Renormalized eddy viscosity and Kolmogorov's constant in forced Navier-Stokes turbulence

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The free-decay renormalization-group theory for Navier-Stokes turbulence [Zhou, Vahala, and Hossain, Phys. Rev. A **37**, 2590 (1988)] is extended to the case of forced turbulence. An eddy-damping function is obtained, which is nonlocal in time and space. Using a multitime scale perturbation analysis, a time-local renormalized eddy viscosity is determined as a fixed point of an integro-difference recursion relation. It exhibits a mild cusp behavior for the particular forcing exponent that gives the Kolmogorov energy spectrum, similar to that for free-decaying turbulence. As in the free-decay theory, the triple nonlinearity in the renormalized Navier-Stokes equation is essential for the cusp to occur near the boundary between the unresolvable and resolvable scales. Unlike the ϵ-expansion theory, however, the renormalized eddy viscosity exhibits a wave-number dependence in the supergrid scales. The standard inertial wave-number scaling of the eddy viscosity is recovered for the Kolmogorov energy spectrum. A numerical value for the Kolmogorov constant is obtained using the Yakhot-Orszag equivalence assumption $C_K = 1.44$.

I. INTRODUCTION

Recently, we[1] have employed the recursive renormalization-group theory (RG) of Rose[2] to free-decaying Navier-Stokes turbulence. The recursion relation for the renormalized eddy viscosity[1] yielded a fixed point with a mild cusp behavior near the wave number separating the subgrid and supergrid scales. Here we extended this recursion technique to the case of forced Navier-Stokes turbulence, but without making the assumption that the time dependence of the subgrid modes can be ignored relative to the large-scale (supergrid) modes as was made in the free-decay case.[1,2] It is also of some interest to consider Navier-Stokes turbulence driven by a Gaussian random force since this is the standard model considered in all ϵ-expansion RG theories.[3-6] One of the consequences of RG theories (among that of other theories) is an eddy viscosity. However, unlike the ϵ-expansion theories in which the eddy viscosity has no supergrid wave-number structure, we will find here that in the recursion RG method the eddy viscosity will have supergrid wave-number structure. Moreover, the eddy viscosity will exhibit a cusp behavior near the wave-number cutoff between subgrid and supergrid scales.

In Sec. II, the recursion RG procedure is outlined. Unlike the treatments in the free-decay problem,[1,2] the time dependence of the subgrid modes is not ignored. In particular, we obtain a nonlocal time (and space) behavior of the eddy-damping function, similar to that found earlier by Kraichnan[7] in his eddy-viscosity theory. After removing the subgrid shells, one obtains an interated Navier-Stokes equation which is very complicated—it contains not only this nonlocal eddy-damping function but also a triple nonlinear coupling.

In Sec. III this equation is simplified by introducing a multitime perturbation analysis based on the assumption that the subgrid modes evolve on a faster time scale than the supergrid modes. The structure of the eddy-viscosity recursion relation is significantly different from that for free decay due to the different structure of the subgrid velocity autocorrelation function.

The recursion relation is renormalized and numerical computations are performed to find the fixed point, the RG eddy viscosity, in Sec. IV. As in free-decay turbulence,[1] a mild cusp behavior is found in the eddy viscosity near the supergrid-subgrid cutoff wave number.

Finally, in Sec. V, the Kolmogorov constant C_K is calculated based on the Yakhot-Orszag[5]–assumed equivalence between the inertial range transfer theory of Kraichnan[8] and that for randomly forced turbulence. In essence, the effects of initial and boundary conditions can be modeled by a Gaussian random force with a specific autocorrelation intensity. We summarize our results in Sec. VI.

II. RECURSION RENORMALIZATION-GROUP PROCEDURE FOR FORCED NAVIER-STOKES TURBULENCE

It is customary[3-5] to introduce a random force **f** into the Navier-Stokes equation so as to model strong turbulence. In this way, the inertial range Kolmogorov energy spectrum can be recovered without having to introduce initial and boundary conditions. In wave-number space, the forced Navier-Stokes equation for the velocity **u** is given by

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_\alpha(\mathbf{k},t) = f_\alpha(\mathbf{k},t)+\int d^3j\, M_{\alpha\beta\gamma}(\mathbf{k})u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)\ , \tag{1}$$

where the coupling coefficient

$$M_{\alpha\beta\gamma}(\mathbf{k})=\frac{1}{2i}[k_\beta D_{\alpha\gamma}(\mathbf{k})+k_\gamma D_{\alpha\beta}(\mathbf{k})]\ , \quad \text{with } D_{\alpha\beta}(\mathbf{k})=\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\ , \tag{2}$$

and ν_0 is the molecular viscosity. The random force **f** is specified by its autocorrelation

$$\langle f_\alpha(\mathbf{k},t)f_\beta(\mathbf{k}',t')\rangle = D_{\alpha\beta}(\mathbf{k})D_0 k^{-y}\delta(\mathbf{k}+\mathbf{k}')\delta(t-t')\ , \tag{3}$$

where y is a suitably chosen exponent and D_0 defines the intensity of the forcing. For example,[3,4] the choice $y=-2$ corresponds to thermal equilibrium, while the choice $y=+3$ yields the Kolmogorov energy spectrum for the velocity fluctuations (see, however, Ref. 6),

$$E(k)=C_k\varepsilon^{-2/3}k^{-5/3}\ . \tag{4}$$

C_K is the Kolmogorov constant and ε is the rate of energy dissipation per unit volume.

A. RG procedure in the removal of the first subgrid shell

In the iterative RG approach one introduces a scale factor h, with $0<h<1$, to partitions wave-number space into

$$(k_*\equiv k_N\equiv h^N k_0,\ k_{N-1}\equiv h^{N-1}k_0,\ \dots,\ k_1\equiv hk_0,\ k_0\equiv k_d)$$

where k_d is the Kolmogorov dissipation wave number and k_* separates the resolvable large scales $(k<k_*)$ from the unresolvable subgrid scales $(k>k_*)$. The RG procedure involves the removal of the subgrid shells, one at a time, starting with the elimination of wave numbers **k** in the shell $k_1<|\mathbf{k}|<k_0$.

To eliminate the first subgrid shell $k_1<k<k_0$, one introduces the superscript notation $>$ to denote the subgrid fields and $<$ to denote the supergrid fields

$$u_\alpha(\mathbf{k},t)=\begin{cases}u_\alpha^>, & \text{for } k_1<k<k_0\\ u_\alpha^<, & \text{for } k<k_1\end{cases}\qquad f_\alpha(\mathbf{k},t)=\begin{cases}f_\alpha^>, & \text{for } k_1<k<k_0\\ f_\alpha^<, & \text{for } k<k_1\ .\end{cases} \tag{5}$$

Furthermore, we introduce an ensemble average over the subgrid modes, so that

$$\langle u_\alpha^<(\mathbf{k},t)\rangle = u_\alpha^<(\mathbf{k},t),\quad \langle u_\alpha^>(\mathbf{k},t)\rangle=0\ ,$$
$$\langle f_\alpha^<(\mathbf{k},t)\rangle = f_\alpha^<(\mathbf{k},t),\quad \langle f_\alpha^>(\mathbf{k},t)\rangle=0\ , \tag{6}$$

There are two mutually exclusive evolutionary equations, depending on whether k lies in the subgrid or supergrid shell. In particular, for the subgrid modes, with $k_1<|\mathbf{k}|<k_0$

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_\alpha^>(\mathbf{k},t)=f_\alpha^>(\mathbf{k},t)+\lambda_0 M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j[u_\beta^<(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)+2u_\beta^>(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t) + u_\beta^>(\mathbf{j},t)u_\gamma^>(\mathbf{k}-\mathbf{j},t)]\ , \tag{7}$$

where the parameter λ_0 is introduced to aid in the perturbation expansion (it will eventually be set equal to unity). The supergrid modes, with **k** below the first subgrid shell, $|\mathbf{k}|<k_1$, evolve according to

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_\alpha^<(\mathbf{k},t)=f_\alpha^<(\mathbf{k},t)+\lambda_0 M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j[u_\beta^<(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)+2u_\beta^>(\mathbf{j},t)u_\gamma^<(\mathbf{k}-j,t) + u^>\beta(\mathbf{j},t)u_\gamma^>(\mathbf{k}-\mathbf{j},t)]\ . \tag{8}$$

Of course, the right-hand sides of (7) and (8) are very different—not only because of the different **k** domains, but also because of the different ranges in the $\int d^3j$ integrals.

To leading order in λ_0, the subgrid velocity satisfies

$$u_\alpha^>(\mathbf{k},t)=\int_{-\infty}^{t}d\tau\, G_0(\mathbf{k},t,\tau)f_\alpha^>(\mathbf{k},\tau)\ , \tag{9}$$

where the zeroth-order Green's function G_0 is given by

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]G_0(k,t,\tau)=\delta(t-\tau)\ , \tag{10}$$

which can be readily solved to yield

$$G_0(\mathbf{k},t,\tau)=\exp(-\nu_0 k^2|t-\tau) \ . \tag{11}$$

On substituting this into the supergrid equation (8) and performing the subgrid ensemble average we find that the third term on the right-hand side of Eq. (8) does not contribute since

$$\langle f_\beta^>(\mathbf{j},\tau)\rangle u_\gamma^<(\mathbf{k}-\mathbf{j}t)=0 \ ,$$

while the fourth term does not contribute since $\langle u^> u^> \rangle$ essentially yields

$$M_{\alpha\beta\gamma}(\mathbf{k})\langle u_\beta^>(\mathbf{j})u_\gamma^>(\mathbf{k}-j)\rangle \to M_{\alpha\beta\gamma}(\mathbf{k})\langle f_\beta^>(\mathbf{j})f_\gamma^>(\mathbf{k}-j)\rangle \to M_{\alpha\beta\gamma}(\mathbf{k})\delta(\mathbf{k})=0 \ .$$

Hence the leading-order subgrid solution does not influence the evolution of the supergrid scales. Proceeding to first order in λ_0.

$$\begin{aligned} u_\alpha^>(\mathbf{k},t)= & \int_{-\infty}^{t} d\tau\, G_0(\mathbf{k},t,\tau) f_\alpha^>(\mathbf{k},\tau) \\ & +\lambda_0 M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\int_{-\infty}^{t} d\tau\, G_0(\mathbf{k},t,\tau)[u_\beta^<(\mathbf{j},\tau)u_\gamma^<(\mathbf{k}-\mathbf{j},\tau)+2u_\beta^>(\mathbf{j},\tau)u_\gamma^<(\mathbf{k}-\mathbf{j},\tau) \\ & +u_\beta^>(j,\tau)u_\gamma^>(\mathbf{k}-j,\tau)] \ . \end{aligned} \tag{12}$$

Substituting Eq. (12) into (8), performing a subgrid ensemble average, and evoking closure by ignoring the triple subgrid moment $\langle u^> u^> u^> \rangle$, we obtain

$$\begin{aligned} \left[\frac{\partial}{\partial t}+\nu_0 k^2\right] u_\alpha^<(\mathbf{k},t)= & f_\alpha^<(\mathbf{k},t)+\lambda_0 M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j \Bigg\{ u_\beta^<(\mathbf{j},t)u_\gamma^<(\mathbf{k}-j,t) \\ & +2\lambda_0 M_{\beta\beta'\gamma'}(\mathbf{j})\int d^3j'\int_{-\infty}^{t} d\tau\, G_0(\mathbf{j},t,\tau)[u_{\beta'}^<(\mathbf{j}',\tau)u_{\gamma'}^<(\mathbf{j}-\mathbf{j}',\tau)u_\gamma^<(\mathbf{k}-\mathbf{j},t) \\ & +2u_{\gamma'}^<(\mathbf{j}-\mathbf{j}',\tau)\langle u_{\beta'}^>(\mathbf{j}',\tau)u_\gamma^>(\mathbf{k}-\mathbf{j},t)\rangle]\Bigg\}+O(\lambda_0^3) . \end{aligned} \tag{13}$$

Consider the last term in Eq. (13). Using Eqs. (9) and (3), this term equals

$$\begin{aligned} & 4\lambda_0^2 M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, M_{\beta\beta'\gamma'}(\mathbf{j})\int d^3j'\int_{-\infty}^{t} d\tau\, G_0(\mathbf{j},t,\tau)u_{\gamma'}^<(\mathbf{j}-\mathbf{j}',\tau)\langle u_{\beta'}^<(j',\tau)u_\gamma^>(\mathbf{k}-\mathbf{j}t)\rangle_0 \\ & =4\lambda_0^2 M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, M_{\beta\beta'\gamma'}(\mathbf{j})\int d^3j'\int_{-\infty}^{t} d\tau\, G_0(\mathbf{j},t,\tau)u_{\gamma'}^<(\mathbf{j}-\mathbf{j}',\tau) \\ & \qquad \times\int_{-\infty}^{\tau} d\tau'\int_{-\infty}^{t} d\tau'' G_0(\mathbf{j}',\tau,\tau')G_0(\mathbf{k}-j,t,\tau'')D_{\beta'\gamma}(\mathbf{k}-\mathbf{j}) \\ & \qquad \times D_0|\mathbf{k}-\mathbf{j}|^{-y}\delta(\mathbf{k}-\mathbf{j}+\mathbf{j}')\delta(\tau'-\tau'') \\ & \equiv -\int_{-\infty}^{t} d\tau\, \eta_0(\mathbf{k},t,\tau)u_\alpha^<(k,\tau) \ , \end{aligned}$$

where $\eta_0(\mathbf{k},t,\tau)$ is a nonlocal generalized eddy-damping function defined by

$$\eta_0(\mathbf{k},t,\tau)\equiv -4\lambda_0^2 M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, M_{\beta\beta'\gamma'}(\mathbf{j})G_0(\mathbf{j},t,\tau)D_{\gamma'\alpha}(\mathbf{k})\int_{-\infty}^{t} d\tau' G_0^2(\mathbf{k}-\mathbf{j},t,\tau')D_{\beta'\gamma}(\mathbf{k}-j)D_0|\mathbf{k}-\mathbf{j}|^{-y} \ . \tag{14}$$

Thus, after removing the first subgrid shell, the renormalized Navier-Stokes equation becomes, for $|\mathbf{k}|<k_1$.

$$\begin{aligned} \left[\frac{\partial}{\partial t}+\nu_0 k^2\right] u_\alpha^<(\mathbf{k},t)+ & \int_{-\infty}^{t} d\tau\, \eta_0(\mathbf{k},t,\tau)u_\alpha^<(\mathbf{k},\tau) \\ & =M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta^<(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t) \\ & +2M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, M_{\beta\beta'\gamma'}(\mathbf{j})\int_{-\infty}^{t} d\tau\, G_0(\mathbf{j},t,\tau)u_{\beta'}^<(\mathbf{j}'\tau)u_{\gamma'}^<(\mathbf{k},\tau)u_\gamma^<(\mathbf{k}-\mathbf{j},t)+f_\alpha^<(\mathbf{k},t) \ . \end{aligned} \tag{15}$$

Thus the RG procedure has generated (a) a triple nonlinearity and (b) a time-nonlocal eddy-damping function $\eta_0(k,t,\tau)$, Eq. (14).

Consider (a). The generation of new types of interaction is quite common in RG theories. Indeed, in the RG (equilibrium) theory for the two-dimensional Ising problem, Wilson[9] finds that after the first spin decimation, not only must one consider the original nearest-neighbor interaction but also the diagonal-nearest-neighbor and four-spin-coupling interactions. For the Ising problem, the new interactions are found to be weaker than the original interaction, and one can neglect the four-spin interaction and still obtain accurate solutions. In our application of dynamical RG to fluid turbulence, we retain the new triple nonlinear interactions, and assume that the higher-order nonlinearities can be neglected. Let us now consider the ϵ-expansion RG theories[3–5] to Navier-Stokes turbulence. For $\epsilon \ll 1$, it can be shown that the higher-order nonlinearities are irrelevant couplings and can be ignored. However, to recover the Kolmogorov energy spectrum, one is forced to choose $\epsilon=4$. For $\epsilon=4$, there are now no valid arguments to show that the higher-order nonlinearities are irrelevant.

The nonlocal behavior in the eddy-damping function, property (b), has been found earlier in the eddy viscosity theory of Kraichnan.[7] In our earlier free-decay[1] RG theory of fluid turbulence, we had assumed that the subgrid scales evolved so rapidly that their time variation[2] could be ignored. Under this approxiation, one loses this nonlocal nature of the eddy-damping function, which then reduces to an eddy-viscosity coefficient. However, this approximation, in forced turbulence, will lead to a different wave-number scaling of the eddy-viscosity coefficient (for more details, see the Appendix). Thus this approximation is not made here.

B. Removal of the $(n+1)$th subgrid shell

We now proceed to eliminate the next subgrid shell $k_2 < k < k_1$ from the renormalized Navier-Stokes equation for which $|\mathbf{k}| < k_1$

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_\alpha(\mathbf{k},t)+\int_{-\infty}^{t} d\tau\,\eta_0(\mathbf{k},t,\tau)u_\alpha(\mathbf{k},\tau)$$
$$=f_\alpha(\mathbf{k},t)+M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)$$
$$+2M_{\alpha\beta\gamma}(\mathbf{k})\int\int d^3j d^3j' M_{\beta\beta'\gamma'}(j)\int_{-\infty}^{t} d\tau\, G_0(\mathbf{j},t,\tau)u_\beta(\mathbf{j}',\tau)u_{\gamma'}(j-j',\tau)u_\gamma(\mathbf{k}-\mathbf{j},t)\ . \quad (16)$$

For modes with $|\mathbf{k}| < k_1$, we again define the subgrid modes as those with wave number in the range $k_2 < k < k_1$ and the supergrid modes have wave numbers $k < k_2$:

$$u_\alpha=\begin{cases} u_\alpha^>, & \text{if } k_2<k<k_1 \\ u_\alpha^<, & \text{if } k<k_2\ , \end{cases}$$

with a similar identification for the forcing function f. To leading order in λ_0 [reintroducing this formal perturbation parameter in the nonlinear terms of Eq. (16)], one obtains the subgrid velocity equation

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_\alpha^>(\mathbf{k},t)+\int_{-\infty}^{t} d\tau\,\eta_0(\mathbf{k},t,\tau)u_\alpha^>(\mathbf{k},\tau)=f_\alpha^>(\mathbf{k},t)\ . \quad (17)$$

To solve Eq. (17) one introduces the Green's function $G_1(\mathbf{k},t,\tau)$,

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]G_1(\mathbf{k},t,\tau)+\int_{\tau}^{t} ds\ \eta_0(\mathbf{k},t,s)G_1(\mathbf{k},s,\tau)=\delta(t-\tau)\ . \quad (18)$$

An analogous Green's function was introduced by Herring and Kraichnan[10] in their comparison of the various theories of isotropic turbulence.

As before, the zeroth-order subgrid solution plays no role in the evolution of the supergrid velocity field. If one proceeds to next order in λ_0.

$$u_\alpha^>(\mathbf{k},t)=\int_{-\infty}^{t} d\tau\, G_1(\mathbf{k},t,\tau)f_\alpha^>(\mathbf{k},\tau)$$
$$+\lambda_0 M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\int_{-\infty}^{t} d\tau\, G_1(\mathbf{k},t,\tau)[u_\beta^<(j,\tau)u_\gamma^<(\mathbf{k}-j,\tau)+2u_\beta^>(j,\tau)u_\gamma^<(\mathbf{k}-\mathbf{j},\tau)+\cdots]\ . \quad (19)$$

where the ellipsis refers to the $u^>u^>$ term, which can be neglected because of our closure approximation.

Following the RG procedure outlined in the removal of the first subgrid shell, one finds that the triple term in Eq. (16) will contribute to the nonlocal eddy-damping function

$$\eta_1^T(\mathbf{k},t,\tau)=-4M_{\alpha\beta\gamma}(\mathbf{k})\int\int d^3j\, d^3j' M_{\beta\beta'\gamma'}(\mathbf{j})G_0(\mathbf{j},t,\tau)D_{\gamma'\alpha}(\mathbf{k})\langle u_{\beta'}^>(j',\tau)u_\gamma^>(\mathbf{k}-\mathbf{j},t)\rangle\ . \quad (20)$$

where the subgrid velocity correlation is to be evaluated using Eq. (19) to zeroth order in λ_0.

The second term in Eq. (19) will contribute a triple nonlinearity to the renormalized Navier-Stokes equation after removal of the second subgrid shell. However, the third term in Eq. (19) will also contribute a term to the eddy-damping function of the form

$$\eta_1^D(\mathbf{k},t,\tau)\equiv-4M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, M_{\beta\beta'\gamma'}(\mathbf{j})G_1(\mathbf{j},t,\tau)D_{\gamma'\alpha}(\mathbf{k})\langle u_{\beta'}^>(\mathbf{j}',\tau)u_\gamma^>(\mathbf{k}-\mathbf{j},t)\rangle\ . \quad (21)$$

where again the subgrid velocity correlation is evaluated using the lowest-order expansion (in λ_0) in Eq. (19).

The final form taken by the renormalized Navier-Stokes equation after removal of the second subgrid shell is

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_\alpha^<(\mathbf{k},t)+\int_{-\infty}^{t}d\tau[\eta_0(\mathbf{k},t,\tau)+\eta_1(\mathbf{k},t,\tau)]u_\alpha^<(\mathbf{k},\tau)$$
$$=f_\alpha^<(\mathbf{k},t)+M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta^<(\mathbf{j},t)u_\gamma^<(\mathbf{k}-\mathbf{j},t)$$
$$+2M_{\alpha\beta\gamma}(\mathbf{k})\int\int d^3j\, d^3j' M_{\beta\beta'\gamma'}(\mathbf{j})\int_{-\infty}^{t}d\tau[G_0(\mathbf{j},t,\tau)+G_1(\mathbf{j},t,\tau)]u_{\beta'}^<(\mathbf{j}',\tau)u_{\gamma'}^<(\mathbf{j}-\mathbf{j}',\tau)u_\gamma^<(\mathbf{k}-\mathbf{j},t)\ , \quad (22)$$

where

$$\eta_1(\mathbf{k},t,\tau)=\eta_1^D(\mathbf{k},t,\tau)+\eta_1^T(\mathbf{k},t,\tau)\ . \quad (23)$$

Proceeding to the removal of the $(n+1)$th subgrid shell, one shell obtains

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]u_\alpha(\mathbf{k},t)+\int_{-\infty}^{t}d\tau\sum_{i=0}^{n}\eta_i(\mathbf{k},t,\tau)u_\alpha(\mathbf{k},\tau)$$
$$=f_\alpha(\mathbf{k},t)+M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)$$
$$+2M_{\alpha\beta\gamma}(\mathbf{k})\int\int d^3j\, d^3j' M_{\beta\beta'\gamma'}(\mathbf{j})\int_{-\infty}^{t}d\tau\sum_{i=0}^{n}G_i(\mathbf{j},t,\tau)u_{\beta'}(\mathbf{j}',\tau)u_{\gamma'}(\mathbf{j}-\mathbf{j}',\tau)u_\gamma(\mathbf{k}-\mathbf{j},t)\ , \quad (24)$$

with the corresponding generalizations for the ith Green's function G_i, Eq. (18), and eddy-damping function η_i, Eqs. (20), (21), and (23). In particular,

$$\left[\frac{\partial}{\partial t}+\nu_0 k^2\right]G_i(\mathbf{k},t,\tau)+\int_{-\infty}^{t}ds\,\eta_{i-1}(\mathbf{k},t,s)G_i(\mathbf{k},s,\tau)=\delta(t-\tau)\ . \quad (25)$$

III. SIMPLIFIED RENORMALIZED NAVIER-STOKES EQUATION

Equation (24) is very complicated. Since the subgrid scales evolve on a faster time scale than the supergrid scales some simplification can be achieved by performing a multitime scale analysis. Under this approximation, Eq. (24) reduces to

$$\left[\frac{\partial}{\partial t}+\nu_{n+1}(k)k^2\right]u_\alpha(\mathbf{k},t)=f_\alpha(\mathbf{k},t)+M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t)$$
$$+2M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j\, d^3j' M_{\beta b'\gamma'}(\mathbf{j})u_{\beta'}(\mathbf{j}',t)u_{\gamma'}(\mathbf{j}-\mathbf{j}',t)u_\gamma(\mathbf{k}-\mathbf{j},t)\int_{-\infty}^{t}d\tau\sum_{i=0}^{n}G_i(\mathbf{j},t,\tau)\ , \quad (26)$$

where the eddy-damping function has now been reduced to a standard eddy-viscosity coefficient

$$\nu_{n+1}(k)k^2=\nu_0 k^2+\int_{-\infty}^{t}d\tau\sum_{i=0}^{n}\eta_i(\mathbf{k},t,\tau)\ , \quad (27)$$

i.e, the nonlocal time behavior is lost under this approximation, but the resulting equations will become more tractable. So that the final recursion relation is more amenable to numerical solution we also make a local approximation in the subgrid velocity autocorrelation which occurs in the evaluation of the eddy-damping function. In particular, in the removal of the first subgrid shell

$$\langle u_{\beta'}^>(\mathbf{j}',\tau)u_\gamma^>(\mathbf{k}-\mathbf{j},\tau')\rangle\approx\langle u_{\beta'}^>(\mathbf{j}',t)u_\gamma^>(\mathbf{k}-\mathbf{j},t)\rangle$$
$$=\int_{-\infty}^{t}d\tau\int_{-\infty}^{t}d\tau'\exp[-\nu_0 j'^2(t-\tau)-\nu_0|\mathbf{k}-\mathbf{j}|^2(t-\tau')]\langle f_{\beta'}^>(\mathbf{j}',\tau)f_\gamma^>(\mathbf{k}-\mathbf{j},\tau')\rangle$$
$$=\frac{D_{\beta'\gamma}(\mathbf{k}-\mathbf{j})D_0|\mathbf{k}-\mathbf{j}|^{-y}\delta(\mathbf{k}-\mathbf{j}+\mathbf{j}')}{2\nu_0|\mathbf{k}-\mathbf{j}|^2} \quad (28)$$

to leading order in λ_0, on using Eqs. (10) and (3). Within the time integration, this is a somewhat mild approximation on the velocity correlation since the correlation function is weighted by an exponentially decaying Green's function

$$G_0(\mathbf{k},t,\tau)=\exp[-\nu_0 k^2|t-\tau|]\ . \quad (29)$$

After removing the first subgrid shell, the eddy viscosity

$$\nu_1(k)=\nu_0+\delta\nu_0(k) \quad (30)$$

where the increment $\delta\nu_0(k)$ to the molecular viscosity ν_0 is given by

$$\delta\nu_0(k)=\frac{D_0}{k^2}\int d^3j\frac{L_{kj}|\mathbf{k}-\mathbf{j}|^{-y}}{\nu_0|\mathbf{k}-\mathbf{j}|^2\nu_0 j^2} , \quad (31)$$

with the coefficient L_{kj} defined by

$$L_{kj}=-2M_{\alpha\beta\gamma}(\mathbf{k})M_{\beta\beta'\gamma'}(j)D_{\beta'\gamma}(\mathbf{k}-\mathbf{j})D_{\gamma'\alpha}(\mathbf{k}) = -\frac{kj(1-\mu^2)[\mu(k^2+j^2)-kj(1+2\mu^2)]}{k^2+j^2-2kj\mu} , \quad (32)$$

$\mathbf{k}\cdot\mathbf{j}=kj\mu$ with $\mu\equiv\cos\theta$. The integration limits in Eq. (31) are $k_1<|\mathbf{k}-\mathbf{j}|$ and $|\mathbf{j}|<k_0$.

Proceeding to the removal of $(n+1)$th subgrid shell, the subgrid autocorrelation (to be weighted by its respective exponentially decaying Green's function) is given by

$$\langle u^{>}_{\beta'}(j',\tau)u^{>}_{\gamma}(\mathbf{k}-\mathbf{j},t)\rangle \approx\frac{D_{\beta'\gamma}(\mathbf{k}-\mathbf{j})D_0|\mathbf{k}-\mathbf{j}|^{-y}\delta(\mathbf{k}-\mathbf{j}+\mathbf{j}')}{2\nu_n(\mathbf{k}-\mathbf{j})|\mathbf{j}-\mathbf{j}|^2} \quad (28')$$

The eddy-viscosity recursion relation is

$$\nu_{n+1}(k)=\nu_n(k)+\delta\nu_n(k) , \quad (33)$$

where

$$\delta\nu_n(k)=\frac{D_0}{k^2}\sum_{i=0}^{n}\int d^3j\frac{L_{kj}|\mathbf{k}-\mathbf{j}|^{-y}}{\nu_i(j)j^2\nu_n(\mathbf{k}-\mathbf{j})|\mathbf{k}-\mathbf{j}|^2} , \quad (34)$$

with the integration limits $k_{n+1}<|\mathbf{k}-\mathbf{j}|<k_n$ and $k_{i+1}<|\mathbf{j}|<k_i, i=0,1,\ldots,n$.

This eddy-viscosity recursion relation for forced Navier-Stokes turbulence differs significantly from that for free-decaying turbulence.[1] In particular, in our earlier free-decay turbulence calculation, the renormalized eddy viscosity satisfied the recursion relation (33), with

$$\delta\nu_n(k)|_{\text{free decay}}=\frac{2}{k^2}\sum_{i=0}^{n}\int d^3j\frac{L_{kj}Q(|\mathbf{k}-\mathbf{j}|)}{\nu_i(j)j^2} . \quad (35)$$

This different structure in the denominator is due to the different form of the subgrid autocorrelation in free-decaying turbulence

$$\langle u^{>}_{\beta'}(\mathbf{j}',t)u^{>}_{\gamma}(\mathbf{k}-\mathbf{j},t)\rangle \approx D_{\beta'\gamma}(\mathbf{k}-\mathbf{j})Q(|\mathbf{k}-\mathbf{j}|)\delta(\mathbf{k}-\mathbf{j}+\mathbf{j}') , \quad (36)$$

where $Q(|\mathbf{k}-\mathbf{j}|)$ is related to the energy spectrum $E(|\mathbf{k}-\mathbf{j}|)$ by

$$Q(|\mathbf{k}-\mathbf{j}|)=\frac{E(|\mathbf{k}-\mathbf{j}|)}{4\pi|\mathbf{k}-\mathbf{j}|^2} . \quad (37)$$

It is not related to the different treatment of the time dependence of the subgrid scales (for more details, see the Appendix).

IV. RENORMALIZED EDDY VISCOSITY

A. Rescaling and the renormalized eddy viscosity

Unlike the ϵ-expansion RG, one performs rescaling transformations in the recursive RG approach. In particular, in the removal of the $(n+1)$th subgrid shell, we make the transformation

$$k\rightarrow k_{n+1}\tilde{k} , \quad (38)$$

and define a renormalized eddy viscosity $\nu^*_n(\tilde{k})$ by

$$\nu^*_n(\tilde{k})\equiv k_{n+1}^{(y+1)/3}\nu_n(k_{n+1}\tilde{k}) \quad \text{for } \tilde{k}\le 1 , \quad (39)$$

so that the renormalized-eddy viscosity recursive relation, Eqs. (33) and (34), become

$$\nu^*_{n+1}(\tilde{k})=h^{(y+1)/3}[\nu^*_n(h\tilde{k})+\delta\nu^*_n(h\tilde{k})] , \quad (40)$$

with

$$\delta\nu^*_n(\tilde{k})=\frac{D_0}{\tilde{k}^2}\sum_{i=0}^{n}h^{-i(y+1)/3}\int d^3\tilde{j}\frac{L_{\tilde{k}\tilde{j}}|\tilde{\mathbf{k}}-\tilde{\mathbf{j}}|^{-y}}{\nu^*_i(h^i\tilde{j})\tilde{j}^2\nu^*_n(|\tilde{\mathbf{k}}-\tilde{\mathbf{j}}|)|\tilde{\mathbf{k}}-\tilde{\mathbf{j}}|^2} , \quad (41)$$

and the integration limits ($\tilde{k}\le 1$)

$$1<|\tilde{\mathbf{k}}-\tilde{\mathbf{j}}|<\frac{1}{h} , \qquad 1<|h^i\tilde{\mathbf{j}}|<\frac{1}{h}, \quad i=0,1,\ldots,n . \quad (42)$$

The parameter $0<h<1$ defines the coarseness of the subgrid shell partition. The $i=0$ contribution in Eq. (41) arises from the standard Navier-Stokes quadratic nonlinearity, while the $i\ge 1$ terms arise from the new triple nonlinearity introduced by the RG procedure.

We now consider the forcing exponent y, Eq. (3), and the energy scaling of the subgrid velocity autocorrelation

$$\langle u^{>}u^{>}\rangle \rightleftarrows \frac{E(k)}{k^2} . \quad (43)$$

But from Eqs. (28′) and (43), we see that

$$k^2\langle u^{>}u^{>}\rangle \rightleftarrows \frac{k^{-y}}{\nu} \rightleftarrows k^{-y}k^{(y+1)/3} \rightleftarrows E(k) \quad (44)$$

since the eddy viscosity scales as [from Eq. (39)]

$$\nu\approx k^{-(y+1)/3} . \quad (45)$$

Hence the choice $y=3$ will recover the Kolmogorov energy spectrum $E(k)\approx k^{-5/3}$, and the eddy-viscosity scaling $\nu\approx k^{-4/3}$. This eddy-viscosity scaling agrees with that arising from Kraichnan's inertial range theory[8] and from dimensional analysis.[11]

B. Numerical results

The renormalized eddy viscosity is defined as the fixed point ($n \rightarrow \infty$) of the recursion relations (40) and (41). Unlike the ϵ-expansion RG, which only gives the eddy viscosity at $\tilde{k}=0$, the recursive RG procedure yields the total $\tilde{k}$ dependence of the eddy viscosity throughout the supergrid scales.

In Fig. 1 the renormalized eddy viscosity $\nu^*(\tilde{k})$ is plotted for the subgrid partition parameter $h=0.7$ for various values of the forcing exponent y. The exponent $y=3$ corresponds to a forcing consistent with the Kolmogorov energy spectrum $E(k)\approx k^{-5/3}$. The reason for considering the exponent $y=\frac{7}{3}$ is given in the Appendix. We see that the RG eddy viscosity exhibits a cusp behavior for $\tilde{k}\approx 1$, i.e., for wave numbers $\tilde{k}$ near the supergrid and/or subgrid cutoff for $y=3$. This is in qualitative agreement with the test-field model of Kraichnan,[7] the eddy-damped quasinormal Markovian approximation of Chollet and Lesieur,[12] as well as with the direct numerical simulation results of Domaradzki *et al.*[13] The cusp behavior is lost for $y=1$.

In Fig. 2, we compare the renormalized eddy viscosity $\nu^*(\tilde{k})$ for free-decaying turbulence,[1] forced turbulence with and without the effects of the triple nonlinearity introduced by the RG transformations, and Kraichnan's test-field model. All relevant parameters are chosen such that the subgrid energy spectrum is just the Kolmogorov spectrum. It should be noticed that the cusp behavior is somewhat stronger for forced turbulence than for free-decaying turbulence. As in free-decaying turbulence, there is no cusp behavior in the eddy viscosity if the triple nonlinearity is dropped from the recursion relations which lead to the fixed point.

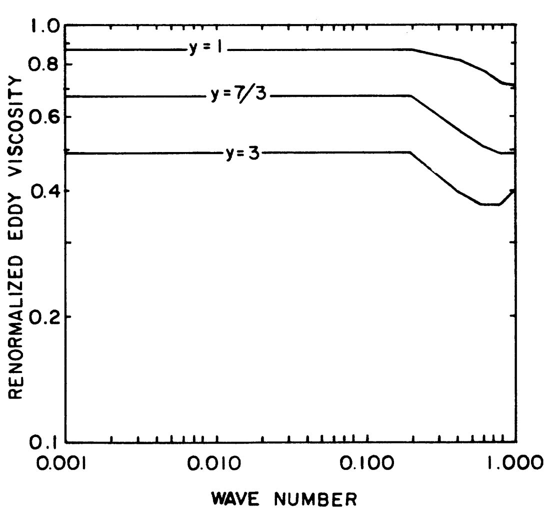

FIG. 1. Renormalized eddy viscosity for various forcing exponents y. Cusp behavior is induced by increased values of the exponent y. $y=3$ recovers the Kolmogorov energy and eddy-viscosity scalings. This choice is required in most ϵ-RG expansion theories.

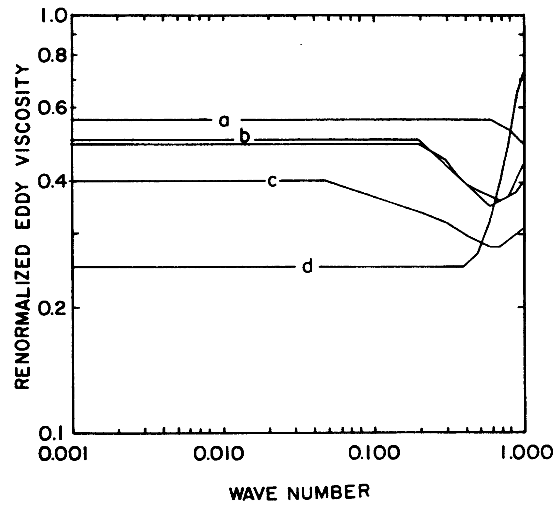

FIG. 2. Renormalized eddy viscosity in the supergrid scales for forced turbulence, including the triple nonlinearity effects with forcing exponent $y=3$. This is contrasted with the corresponding eddy viscosity for (a) forced turbulence, but dropping the triple nonlinearity terms: (b) forced turbulence with triple nonlinearity effects, but the time scales of the subgrid modes ignored (and this requires the choice of forcing exponent $y=\frac{7}{3}$: see the Appendix): (c) free-decaying eddy viscosity as found in Ref. 1; and (d) that from Kraichnan's test field model (Ref. 8). Note that if the triple term is neglected, case (a), there is no cusp behavior in the eddy viscosity near the supergrid-subgrid wave-number cutoff.

If one had assumed that the subgrid scales evolved so rapidly that one could treat all their time dependences as irrelevant, then it is shown in the Appendix that a forcing exponent of $y=\frac{7}{3}$ will yield a Kolmogorov $k^{-5/3}$ energy spectrum and an eddy-viscosity scaling of $k^{-4/3}$. However, other theories[3-5,11] have indicated that $y=3$ is the required choice in order to recover the Kolmogorov energy spectrum. In Fig. 3 we plot the results of the Appendix for the forcing exponent $y=3$ (giving a forced energy spectrum k^{-2} and an eddy viscosity scaling $k^{-3/2}$), $y=\frac{7}{3}$, and $y=1$.

The effect of a spectral gap on the eddy viscosity and the direct effect of the triple nonlinearity in the renormalized Navier-Stokes equation are similar to that for free-decaying turbulence. These effects have been discussed at some length in our earlier paper on free-decaying turbulence[1] and we refer the reader to that paper for details. If there is a spectral gap between the subgrid and supergrid modes, then there will be no triple nonlinearity term present into the renormalized Navier-Stokes equation. Thus the resulting RG eddy viscosity does not exhibit a cusp behavior near the subgrid-supergrid cutoff.[1] The triple nonlinearity in the RG Navier-Stokes equation yields an extra damping effect near the subgrid-supergrid cutoff. This could account for the weaker cusp seen in the RG theory over that of the theories of Kraichnan[7] and Chollet and Lesieur[12] in which there are no triple nonlinearities present in the evolution equation.

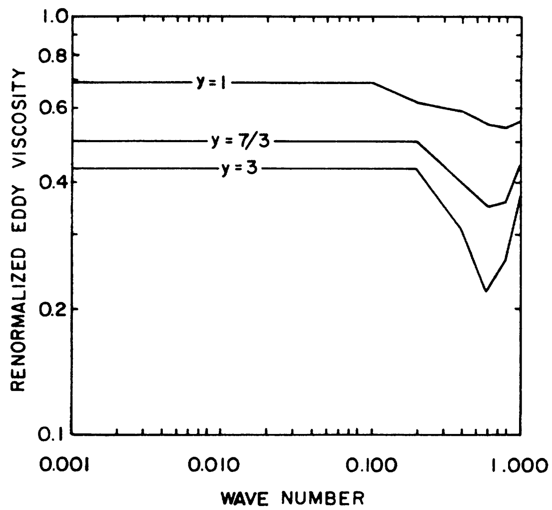

FIG. 3. Renormalized eddy viscosity for various forcing exponents j, but with the $du^{>}/dt$ neglected. In this case the choice $y=\frac{7}{3}$ is required if one wanted to recover the accepted energy and viscosity scalings.

V. EVALUATION OF THE KOLMOGOROV CONSTANT

We now use these numerically determined values of the renormalized eddy viscosity to calculate the Kolmogorov constant. We shall invoke the Yakhot-Orszag equivalence assumption[5] between the inertial-range structures satisfying initial and boundary conditions and the turbulence driven by a random force of appropriate strength D_0.

The discussion will be restricted to the case of three dimensions. The energy spectrum

$$E(\mathbf{k},t)=2\pi k^2 U(\mathbf{k},t) . \tag{46}$$

where the velocity covariance

$$U(\mathbf{k},t)=D_{\alpha\beta}(\mathbf{k})\langle u_\alpha(\mathbf{k},t)u_\beta(-\mathbf{k},t)\rangle \tag{47}$$

since

$$[D_{\alpha\beta}(\mathbf{k})]^2=2 .$$

At the fixed point of the renormalized eddy viscosity, the covariance is determined from Eq. (28) and (28′),

$$\langle u_\alpha(\mathbf{k},t)u_\beta(-\mathbf{k},t)\rangle=\frac{D_0 D_{\alpha\beta}(\mathbf{k})|\mathbf{k}|^{-y}}{2\nu(\mathbf{k})k^2} , \tag{48}$$

where D_0 defines the intensity of the forcing. Eq. (3), and $y=3$ in three dimensions. Substituting Eq. (48) into (47) and (46), we obtain

$$E(\mathbf{k},t)=\frac{2\pi^2 D_0 k^{-y-2}}{\nu(\mathbf{k})} . \tag{49}$$

To relate this eddy viscosity $\nu(\mathbf{k})$ to the numerical results in Figs. 1–3, we must fold back the rescaling transformation used in Sec. IV. Letting $\nu_*(\mathbf{k})$ denote the numerical renormalized eddy viscosity, we have

$$\nu(\mathbf{k})=\nu_*(\mathbf{k})(2\pi D_0)^{3/2}k^{-4/3} , \tag{50}$$

so that

$$E(\mathbf{k},t)=\frac{(2\pi D_0)^{2/3}}{\nu_*}k^{-5/3} . \tag{51}$$

Following Yakhot and Orszag,[5] the Kolmogorov constant C_K is evaluated by relating this energy spectrum to that determined by Kraichnan[8] in his inertial-range transfer turbulence theory. In particular, Kraichnan[8] has shown that

$$\nu(\mathbf{k})=0.1904C_K^2\varepsilon^{1/3}k^{-4/3} \tag{52}$$

with

$$E(\mathbf{k})=C_K\varepsilon^{2/3}k^{-5/3} . \tag{53}$$

Equating (51) and (53), we find that the Kolmogorov constant is given by

$$C_K=\frac{1}{(0.1904)^{2/3}}\nu_*=3.02\nu_* . \tag{54}$$

Now the effect of the eddy viscosity is usually made in its asymptotic form[14] $\nu_*(k|k_*=1)$ with $k \ll k_*$ where k_* is the wave number separating the supergrid and subgrid scales. Hence we predict a Kolmogorov constant of

$$C_K=1.44 . \tag{55}$$

VI. CONCLUSIONS

Recursion RG has been applied to turbulence driven by Gaussian random forces. The eddy-damping function exhibits nonlocal behavior in time and space. To proceed further, we assume that the subgrid modes evolve on a faster time scale than the large-scale (supergrid) modes. Under this assumption, we lose the nonlocal time behavior in the damping function, and after RG obtain a fixed-for-the-eddy viscosity. A mild cusp behavior is found near the subgrid-supergrid cutoff wave number, as in free-decaying turbulence.[1] The triple nonlinearity has the same qualitative effect on both the eddy-viscosity cusp and on dissipation properties in the renormalized Navier-Stokes equation itself as in free-decaying turbulence,[1] and the reader is referred to that paper for further discussions on these points.

However, we do not assume that the one can ignore the time dependence of the subgrid modes as has been done in free-decaying turbulence.[1,2] This question is examined in the Appendix. We find that if the subgrid time scales are ignored, then one recovers the Kolmogorov energy spectrum scaling $E(k)\approx k^{-5/3}$ and eddy-viscosity scaling $\nu(k)\approx k^{-4/3}$ only if the forcing exponent $y=\frac{7}{3}$. This is considered inappropriate, since in most ϵ-expansion RG theories[3–5] the forcing exponent must be chosen to be $y=3$.

The Kolmogorov constant is evaluated using the Yakhot-Orszag[5] assumption that the inertial range structures can be modeled by randomly driven turbulence provided the intensity of the random force is suitably chosen. We find $C_K = 1.44$, which is an acceptable value.[15,16] A more sophisticated theory was attempted along the lines of Dannevik *et al.*[14] However, the emergence of the triple nonlinearity in the renormalized Navier-Stokes equation forces one into further assumptions which would cloud any claims to a successful value for C_K.

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APPENDIX: EFFECT OF THE TIME EVOLUTION OF THE SUBGRID MODES

Here we shall assume that the subgrid modes evolve on a much faster time scale than the supergrid modes so that we can make the approximation

$$\frac{\partial}{\partial t} u^{>}(\mathbf{k},t) \to 0 \; . \tag{A1}$$

This approximation had been used by Rose in his RG analysis of passive scalar advection and by us in our free-decaying turbulence calculation. The Green's function G_0, Eq. (11), will reduce to

$$G_0(\mathbf{k},t,\tau) \to \frac{1}{\nu_0 k^2}\delta(t-\tau) \; , \tag{A2}$$

so that to leading order the subgrid autocorrelation

$$\langle u_\gamma^{>}(\mathbf{k}-\mathbf{j},t) u_\beta^{>}(\mathbf{j}',t)\rangle = \frac{\langle f_\gamma^{>}(\mathbf{k}-\mathbf{j},t) f_{\beta'}^{>}(\mathbf{j}',t)\rangle}{\nu_0|\mathbf{k}-\mathbf{j}|^2 \nu_0 j'^2} \; . \tag{A3}$$

Under these approximations, the eddy viscosity arising from the removal of the first subgrid shell is

$$\nu_1(k) = \nu_0 + \delta\nu_0(k) \; , \tag{A4}$$

with

$$\delta\nu_0(k) = \frac{2D_0}{k_0^2} \int d^3 j \frac{L_{kj}|\mathbf{k}-\mathbf{j}|^{-y}}{\nu_0 j^2 \nu_0^2 |\mathbf{k}-\mathbf{j}|^4} \; . \tag{A5}$$

Equation (A5) should be compared to Eq. (31). Note the different dependence on the viscosity in the integrand.

Proceeding to the removal of the nth subgrid shell, we obtain the eddy-viscosity recursion relation

$$\nu_{n+1}(k) = \nu_n(k) + \delta\nu_n(k) \; , \tag{A6}$$

with

$$\delta\nu_n(k) = \frac{2D_0}{k^2} \sum_{i=0}^{n} \int d^3 j \frac{L_{kj}|\mathbf{k}-\mathbf{j}|^{-y}}{\nu_i(j) j^2 \nu_n^2(\mathbf{k}-\mathbf{j})|\mathbf{k}-\mathbf{j}|^4} \; . \tag{A7}$$

Equation (A7) should be compared to Eq. (34).

The RG transformation differs from Eq. (39) due to the extra viscosity dependence in Eq. (A7). Indeed, under the transformation

$$k \to k_{n+1}\tilde{k} \; , \tag{A8}$$

we define the renormalized eddy viscosity $\nu_n^*(\tilde{k})$ by

$$\nu_n^*(\tilde{k}) = k_{n+1}^{(y+3)/4} \nu_n(k_{n+1}\tilde{k}) \; , \tag{A9}$$

so that the renormalized eddy-viscosity recursive relation becomes

$$\nu_{n+1}^*(\tilde{k}) = h^{(y+3)/4}[\nu_n^*(h\tilde{k}) + \delta\nu_n^*(h\tilde{k})] \; . \tag{A10}$$

The energy scaling of the subgrid modes is

$$\langle u^{>} u^{>}\rangle \rightleftarrows \frac{E(k)}{k^2} \; , \tag{A11}$$

so that from Eqs. (A3) and (A7) we have

$$k^2\langle u^{>} u^{>}\rangle \rightleftarrows \frac{k^{-y}}{\nu^2 k^2} \rightleftarrows k^{-y-2} k^{(y+3)/2} \rightleftarrows E(k) \; , \tag{A12}$$

with the eddy-viscosity scaling as

$$\nu \approx k^{-(y+3)/4} \; . \tag{A13}$$

The Kolmogorov energy spectrum $E(k) \approx k^{-5/3}$ and the eddy-viscosity scaling $\nu \approx k^{-4/3}$ are recovered by the choice of $y = \frac{7}{3}$ for the forcing exponent. The standard choice of $y = 3$, however, leads to energy scaling $E(k) \approx k^{-2}$ and eddy-viscosity scaling $\nu \approx k^{-3/2}$.

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