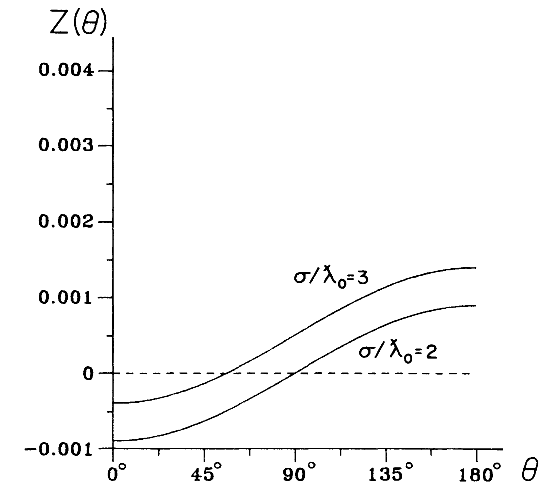

FIG. 6. Plots of the frequency shift parameter $z(\theta)$ with $\Gamma_0/\omega_0=0$ and $\Gamma_\tau/\Gamma_\sigma=0.03$, for two selected values of $\sigma/\lambda\!\!\!^{-}_0$.

It then follows from Eqs. (3.19) and (3.20) that[16]

$$H(\theta,\psi)\approx\sin^2\psi\exp\left[-2\left[\frac{\sigma}{\lambda\!\!\!^{-}_0}\right]^2\sin^2(\theta/2)\right] . \qquad (4.20)$$

Equation (4.20) shows that, for a fixed value of ψ, the strength of the scattered light decreases with increasing θ, and may be small at the values of θ where large red shifts occur. Equation (4.20) also shows that, for fixed value of ψ, the decrease of the strength of the scattered light with increasing θ is larger for more highly spatially correlated media (media with larger σ).

C. Numerical results

Figures 2–6 and Fig. 7 show plots of $z(\theta)$ and $H(\theta,90°)$, respectively, as functions of θ, for some selected values of the parameters. They were calculated from Eqs. (4.7) and (3.16). In all the cases presented below, conditions (4.8) are fulfilled, and hence $z(\theta)$ and $H(\theta,\psi)$ are well described by expressions (4.13) [or (4.18)] and (4.20), respectively.

In Fig. 2, $z(\theta)$ versus θ is plotted for the cases[17] $\Gamma_0/\omega_0=0.01$, $\Gamma_\tau/\Gamma_\sigma=0$, and $\sigma/\lambda\!\!\!^{-}_0=2,3$. The expected $\sin^2(\theta/2)$ behavior [cf. Eq. (4.13)] is evident, and it is clear from the curves that an increase of the spatial correlation length of the medium produces larger shifts in the spectrum of the scattered light. In Figs. 3 and 4, $\Gamma_0/\omega_0=0.01$, $\sigma/\lambda\!\!\!^{-}_0=2,3$, and $\Gamma_\tau/\Gamma_\sigma$ is increased to 0.02, 0.03. It is clear from the figures that this increase of the ratio $\Gamma_\tau/\Gamma_\sigma$ produces larger shifts.

In Fig. 5, $z(\theta)$ is plotted as a function of θ for the cases $\Gamma_0/\omega_0=0$ (monochromatic incident light), $\Gamma_\tau/\Gamma_\sigma=0.02$,

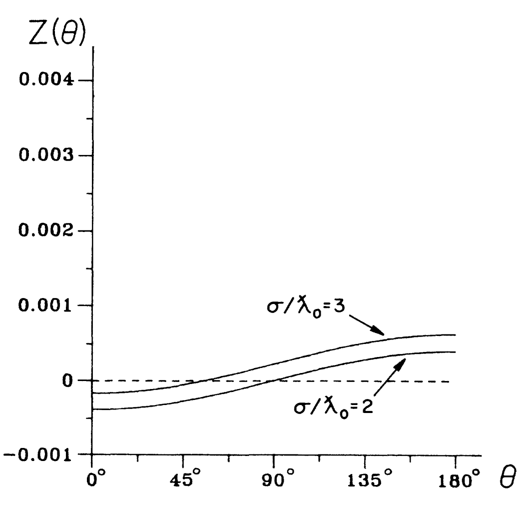

FIG. 5. Plots of the frequency shift parameter $z(\theta)$ with $\Gamma_0/\omega_0=0$ and $\Gamma_\tau/\Gamma_\sigma=0.02$, for two selected values of $\sigma/\lambda\!\!\!^{-}_0$.

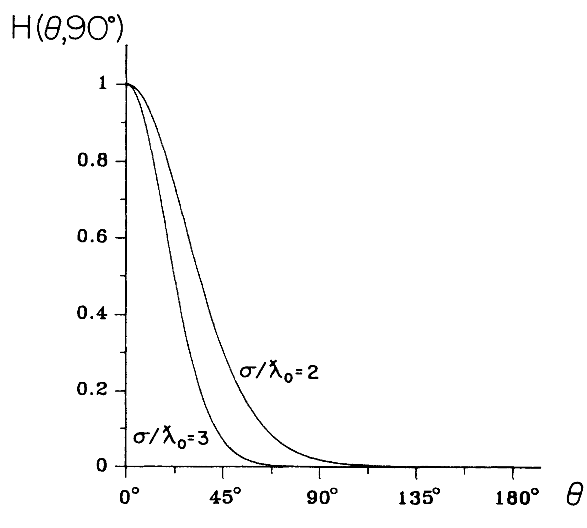

FIG. 7. Plots of the factor $H(\theta,90°)$ with $\Gamma_0/\omega_0=0.01$ and $\Gamma_\tau/\Gamma_\sigma=0.02$, for two selected values of $\sigma/\lambda\!\!\!^{-}_0$.

and $\sigma/\lambda_0=2,3$. Upon comparing Fig. 5 to Fig. 3, we see that the shifts are greatly reduced when the linewidth of the incident light is reduced. In Fig. 6, $\Gamma_0/\omega_0=0$, $\sigma/\lambda_0=2,3$, and $\Gamma_\tau/\Gamma_\sigma$ is increased to 0.03. Upon comparing this figure to Fig. 5, it is again clear that an increase in the ratio $\Gamma_\tau/\Gamma_\sigma$ produces larger shifts. Furthermore, if we compare Fig. 6 to Fig. 4, we see again that the shift is significantly smaller when the linewidth of the incident light is reduced.

In Fig. 7 the factor $H(\theta,90°)$ is plotted versus θ for the same values of the parameters as were used in Fig. 3. It is evident from this curve that the scattering is strongest in the forward direction and is weak when $90° < \theta \le 180°$. Since, for all the sets of parameter values used in Figs. 2–6, $H(\theta,\psi)$ is approximately given by Eq. (4.20), the plots of $H(\theta,90°)$ for the parameter sets used in Fig. 2 and Figs. 4–6 are indistinguishable from those of Fig. 7 and are not presented.

ACKNOWLEDGMENTS

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APPENDIX A: EVALUATION OF $S_{+}^{(s)}(r\mathbf{u},\omega)$

In the subsequent calculations presented in this Appendix, the following product theorem for Gaussian functions (Ref. 11, Appendix A) will be needed.

Theorem. If

$$g(\omega-\omega_j;\Gamma_j)=e^{-(\omega-\omega_j)^2/2\Gamma_j^2} \quad (j=1,2)\ , \tag{A1}$$

then

$$g(\omega-\omega_1;\Gamma_1)g(\omega-\omega_2;\Gamma_2) =g[\omega_1-\omega_2;(\Gamma_1^2+\Gamma_2^2)^{1/2}]g(\omega-\tilde{\omega};\tilde{\Gamma})\ , \tag{A2}$$

where

$$\tilde{\omega}=\frac{\omega_1\Gamma_2^2+\omega_2\Gamma_1^2}{\Gamma_1^2+\Gamma_2^2}\ , \tag{A3}$$

$$\frac{1}{\tilde{\Gamma}^2}=\frac{1}{\Gamma_1^2}+\frac{1}{\Gamma_2^2}\ . \tag{A4}$$

1. Derivation of an alternative form for the generalized structure function (3.13)

It follows from the theorem we just stated that

$$g(\omega'-\omega\cos\theta;\Gamma_\sigma)g(\omega'-\omega;\Gamma_\tau) =g[\omega(1-\cos\theta);(\Gamma_\sigma^2+\Gamma_\tau^2)^{1/2}]g(\omega'-\tilde{\omega};\tilde{\Gamma})\ , \tag{A5}$$

where[18]

$$\tilde{\omega}=\omega\frac{1+(\Gamma_\tau/\Gamma_\sigma)^2\cos\theta}{1+(\Gamma_\tau/\Gamma_\sigma)^2}\ , \tag{A6}$$

$$\frac{1}{\tilde{\Gamma}^2}=\frac{1}{\Gamma_\sigma^2}+\frac{1}{\Gamma_\tau^2}\ . \tag{A7}$$

For later reference we note that Eq. (A6) can be rewritten as

$$\tilde{\omega}=\omega/\beta\ , \tag{A8}$$

where

$$\beta=\frac{\alpha_\tau^2}{f}\ , \tag{A9}$$

$$\alpha_\tau^2=1+(\Gamma_\tau/\Gamma_\sigma)^2\ , \tag{A10}$$

$$f=1+(\Gamma_\tau/\Gamma_\sigma)^2\cos\theta\ . \tag{A11}$$

Upon substituting Eq. (A5) into Eq. (3.13) we find that

$$\mathcal{S}(\mathbf{k}-\mathbf{k}',\omega-\omega';\omega_0) =\frac{B}{(2\pi)^4}(2\pi/\Gamma_\tau^2)^{1/2}g(\omega\sin\theta;\Gamma_\sigma) \times g[\omega(1-\cos\theta);(\Gamma_\sigma^2+\Gamma_\tau^2)^{1/2}]g(\omega'-\tilde{\omega};\tilde{\Gamma})\ . \tag{A12}$$

By using the definition of g [cf. Eq. (A1)], the first two Gaussian functions on the right-hand side of Eq. (A12) can be combined, and we find that

$$\mathcal{S}(\mathbf{k}-\mathbf{k}',\omega-\omega';\omega_0) =\frac{B}{(2\pi)^4}(2\pi/\Gamma_\tau^2)^{1/2}g(\omega;\Gamma_1)g(\omega'-\tilde{\omega};\tilde{\Gamma})\ , \tag{A13}$$

where

$$\frac{1}{\Gamma_1^2}=\frac{\sin^2\theta}{\Gamma_\sigma^2}+\frac{(1-\cos\theta)^2}{\Gamma_\sigma^2+\Gamma_\tau^2}\ , =\frac{\sin^2\theta}{\Gamma_\sigma^2}+\frac{(1-\cos\theta)^2}{\Gamma_\sigma^2\alpha_\tau^2}\ . \tag{A14}$$

2. Calculation of $S_{+}^{(s)}(r\mathbf{u},\omega)$

Upon substituting Eqs. (3.7) and (A13) into Eq. (3.6), we find that

$$S_{+}^{(s)}(r\mathbf{u},\omega)=M(r)(\sin^2\psi)\omega^4 g(\omega;\Gamma_1) \times\int_{-\infty}^{\infty}g(\omega'-\tilde{\omega};\tilde{\Gamma})g(\omega'-\omega_0;\Gamma_0)d\omega'\ , \tag{A15}$$

with

$$M(r)=\frac{V}{2\pi c^4 r^2}\frac{AB}{\Gamma_0\Gamma_\tau}\ . \tag{A16}$$

By a straightforward calculation, one can show that

$$\int_{-\infty}^{\infty}g(\omega'-\tilde{\omega};\tilde{\Gamma})g(\omega'-\omega_0;\Gamma_0)d\omega' =\left[\frac{2\pi\tilde{\Gamma}^2\Gamma_0^2}{\tilde{\Gamma}^2+\Gamma_0^2}\right]^{1/2}g[\tilde{\omega}-\omega_0;(\tilde{\Gamma}^2+\Gamma_0^2)^{1/2}] =\left[\frac{2\pi\tilde{\Gamma}^2\Gamma_0^2}{\tilde{\Gamma}^2+\Gamma_0^2}\right]^{1/2}g(\omega-\omega_0\beta;\Gamma_2)\ , \tag{A17}$$

where Eq. (A8) was used in the last step and

$$\Gamma_2=\beta(\tilde{\Gamma}^2+\Gamma_0^2)^{1/2} . \quad (A18)$$

Upon substituting Eq. (A17) into Eq. (A15) we find that

$$S_+^{(s)}(r\mathbf{u},\omega)=N(r)(\sin^2\psi)\omega^4 g(\omega;\Gamma_1)g(\omega-\omega_0\beta;\Gamma_2) , \quad (A19)$$

where

$$N(r)=\frac{ABV}{2\pi c^4 r^2}\left[\frac{2\pi\tilde{\Gamma}^2}{\Gamma_\tau^2(\tilde{\Gamma}^2+\Gamma_0^2)}\right]^{1/2} . \quad (A20)$$

It follows from Eqs. (A7) and (A10) that

$$\tilde{\Gamma}=\Gamma_\tau/\alpha_\tau . \quad (A21)$$

Straightforward algebraic manipulations show that Eq. (A20) can be rewritten as

$$N(r)=\frac{ABV}{2\pi c^4 r^2}\left[\frac{2\pi}{\Gamma_\tau^2+\Gamma_0^2\alpha_\tau^2}\right]^{1/2} . \quad (A22)$$

Upon again using the product theorem for Gaussian functions [cf. Eqs. (A1)–(A4)], we find that Eq. (A19) can be rewritten as

$$\begin{aligned} S_+^{(s)}(r\mathbf{u},\omega)&=N(r)(\sin^2\psi)g[\omega_0\beta;(\Gamma_1^2+\Gamma_2^2)^{1/2}] \\ &\quad\times\omega^4 g[\omega-\hat{\omega}(\theta);\hat{\Gamma}(\theta)] \\ &=N(r)(\sin^2\psi)g[\omega_0;\Gamma_d(\theta)] \\ &\quad\times\omega^4 g[\omega-\hat{\omega}(\theta);\hat{\Gamma}(\theta)] , \end{aligned} \quad (A23)$$

where

$$\frac{1}{\Gamma_d^2(\theta)}=\frac{\beta^2(\theta)}{\Gamma_1^2(\theta)+\Gamma_2^2(\theta)} , \quad (A24)$$

$$\hat{\omega}(\theta)=\omega_0\frac{\beta(\theta)\Gamma_1^2(\theta)}{\Gamma_1^2(\theta)+\Gamma_2^2(\theta)} , \quad (A25)$$

$$\frac{1}{\hat{\Gamma}^2(\theta)}=\frac{1}{\Gamma_1^2(\theta)}+\frac{1}{\Gamma_2^2(\theta)} . \quad (A26)$$

The function $\beta(\theta)$ is given by Eqs. (A9)–(A11), $\Gamma_1(\theta)$ by Eq. (A14), and $\Gamma_2(\theta)$ by Eq. (A18).

It is shown in Appendix B that $\hat{\omega}(\theta)$, $\hat{\Gamma}(\theta)$, and $\Gamma_d(\theta)$ can be rewritten as [see Eqs. (B7), (B11), and (B13)],

$$\hat{\omega}(\theta)=\frac{\omega_0}{\alpha^2(\theta)}\left[1+\frac{\Gamma_\tau^2}{\Gamma_\sigma^2}\cos\theta\right] , \quad (A27)$$

$$\hat{\Gamma}(\theta)=\frac{(\Gamma_\tau^2+\alpha_\tau^2\Gamma_0^2)^{1/2}}{\alpha(\theta)} , \quad (A28)$$

$$\frac{1}{\Gamma_d^2(\theta)}=\frac{\alpha_\tau^2}{\alpha^2(\theta)}\frac{1}{\Gamma_\sigma^2}\left[\sin^2\theta+\frac{(1-\cos\theta)^2}{\alpha_\tau^2}\right] , \quad (A29)$$

where $\alpha^2(\theta)$ is as given in Eq. (B8). Using Eq. (A29) and the definition of g in Eq. (A23) we find that

$$S_+^{(s)}(r\mathbf{u},\omega)=N(r)H(\theta,\psi)\omega^4\exp\left[-\frac{1}{2}\left[\frac{\omega-\hat{\omega}(\theta)}{\hat{\Gamma}(\theta)}\right]^2\right] \quad (A30)$$

where

$$H(\theta,\psi)=\sin^2\psi\exp\left[-\frac{1}{2}\left[\frac{\omega_0}{\Gamma_\sigma}\right]^2\frac{\alpha_\tau^2}{\alpha^2(\theta)}\times\left[\sin^2\theta+\frac{(1-\cos\theta)^2}{\alpha_\tau^2}\right]\right] . \quad (A31)$$

APPENDIX B: SIMPLIFICATION OF THE EXPRESSIONS FOR $\hat{\omega}(\theta)$, $\hat{\Gamma}(\theta)$, and $\Gamma_d(\theta)$

1. Simplification of $\hat{\omega}(\theta)$

Equation (A25) can be rewritten as

$$\hat{\omega}(\theta)=\frac{\omega_0\beta(\theta)}{1+[\Gamma_2(\theta)/\Gamma_1(\theta)]^2} . \quad (B1)$$

Upon substituting Eqs. (A9), (A14), (A18), and (A21) into Eq. (B1) we find, after some algebra, that

$$\hat{\omega}(\theta)=\frac{\omega_0\alpha_\tau^2 f(\theta)}{h(\theta)+(\alpha_\tau\Gamma_0/\Gamma_\sigma)^2[\alpha_\tau^2\sin^2\theta+(1-\cos\theta)^2]} , \quad (B2)$$

where

$$h(\theta)=f^2(\theta)+\left[\frac{\Gamma_\tau}{\Gamma_\sigma}\right]^2[\alpha_\tau^2\sin^2\theta+(1-\cos\theta)^2] . \quad (B3)$$

It follows directly from Eq. (A11) that

$$f^2(\theta)=1+2\left[\frac{\Gamma_\tau}{\Gamma_\sigma}\right]^2\cos\theta+\left[\frac{\Gamma_\tau}{\Gamma_\sigma}\right]^4\cos^2\theta . \quad (B4)$$

Furthermore, upon using Eq. (A10) we find that

$$\left[\frac{\Gamma_\tau}{\Gamma_\sigma}\right]^2[\alpha_\tau^2\sin^2\theta+(1-\cos\theta)^2] =2\left[\frac{\Gamma_\tau}{\Gamma_\sigma}\right]^2-2\left[\frac{\Gamma_\tau}{\Gamma_\sigma}\right]^2\cos\theta+\left[\frac{\Gamma_\tau}{\Gamma_\sigma}\right]^4\sin^2\theta . \quad (B5)$$

Upon substituting Eqs. (B4) and (B5) into Eq. (B3) we obtain, after straightforward calculations, the following expression for $h(\theta)$:

$$h(\theta)=\alpha_\tau^4 , \quad (B6)$$

where α_τ^2 is given by Eq. (A10). It follows from Eq. (B2) that

$$\hat{\omega}(\theta)=\omega_0\frac{f(\theta)}{\alpha^2(\theta)} , \quad (B7)$$

where $f(\theta)$ is given by Eq. (A11) and

$$\alpha^2(\theta)=\alpha_\tau^2+\left[\frac{\Gamma_0}{\Gamma_\sigma}\right]^2[\alpha_\tau^2\sin^2\theta+(1-\cos\theta)^2]$$
$$=1+\left[2\frac{\Gamma_0}{\Gamma_\sigma}\sin(\theta/2)\right]^2+\left[\frac{\Gamma_\tau}{\Gamma_\sigma}\right]^2\left[1+\frac{\Gamma_0^2}{\Gamma_\sigma^2}\sin^2\theta\right]. \quad \text{(B8)}$$

2. Simplification of the expression for $\hat{\Gamma}(\theta)$

It follows from Eq. (A26) that

$$\hat{\Gamma}^2(\theta)=\frac{\Gamma_1^2(\theta)\Gamma_2^2(\theta)}{\Gamma_1^2(\theta)+\Gamma_2^2(\theta)}, \quad \text{(B9)}$$

and from Eq. (A25) that Eq. (B9) can be rewritten as

$$\hat{\Gamma}^2(\theta)=\frac{\hat{\omega}(\theta)}{\omega_0\beta(\theta)}\Gamma_2^2(\theta). \quad \text{(B10)}$$

We then find from Eqs. (B7) and (A18) that Eq. (B10) can be rewritten as

$$\hat{\Gamma}^2(\theta)=\frac{f(\theta)}{\alpha^2(\theta)\beta(\theta)}\beta^2(\theta)(\tilde{\Gamma}^2+\Gamma_0^2),$$
$$=\frac{f(\theta)}{\alpha^2(\theta)}\beta(\theta)\left[\frac{\Gamma_\tau^2}{\alpha_\tau^2}+\Gamma_0^2\right]$$
$$=\frac{f(\theta)}{\alpha^2(\theta)}\frac{\alpha_\tau^2}{f(\theta)}\left[\frac{\Gamma_\tau^2}{\alpha_\tau^2}+\Gamma_0^2\right],$$
$$=\frac{\Gamma_\tau^2+\alpha_\tau^2\Gamma_0^2}{\alpha^2(\theta)}.$$

Therefore

$$\hat{\Gamma}(\theta)=\frac{(\Gamma_\tau^2+\alpha_\tau^2\Gamma_0^2)^{1/2}}{\alpha(\theta)}. \quad \text{(B11)}$$

3. Simplification of the expression for $\Gamma_d(\theta)$

From Eqs. (A24) and (A25), it follows that

$$\frac{1}{\Gamma_d^2(\theta)}=\frac{\hat{\omega}(\theta)}{\omega_0\Gamma_1^2(\theta)}\beta(\theta). \quad \text{(B12)}$$

Upon using Eqs. (B7) and (A9) in Eq. (B12), we find that

$$\frac{1}{\Gamma_d^2(\theta)}=\frac{f(\theta)}{\alpha^2(\theta)}\frac{1}{\Gamma_1^2(\theta)}\frac{\alpha_\tau^2}{f(\theta)}$$
$$=\frac{\alpha_\tau^2}{\alpha^2(\theta)}\frac{1}{\Gamma_\sigma^2}\left[\sin^2\theta+\frac{(1-\cos\theta)^2}{\alpha_\tau^2}\right], \quad \text{(B13)}$$

where Eq. (A13) was used in the last step.

APPENDIX C: THE BEHAVIOR OF $\hat{\omega}(\theta)$

It follows from Eqs. (3.17) and (3.20) that

$$\omega_0-\hat{\omega}(\theta)=\frac{\omega_0}{\alpha^2(\theta)}\left[\alpha^2(\theta)-\left[1+\frac{\Gamma_\tau^2}{\Gamma_\sigma^2}\cos\theta\right]\right]$$
$$=\frac{\omega_0}{\alpha^2(\theta)}\left[4\left[\frac{\Gamma_0}{\Gamma_\sigma}\right]^2\sin^2(\theta/2)+\left[\frac{\Gamma_\tau}{\Gamma_\sigma}\right]^2\left[1+\frac{\Gamma_0^2}{\Gamma_\sigma^2}\sin^2\theta\right]-\left[\frac{\Gamma_\tau}{\Gamma_\sigma}\right]^2\cos\theta\right]. \quad \text{(C1)}$$

Since $1-\cos\theta=2\sin^2(\theta/2)$, Eq. (C1) can be rewritten as

$$\omega_0-\hat{\omega}(\theta)=\frac{\omega_0}{\alpha^2(\theta)}\left\{\left[4\left[\frac{\Gamma_0}{\Gamma_\sigma}\right]^2+2\left[\frac{\Gamma_\tau}{\Gamma_\sigma}\right]^2\right]\sin^2(\theta/2)+\left[\frac{\Gamma_0}{\Gamma_\sigma}\right]^2\left[\frac{\Gamma_\tau}{\Gamma_\sigma}\right]^2\sin^2\theta\right\}. \quad \text{(C2)}$$

Since according to Eq. (3.20) $\alpha^2(\theta)\geq 1$, it follows from Eq. (C2) that

$$\hat{\omega}(\theta)\leq\omega_0, \quad \text{(C3)}$$

with the equality holding only in the following three special cases: (a) $\theta=0$, i.e., for forward scattering; (b) $\Gamma_\sigma\to\infty$ $(\sigma\to 0)$, i.e., when the dielectric susceptibility fluctuations are spatially uncorrelated; (c) $\Gamma_0\to 0$ and $\Gamma_\tau\to 0$ $(\tau\to\infty)$, i.e., when the incident light monochromatic and the dielectric susceptibility fluctuations are static.

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[18]For the sake of economy of notation we do not show explicitly the dependence of $\tilde{\omega}$ (and also of the quantities α, β, f, Γ_1, and Γ_2 defined below) on θ throughout the main part of this appendix.