Scattering of electromagnetic fields of any state of coherence from space-time fluctuations

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(Received 23 January 1989)

A theory is developed, valid within the accuracy of the first-order Born approximation, of the scattering of electromagnetic fields that applies under much more general conditions than do the current theories. The incident field is assumed to be statistically homogeneous and stationary, of arbitrary state of coherence, of arbitrary state of polarization, and to have arbitrary spectrum. The medium is assumed to be linear and spatially and temporally random, with the randomness being characterized by an ensemble that is statistically homogeneous, isotropic, and stationary. The analysis is not restricted to scattering from a medium in thermal equilibrium and applies in both off-resonance and near-resonance regions of the medium.

I. INTRODUCTION

As is well known, light scattering phenomena are of great importance in physics, astronomy, chemistry, meteorology, biology, and in other fields. Systematic studies of light scattering have spanned a long period of time, dating back to the pioneering researches of Tyndall and Rayleigh, made around 1870. Since then, other distinguished scientists, such as Smoluchowski, Einstein, Brillouin, Raman, and many others, have made substantial contributions to this field.[1] A new chapter in the development of the subject came with the development of the laser in 1960, which has produced light whose spectrum is exceedingly narrow and which is highly directional. These properties of laser light have made it possible, in recent years, to obtain important detailed information not previously accessible to experiment about the interaction of light with liquid, solids, and gases and about microscopic properties of matter.[2]

Much of the existing literature on the theory of light scattering is restricted to scattering of monochromatic light[3] and assumes that the density fluctuations of the scattering medium may be described by equilibrium thermodynamics. The first of these restrictions makes it impossible to explain some phenomena, such as the recently discovered frequency shifts of spectral lines that may be produced under appropriate circumstances by the scattering of partially coherent light from a spatially random medium;[4] and the assumption of thermodynamic equilibrium has in some cases led to disagreement of at least one order of magnitude between theory and experiment.[5]

In this paper we present a theory of scattering of electromagnetic waves, valid within the accuracy of the first-order Born approximation, that applies under much more general conditions than do the current theories. In particular, the incident field can be any statistically homogeneous and stationary field and be of arbitrary state of coherence, of arbitrary state of polarization, and can have arbitrary spectrum. The medium is assumed to be linear and spatially and temporally random, with the randomness being characterized by a statistical ensemble that is homogeneous, isotropic, and stationary. Our analysis is based on statistical continuum theory and is not restricted to scattering under equilibrium conditions. Moreover, the results apply in both off-resonance and near-resonance regions of the medium.[6] The central quantity of this theory is a generalized analogue of the Van Hove correlation function, well known in the theory of neutron scattering.

We illustrate the generality of our theory by showing that, when appropriate assumptions are made about the incident field and the scattering medium, our main formula for the spectrum of the scattered field yields several well-known results of conventional scattering theory. In the accompanying paper[7] we show that, under appropriate circumstances, our theory also predicts frequency shifts that depend on the state of coherence of the incident light and on the correlation properties of the fluctuating medium, a fact that may be of particular interest for astronomy.

II. DIELECTRIC RESPONSE OF A LINEAR, INHOMOGENEOUS, ISOTROPIC MEDIUM

It will be useful to begin by recalling some standard results relating to an electromagnetic field in a linear, inhomogeneous, isotropic, and nonmagnetic medium, whose macroscopic properties do not depend on time. In such a medium the Fourier transforms

$$\tilde{\mathbf{E}}(\mathbf{r},\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{E}(\mathbf{r},t)e^{i\omega t}dt\ , \tag{2.1a}$$

$$\tilde{\mathbf{P}}(\mathbf{r},\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{P}(\mathbf{r},t)e^{i\omega t}dt \tag{2.1b}$$

of the (real) electric field $\mathbf{E}(\mathbf{r},t)$ and the induced polarization $\mathbf{P}(\mathbf{r},t)$, respectively,[8] ($\mathbf{r}$ denotes a position vector of a field point and t denotes the time) are connected by the simple constitutive relation

$$\tilde{\mathbf{P}}(\mathbf{r},\omega)=\tilde{\eta}(\mathbf{r},\omega)\tilde{\mathbf{E}}(\mathbf{r},\omega)\ , \qquad (2.2)$$

where $\tilde{\eta}(\mathbf{r},\omega)$ is the dielectric susceptibility that describes the response of the medium at frequency ω.

It is known from microscopic theory that for many media $\tilde{\eta}(\mathbf{r},\omega)$ may be expressed in terms of the average number $N(\mathbf{r})$ of molecules per unit volume and the mean polarizability $\alpha(\omega)$ of each molecule by the formula

$$\tilde{\eta}(\mathbf{r},\omega)=\frac{N(\mathbf{r})\alpha(\omega)}{1-(4\pi/3)N(\mathbf{r})\alpha(\omega)}\ . \qquad (2.3)$$

Equation (2.3) is one form of the well-known Lorentz-Lorenz relation (Ref. 9, Sec. 2.3.3).

On taking the Fourier transform of Eq. (2.2) and making use of the fact that the response of the medium must necessarily be causal, we obtain the following constitutive relation valid in the space-time domain:

$$\mathbf{P}(\mathbf{r},t)=\frac{1}{2\pi}\int_0^\infty \eta(\mathbf{r},t')\mathbf{E}(\mathbf{r},t-t')dt'\ , \qquad (2.4)$$

where

$$\eta(\mathbf{r},t)=\int_{-\infty}^{\infty}\tilde{\eta}(\mathbf{r},\omega)e^{-i\omega t}d\omega\ . \qquad (2.5)$$

Suppose next that the macroscopic properties of the medium change in the course of time, in either a deterministic or a random way, but that the response is still linear and isotropic. Then, in place of Eq. (2.4) we have the more general causal, linear relationship[10]

$$\mathbf{P}(\mathbf{r},t)=\frac{1}{2\pi}\int_0^\infty \eta(\mathbf{r},t;t')\mathbf{E}(\mathbf{r},t-t')dt' \qquad (2.6)$$

that contains a generalized dielectric susceptibility function $\eta(\mathbf{r},t;t')$ which depends on two time arguments. Its dependence on the second temporal argument t' characterizes the response of the medium to sufficiently short pulses.

To obtain some insight into the behavior of the generalized dielectric susceptibility function let us take its Fourier transform with respect to its second argument:

$$\hat{\eta}(\mathbf{r},t;\omega')=\frac{1}{2\pi}\int_0^\infty \eta(\mathbf{r},t;t')e^{i\omega't'}dt'\ . \qquad (2.7)$$

If the response of the medium is time independent, $\hat{\eta}=\tilde{\eta}$, and for many media $\tilde{\eta}$ is related to the time-independent number density $N(\mathbf{r})$ of the molecules by Eq. (2.3). However, if the response of the medium changes in time, the number density also becomes a function of time, $N(\mathbf{r},t)$, say. If the temporal variations are not too rapid, we may expect that the following generalization of Eq. (2.3) will hold:

$$\hat{\eta}(\mathbf{r},t;\omega')=\frac{N(\mathbf{r},t)\alpha(\omega')}{1-\frac{4\pi}{3}N(\mathbf{r},t)\alpha(\omega')}\ . \qquad (2.8)$$

Although the precise range of validity of this equation can only be determined from detailed microscopic considerations, these remarks indicate the physical origin of the generalized dielectric susceptibility, which depends [via Eqs. (2.8) and (2.7)] on two, rather than one, temporal arguments.

When the effective frequencies of the electric field are not too close to any of the resonance frequencies of the medium, the constitutive relation (2.6) may be aproximated by a more familiar formula, as we will now show. For this purpose we first rewrite Eq. (2.6) in the following form that readily follows by the use of Fourier integral representations of $\eta(\mathbf{r},t;t')$ and $\mathbf{E}(\mathbf{r},t-t')$ in the integral of Eq. (2.6):

$$\mathbf{P}(\mathbf{r},t)=\int_{-\infty}^{\infty}\hat{\eta}(\mathbf{r},t;\omega')\tilde{\mathbf{E}}(\mathbf{r},\omega')e^{-i\omega' t}d\omega'\ . \qquad (2.9)$$

Suppose that $|\tilde{\mathbf{E}}(\mathbf{r},\omega)|$ is appreciable only in the neighborhood of frequencies $\pm\omega_0$. If these frequencies are not close to any of the resonance frequencies of the medium, the variation of $\hat{\eta}(\mathbf{r},t;\omega')$ with ω' in the neighborhood of $\omega'=\pm\omega_0$ $(\omega_0>0)$ may be neglected and Eq. (2.9) then gives

$$\mathbf{P}(\mathbf{r},t)\approx\hat{\eta}(\mathbf{r},t;+\omega_0)\int_0^\infty\tilde{\mathbf{E}}(\mathbf{r},\omega')e^{-i\omega' t}d\omega' + \hat{\eta}(\mathbf{r},t;-\omega_0)\int_{-\infty}^{0}\tilde{\mathbf{E}}(\mathbf{r},\omega')e^{-i\omega' t}d\omega'\ . \qquad (2.10)$$

Now, since $\eta(\mathbf{r},t;t')$ and $\mathbf{E}(\mathbf{r},t)$ are real, it follows at once from Eqs. (2.7) and (2.1a) that

$$\hat{\eta}(\mathbf{r},t;-\omega_0)=\hat{\eta}^*(\mathbf{r},t;\omega_0) \qquad (2.11)$$

and

$$\tilde{\mathbf{E}}(\mathbf{r},\omega')=\tilde{\mathbf{E}}^*(\mathbf{r},-\omega')\ . \qquad (2.12)$$

Hence, Eq. (2.10) may be rewritten as

$$\mathbf{P}(\mathbf{r},t)\approx 2\,\mathrm{Re}\left[\hat{\eta}(\mathbf{r},t;\omega_0)\int_0^\infty\tilde{\mathbf{E}}(\mathbf{r},\omega')e^{-i\omega' t}d\omega'\right]\ , \qquad (2.13)$$

where Re denotes the real part. Now, since we assumed that ω_0 is not too close to any of the resonance frequencies of the medium, the imaginary part of $\hat{\eta}(\mathbf{r},t;\omega_0)$ will be negligible and hence Eq. (2.13) may be rewritten as

$$\mathbf{P}(\mathbf{r},t)=\hat{\eta}(\mathbf{r},t;\omega_0)\left[2\,\mathrm{Re}\left[\int_0^\infty\tilde{\mathbf{E}}(\mathbf{r},\omega')e^{-i\omega' t}d\omega'\right]\right]\ , \qquad (2.14a)$$

or, using Eq. (2.12) again,

$$\mathbf{P}(\mathbf{r},t)=\hat{\eta}(\mathbf{r},t;\omega_0)\mathbf{E}(\mathbf{r},t)\ . \qquad (2.14b)$$

Frequently the constitutive relation of a time-dependent medium is written, without any justification, in the form $\mathbf{P}(\mathbf{r},t)=\eta(\mathbf{r},t)\mathbf{E}(\mathbf{r},t)$. Comparison of this formula with Eq. (2.14b) reveals the real significance and the approximate nature of the time-dependent response function $\eta(\mathbf{r},t)$. Moreover, our analysis indicates that, unlike Eq. (2.6), such a constitutive relation will not describe adequately the response of a medium under circumstances when the field contains frequencies that are in the resonance region of the medium.

III. DETERMINISTIC SCATTERING IN THE FIRST-ORDER BORN APPROXIMATION

In Sec. IV we will investigate the scattering of electromagnetic waves of arbitrary state of coherence from a medium which fluctuates both in space and in time. The incident and the scattered fields, and the dielectric susceptibility, will then be random functions of position and time. It will be useful, however, to derive first some general formulas that we will then need, pertaining to situations where the fields and the response of the medium are deterministic.

Let $\mathbf{E}^{(i)}(\mathbf{r},t)$ and $\mathbf{H}^{(i)}(\mathbf{r},t)$ denote the electric and the magnetic field vectors of a deterministic electromagnetic field incident on a deterministic medium that occupies a finite volume V in free space. We assume that the medium is of the kind considered in Sec. II. Its dielectric response is then characterized by formula (2.6). Further, let the scattered field vectors produced by the interaction of the incident field with the medium be denoted by $\mathbf{E}^{(s)}(\mathbf{r},t)$ and $\mathbf{H}^{(s)}(\mathbf{r},t)$. The total field (inside and outside the medium) may then be written as

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}^{(i)}(\mathbf{r},t)+\mathbf{E}^{(s)}(\mathbf{r},t) , \tag{3.1a}$$

$$\mathbf{H}(\mathbf{r},t)=\mathbf{H}^{(i)}(\mathbf{r},t)+\mathbf{H}^{(s)}(\mathbf{r},t) . \tag{3.1b}$$

It will satisfy Maxwell's equations

$$\nabla\times\mathbf{E}(\mathbf{r},t)=-\frac{1}{c}\frac{\partial\mathbf{H}(\mathbf{r},t)}{\partial t} , \tag{3.2}$$

$$\nabla\times\mathbf{H}(\mathbf{r},t)=\frac{1}{c}\frac{\partial\mathbf{E}(\mathbf{r},t)}{\partial t}+\frac{4\pi}{c}\frac{\partial\mathbf{P}(\mathbf{r},t)}{\partial t} , \tag{3.3}$$

$$\nabla\cdot\mathbf{E}(\mathbf{r},t)=-4\pi\nabla\cdot\mathbf{P}(\mathbf{r},t) , \tag{3.4}$$

$$\nabla\cdot\mathbf{H}(\mathbf{r},t)=0 , \tag{3.5}$$

where c is the speed of light *in vacuo*. If we make use of the fact that the incident field satisfies the source-free Maxwell equations [Eqs. (3.2)–(3.5) with $\mathbf{P}\equiv\mathbf{0}$], subtract these equations from Eqs. (3.2)–(3.5) and use Eqs. (2.6) and (3.1), we obtain the equations

$$\nabla\times\mathbf{E}^{(s)}(\mathbf{r},t)=-\frac{1}{c}\frac{\partial\mathbf{H}^{(s)}(\mathbf{r},t)}{\partial t} , \tag{3.6}$$

$$\nabla\times\mathbf{H}^{(s)}(\mathbf{r},t)=\frac{1}{c}\frac{\partial\mathbf{E}^{(s)}(\mathbf{r},t)}{\partial t}+\frac{4\pi}{c}\frac{\partial[\mathbf{P}_1(\mathbf{r},t)+\mathbf{P}_2(\mathbf{r},t)]}{\partial t} , \tag{3.7}$$

$$\nabla\cdot\mathbf{E}^{(s)}(\mathbf{r},t)=-4\pi\nabla\cdot[\mathbf{P}_1(\mathbf{r},t)+\mathbf{P}_2(\mathbf{r},t)] , \tag{3.8}$$

$$\nabla\cdot\mathbf{H}^{(s)}(\mathbf{r},t)=0 , \tag{3.9}$$

where

$$\mathbf{P}_1(\mathbf{r},t)=\frac{1}{2\pi}\int_0^\infty\eta(\mathbf{r},t;t')\mathbf{E}^{(i)}(\mathbf{r},t-t')dt' , \tag{3.10}$$

$$\mathbf{P}_2(\mathbf{r},t)=\frac{1}{2\pi}\int_0^\infty\eta(\mathbf{r},t;t')\mathbf{E}^{(s)}(\mathbf{r},t-t')dt' . \tag{3.11}$$

Suppose now that the scattering medium is weak in the sense that for all values of its arguments the polarization induced by the scattered field is much smaller than the polarization induced by the incident field, i.e., that

$$|\mathbf{P}_2(\mathbf{r},t)|\ll|\mathbf{P}_1(\mathbf{r},t)| , \tag{3.12}$$

and also that

$$\left|\frac{\partial\mathbf{P}_2(\mathbf{r},t)}{\partial t}\right|\ll\left|\frac{\partial\mathbf{P}_1(\mathbf{r},t)}{\partial t}\right| , \tag{3.13a}$$

$$|\nabla\cdot\mathbf{P}_2(\mathbf{r},t)|\ll|\nabla\cdot\mathbf{P}_1(\mathbf{r},t)| . \tag{3.13b}$$

Under these circumstances, Eqs. (3.6)–(3.9) give the following equations for the scattered field:

$$\nabla\times\mathbf{E}^{(s)}(\mathbf{r},t)=-\frac{1}{c}\frac{\partial\mathbf{H}^{(s)}(\mathbf{r},t)}{\partial t} , \tag{3.14}$$

$$\nabla\times\mathbf{H}^{(s)}(\mathbf{r},t)=\frac{1}{c}\frac{\partial\mathbf{E}^{(s)}(\mathbf{r},t)}{\partial t}+\frac{4\pi}{c}\frac{\partial\mathbf{P}_1(\mathbf{r},t)}{\partial t} , \tag{3.15}$$

$$\nabla\cdot\mathbf{E}^{(s)}(\mathbf{r},t)=-4\pi\nabla\cdot\mathbf{P}_1(\mathbf{r},t) , \tag{3.16}$$

$$\nabla\cdot\mathbf{H}^{(s)}(\mathbf{r},t)=0 , \tag{3.17}$$

where $\mathbf{P}_1(\mathbf{r},t)$ is given by Eq. (3.10). Equations (3.14)–(3.17), together with Eq. (3.10), characterize the behavior of the scattered electromagnetic field with the same kind of accuracy as the first-order Born approximation characterizes the scattered field in quantum collision theory.

The solutions of Eqs. (3.14)–(3.17), which behave as outgoing waves at infinity, can be expressed in the form (Ref. 9, Sec. 2.2.2)

$$\mathbf{E}^{(s)}(\mathbf{r},t)=\nabla\times\nabla\times\mathbf{\Pi}(\mathbf{r},t) , \tag{3.18a}$$

$$\mathbf{H}^{(s)}(\mathbf{r},t)=\frac{1}{c}\nabla\times\frac{\partial\mathbf{\Pi}(\mathbf{r},t)}{\partial t} , \tag{3.18b}$$

where $\mathbf{\Pi}(\mathbf{r},t)$ is the Hertz vector

$$\mathbf{\Pi}(\mathbf{r},t)=\int_V\frac{\mathbf{P}_1(\mathbf{r}',t-R/c)}{R}d^3r' , \tag{3.19}$$

with

$$R=|\mathbf{r}-\mathbf{r}'| . \tag{3.20}$$

For later purposes it will be more useful to consider the fields in the space-frequency domain, rather than in the space-time domain. We, therefore, introduce Fourier transforms, defined by formulas of the form

$$\tilde{f}(\mathbf{r},\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(\mathbf{r},t)e^{i\omega t}dt . \tag{3.21}$$

Equations (3.18) then imply that

$$\tilde{\mathbf{E}}^{(s)}(\mathbf{r},\omega)=\nabla\times\nabla\times\tilde{\mathbf{\Pi}}(\mathbf{r},\omega) , \tag{3.22a}$$

$$\tilde{\mathbf{H}}^{(s)}(\mathbf{r},\omega)=-ik\nabla\times\tilde{\mathbf{\Pi}}(\mathbf{r},\omega) , \tag{3.22b}$$

where

$$\tilde{\mathbf{\Pi}}(\mathbf{r},\omega)=\int_V\tilde{\mathbf{P}}_1(\mathbf{r}',\omega)\frac{e^{ikR}}{R}d^3r' \tag{3.23}$$

and

$$k=\omega/c . \tag{3.24}$$

We are mainly interested in the far field. Therefore, we will set

$$\mathbf{r}=r\mathbf{u}, \quad \mathbf{k}\equiv k\mathbf{u}=(\omega/c)\mathbf{u}, \quad |\mathbf{u}|=1 , \tag{3.25}$$

and consider the asymptotic behavior of Eqs. (3.22) as $kr\to\infty$, with $\mathbf{u}$ being kept fixed. By elementary geometry, we have, in this limit, $R\sim r-\mathbf{u}\cdot\mathbf{r}'$ and, with this approximation, the expression (3.23) becomes

$$\tilde{\mathbf{\Pi}}(r\mathbf{u},\omega)\sim(2\pi)^3\mathcal{P}_1(\mathbf{k},\omega)\frac{e^{ikr}}{r}, \quad kr\to\infty \tag{3.26}$$

where

$$\mathcal{P}_1(\mathbf{k},\omega)=\frac{1}{(2\pi)^3}\int_V \tilde{\mathbf{P}}_1(\mathbf{r}',\omega)e^{-i\mathbf{k}\cdot\mathbf{r}'}d^3r' , \tag{3.27a}$$

or, in terms of $\mathbf{P}_1(\mathbf{r},t)$ rather than $\tilde{\mathbf{P}}_1(\mathbf{r},\omega)$,

$$\mathcal{P}_1(\mathbf{k},\omega)=\frac{1}{(2\pi)^4}\int_V d^3r'e^{-i\mathbf{k}\cdot\mathbf{r}'}\int_{-\infty}^{\infty}\mathbf{P}_1(\mathbf{r}',t')e^{i\omega t'}dt' . \tag{3.27b}$$

With $\tilde{\mathbf{\Pi}}$ given by Eq. (3.26), Eqs. (3.22) may be shown to give the following expressions for the scattered field in the far zone:[11]

$$\tilde{\mathbf{E}}^{(s)}(r\mathbf{u},\omega)\sim-(2\pi)^3k^2\mathbf{u}\times[\mathbf{u}\times\mathcal{P}_1(\mathbf{k},\omega)]\frac{e^{ikr}}{r} , \tag{3.28a}$$

$$\tilde{\mathbf{H}}^{(s)}(r\mathbf{u},\omega)\sim(2\pi)^3k^2[\mathbf{u}\times\mathcal{P}_1(\mathbf{k},\omega)]\frac{e^{ikr}}{r} . \tag{3.28b}$$

For later purposes it will be useful to express the quantity $\mathcal{P}_1(\mathbf{k},\omega)$ in a different form. We find, after using the Fourier integral representations of $\eta(\mathbf{r},t;t')$ and $\mathbf{E}^{(i)}(\mathbf{r},t-t')$ in Eq. (3.10), that

$$\mathbf{P}_1(\mathbf{r},t)=\int_{-\infty}^{\infty}\hat{\eta}(\mathbf{r},t;\omega')\tilde{\mathbf{E}}^{(i)}(\mathbf{r},\omega')e^{-i\omega' t}d\omega' , \tag{3.29}$$

and hence, on taking the Fourier transform, that

$$\tilde{\mathbf{P}}_1(\mathbf{r},\omega)=\int_{-\infty}^{\infty}\bar{\eta}(\mathbf{r},\omega-\omega';\omega')\tilde{\mathbf{E}}^{(i)}(\mathbf{r},\omega')d\omega' , \tag{3.30}$$

where

$$\bar{\eta}(\mathbf{r},\omega;\omega')=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\eta}(\mathbf{r},t;\omega')e^{i\omega t}dt . \tag{3.31}$$

On substituting from Eq. (3.30) into Eq. (3.27a) we obtain the following expression for the vector function $\mathcal{P}_1(\mathbf{k},\omega)$ that enters the expression (3.28) for the scattered field in the far zone, within the accuracy of the first-order Born approximation:

$$\mathcal{P}_1(\mathbf{k},\omega)=\frac{1}{(2\pi)^3}\int_V d^3r'e^{-i\mathbf{k}\cdot\mathbf{r}'}\times\int_{-\infty}^{\infty}\bar{\eta}(\mathbf{r}',\omega-\omega';\omega')\tilde{\mathbf{E}}^{(i)}(\mathbf{r}',\omega')d\omega' . \tag{3.32}$$

IV. SCATTERING FROM SPACE-TIME FLUCTUATIONS IN THE FIRST-ORDER BORN APPROXIMATION

We will now consider the problem of the scattering of a randomly fluctuating electromagnetic field from a randomly fluctuating medium. We will retain the assumptions that the medium is linear, isotropic, and nonmagnetic, but we will allow its generalized dielectric susceptibility function $\hat{\eta}(\mathbf{r},t;\omega')$ [cf. Eqs. (2.6) and (2.7)] to be, at each frequency ω', a random function of both position ($\mathbf{r}$) and time (t). More specifically we will assume the following.

(i) The dielectric susceptibility function is statistically homogeneous and stationary, at least in the wide sense,[12] and of approximately zero mean. Then the correlation function $\langle\hat{\eta}^*(\mathbf{r}_1,t_1;\omega')\hat{\eta}(\mathbf{r}_2,t_2,\omega')\rangle$, where the angular brackets denote average over an ensemble of the random medium, depends on $\mathbf{r}_1$ and $\mathbf{r}_2$ and on t_1 and t_2 only through the differences $\mathbf{r}_2-\mathbf{r}_1$ and t_2-t_1, respectively, i.e., it has the form

$$\langle\hat{\eta}^*(\mathbf{r}_1,t_1;\omega')\hat{\eta}(\mathbf{r}_2,t_2;\omega')\rangle=G(\mathbf{r}_2-\mathbf{r}_1,t_2-t_1;\omega') . \tag{4.1}$$

(ii) The spatial extent of the susceptibility correlation is small compared to the size of the scattering volume V, i.e., the separation distances $R=|\mathbf{R}|=|\mathbf{r}_2-\mathbf{r}_1|$ for which $|G(\mathbf{R},T;\omega')|$ has appreciable values are much smaller than the linear dimensions of V.

(iii) The scattering is so weak that, to a good approximation, it may be described within the framework of the first-order Born approximation.

As regards the incident electric field we will assume that it is statistically homogeneous and stationary, at least in the wide sense,[12] which implies that the correlation tensor $\langle E_l^{(i)}(\mathbf{r}_1,t_1)E_m^{(i)}(\mathbf{r}_2,t_2)\rangle$ (the subscripts l and m labeling Cartesian components) is a function of the differences $\mathbf{r}_2-\mathbf{r}_1$ and of t_2-t_1 only. Hence the correlation tensor of the incident electric field has the form

$$\langle E_l^{(i)}(\mathbf{r}_1,t_1)E_m^{(i)}(\mathbf{r}_2,t_2)\rangle=\mathcal{E}_{lm}^{(i)}(\mathbf{r}_2-\mathbf{r}_1,t_2-t_1) . \tag{4.2}$$

Here the angular brackets denote the average taken over the ensemble that characterizes the statistical properties of the incident field. We will also need the cross-spectral density tensor $W_{lm}^{(i)}(\mathbf{r}_2-\mathbf{r}_1,\omega)$ of the incident electric field. It may be defined formally by the equation

$$\langle[\tilde{E}_l^{(i)}(\mathbf{r}_1,\omega)]^*\tilde{E}_m^{(i)}(\mathbf{r}_2,\omega')\rangle=\mathbf{W}_{lm}^{(i)}(\mathbf{r}_2-\mathbf{r}_1,\omega)\delta(\omega-\omega') , \tag{4.3}$$

where $\tilde{E}_l^{(i)}$ and $\tilde{E}_m^{(i)}$ are the Fourier transforms of $E_l^{(i)}$ and $E_m^{(i)}$, respectively,[13] and δ is the Dirac delta functon. According to the Wiener-Khintchine theorem[14] the electric cross-spectral density tensor is the Fourier transform of the electric correlation tensor, i.e.,

$$W_{lm}^{(i)}(\mathbf{R},\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{E}_{lm}^{(i)}(\mathbf{R},T)e^{i\omega T}dT . \tag{4.4}$$

We stress that the expectation values in Eqs. (4.1) and (4.2) [or (4.3)] are taken over two different ensembles. In Eq. (4.1) it is taken over the ensemble that characterizes the fluctuations of the dielectric susceptibility, whereas in Eq. (4.2) it is taken over the ensemble that characterizes the fluctuations of the incident field. We will assume that these two kinds of fluctuations are statistically independent. The assumption is evidently reasonable if the incident field is not exceptionally strong.

We will now derive expressions for the angular and spectral distribution of energy of the scattered field in the far zone. For this purpose we first note that the quantity $\mathcal{P}_1(\mathbf{k},\omega)$, given by Eq. (3.32), is now a random variable because both $\tilde{\eta}$ and $\tilde{\mathbf{E}}^{(i)}$ are themselves random variables. Hence the (generalized) Fourier transforms $\tilde{\mathbf{E}}^{(s)}$ and $\tilde{\mathbf{H}}^{(s)}$ of the scattered field in the far zone, given by Eqs. (3.28), are also random variables. For each realization we have from Eq. (3.28a),

$$[\tilde{\mathbf{E}}^{(s)}(r\mathbf{u},\omega)]^*\cdot\tilde{\mathbf{E}}^{(s)}(r\mathbf{u},\omega')=(2\pi)^6\frac{k^2k'^2}{r^2}e^{i(k'-k)r}\{\mathbf{u}\times[\mathbf{u}\times\mathcal{P}_1^*(\mathbf{k},\omega)]\}\cdot\{\mathbf{u}\times[\mathbf{u}\times\mathcal{P}_1(\mathbf{k}',\omega')]\}\ , \tag{4.5}$$

where

$$\mathbf{k}=k\mathbf{u}=\frac{\omega}{c}\mathbf{u},\quad \mathbf{k}'=k'\mathbf{u}=\frac{\omega'}{c}\mathbf{u}\ . \tag{4.6}$$

It is shown in Appendix A that the right-hand side of Eq. (4.5) may be simplified by the use of an elementary vector identity. Equation (4.5) then reduces to

$$[\tilde{\mathbf{E}}^{(s)}(r\mathbf{u},\omega)]^*\cdot\tilde{\mathbf{E}}^{(s)}(r\mathbf{u},\omega')=(2\pi)^6\frac{k^2k'^2}{r^2}e^{i(k'-k)r}\times(\delta_{lm}-u_lu_m)\mathcal{P}_{1l}^*(\mathbf{k},\omega)\mathcal{P}_{1m}(\mathbf{k}',\omega')\ , \tag{4.7}$$

where δ_{lm} denotes the Kronecker symbol, and summation over repeated suffices is implied.

Let us now take averages of Eq. (4.7) over the ensembles of the incident field and of the fluctuating medium. Denoting this double average by double angular brackets, we have at once from Eq. (4.7) that

$$\langle\langle[\tilde{\mathbf{E}}^{(s)}(r\mathbf{u},\omega]^*\cdot\tilde{\mathbf{E}}^{(s)}(r\mathbf{u},\omega')\rangle\rangle=(2\pi)^6\frac{k^2k'^2}{r^2}e^{i(k'-k)r}(\delta_{lm}-u_lu_m)\times\langle\langle\mathcal{P}_{1l}^*(\mathbf{k},\omega)\mathcal{P}_{1m}(\mathbf{k}',\omega')\rangle\rangle\ . \tag{4.8}$$

We show in Appendix B that under the assumptions stated at the beginning of this section the double average on the right-hand side of Eq. (4.8) is given by

$$\langle\langle\mathcal{P}_{1l}^*(\mathbf{k},\omega)\mathcal{P}_{1m}(\mathbf{k}',\omega')\rangle\rangle=\frac{V\delta(\omega-\omega')}{(2\pi)^6}\int_V d^3R e^{-i\mathbf{k}\cdot\mathbf{R}}\times\int_{-\infty}^{\infty}\bar{G}(\mathbf{R},\omega-\omega_1;\omega_1)\times W_{lm}^{(i)}(\mathbf{R},\omega_1)d\omega_1\ , \tag{4.9}$$

where $\bar{G}$ is the Fourier transform, defined by the formula

$$\bar{G}(\mathbf{R},\Omega;\omega')=\frac{1}{2\pi}\int_{-\infty}^{\infty}G(\mathbf{R},T;\omega')e^{i\Omega T}dT\ , \tag{4.10}$$

of the correlation function (4.1) of the generalized susceptibility function $\hat{\eta}(\mathbf{r},t;\omega')$ of the scattering medium.

Finally, on substituting from Eq. (4.9) into Eq. (4.8) we find that

$$\langle\langle[\tilde{\mathbf{E}}^{(s)}(r\mathbf{u},\omega)]^*\cdot\tilde{\mathbf{E}}^{(s)}(r\mathbf{u},\omega')\rangle\rangle=S^{(s)}(r\mathbf{u},\omega)\delta(\omega-\omega')\ , \tag{4.11}$$

where

$$S^{(s)}(r\mathbf{u},\omega)=\frac{Vk^4}{r^2}(\delta_{lm}-u_lu_m)\int_{-\infty}^{\infty}d\omega_1\int_V\bar{G}(\mathbf{R},\omega-\omega_1;\omega_1)W_{lm}^{(i)}(\mathbf{R},\omega_1)e^{-i\mathbf{k}\cdot\mathbf{R}}d^3R\ , \tag{4.12}$$

with k and $\mathbf{k}$ being defined in Eq. (4.6).

It is clear from (4.11) that the function $S^{(s)}(r\mathbf{u},\omega)$ is proportional to the expectation value of the electric energy density at frequency ω, of the scattered electric field at a typical point $r\mathbf{u}$ in the far zone. According to well-known properties of the far field [cf. Ref. 11, Eqs. (4.10) and (4.11)] it is also proportional to the expectation values of the magnetic energy density and of the magnitude of the Poynting vector at frequency ω in the far zone. We will therefore refer to $S^{(s)}(r\mathbf{u},\omega)$ as the *spectral density (spectrum) of the scattered field.*

Formula (4.12) is the main result of this investigation. It expresses the spectral density of the scattered field throughout the far zone as a linear transform of the cross-spectral density tensor $W_{lm}^{(i)}$ [defined by Eq. (4.3)] of the fluctuating incident field. The kernel of the transform is, apart from a simple geometrical factor, the Fourier transform [Eq. (4.10)] of the two-point correlation function G [defined by Eq. (4.1)] of the generalized dielectric susceptibility of the scattering medium. This two-point correlation function is somewhat analogous to the well-known Van Hove time-dependent two-particle correlation function[15] (known also as the pair distribution function), frequently employed in the theory of neutron scattering. We will now specialize formula (4.12) to some special cases of practical interest.

V. SOME SPECIAL CASES

A. Plane, polychromatic, linearly polarized incident wave

Suppose that the incident field is a fluctuating polychromatic plane wave, propagating in a direction specified by a real unit vector $\mathbf{u}_0$, with its electric vector linearly polarized along a direction specified by a unit vector $\mathbf{e}_0$ ($\mathbf{u}_0\cdot\mathbf{e}_0=0$). Then each realization of the incident electric field may be represented in the form

$$\mathbf{E}^{(i)}(\mathbf{r},t)=\mathbf{e}_0\int_{-\infty}^{\infty}A(\omega)e^{i(k\mathbf{u}_0\cdot\mathbf{r}-\omega t)}d\omega\ , \tag{5.1}$$

where $A(\omega)$ is, for each frequency ω, a random variable.[13]

The Fourier transform of $\mathbf{E}^{(i)}(\mathbf{r},t)$ evidently is

$$\tilde{\mathbf{E}}^{(i)}(\mathbf{r},\omega)=A(\omega)e^{ik\mathbf{u}_0\cdot\mathbf{r}}\mathbf{e}_0 \ . \tag{5.2}$$

Hence, Eq. (4.3) gives, in this case,

$$\langle A^*(\omega)A(\omega')\rangle e^{i\mathbf{u}_0\cdot(k'\mathbf{r}_2-k\mathbf{r}_1)}e_{0l}e_{0m} = W_{lm}^{(i)}(\mathbf{r}_2-\mathbf{r}_1,\omega)\delta(\omega-\omega') \ . \tag{5.3}$$

Now the spectral density $S^{(i)}(\omega)$, say, of the incident field is just the trace of $W_{lm}^{(i)}(0,\omega)$, and hence we have from Eq. (5.3)

$$\langle A^*(\omega)A(\omega')\rangle = S^{(i)}(\omega)\delta(\omega-\omega') \ . \tag{5.4}$$

It follows from Eqs. (5.3) and (5.4) that

$$W_{lm}^{(i)}(\mathbf{R},\omega)=S^{(i)}(\omega)e^{ik\mathbf{u}_0\cdot\mathbf{R}}e_{0l}e_{0m} \ . \tag{5.5}$$

On substituting from Eq. (5.5) into Eq. (4.12) we obtain for the expectation value of the spectral density of the scattered field the expression

$$S^{(s)}(r\mathbf{u},\omega)=\frac{Vk^4\sin^2\psi}{r^2}\int_{-\infty}^{\infty}d\omega'\int_V \bar{G}(\mathbf{R},\omega-\omega';\omega')S^{(i)}(\omega')e^{-i(\mathbf{k}-\mathbf{k}_0')\cdot\mathbf{R}}d^3R \ , \tag{5.6}$$

where

$$\mathbf{k}=k\mathbf{u}=\frac{\omega}{c}\mathbf{u} \tag{5.7a}$$

is the wave vector of the ω component of the scattered field,

$$\mathbf{k}_0'=k'\mathbf{u}_0=\frac{\omega'}{c}\mathbf{u}_0 \tag{5.7b}$$

is the wave vector of the ω' component of the incident field, and ψ is the angle between the direction of observation ($\mathbf{u}$) and the direction of polarization ($\mathbf{e}_0$) of the incident electric field, i.e., $\cos\psi=\mathbf{u}\cdot\mathbf{e}_0$. We have also made use here of the identity

$$(\delta_{lm}-u_lu_m)e_{0l}e_{0m}=1-(\mathbf{u}\cdot\mathbf{e}_0)^2=1-\cos^2\psi=\sin^2\psi \ . \tag{5.8}$$

If we define a function $\mathcal{S}(\mathbf{K},\Omega;\omega')$ by the formula

$$\mathcal{S}(\mathbf{K},\Omega;\omega')\equiv\frac{1}{(2\pi)^3}\int_V \bar{G}(\mathbf{R},\Omega;\omega')e^{-i\mathbf{K}\cdot\mathbf{R}}d^3R \ , \tag{5.9}$$

Eq. (5.6) reduces to

$$S^{(s)}(r\mathbf{u},\omega)=\frac{(2\pi)^3Vk^4\sin^2\psi}{r^2}\times\int_{-\infty}^{\infty}\mathcal{S}(\mathbf{k}-\mathbf{k}_0',\omega-\omega';\omega')S^{(i)}(\omega')d\omega' \ . \tag{5.10}$$

The function $\mathcal{S}(\mathbf{K},\Omega;\omega')$, introduced by the formula (5.9), has a simple meaning. If in Eq. (5.9) we substitute for $\bar{G}$ from Eq. (4.10) we see at once that

$$\mathcal{S}(\mathbf{K},\Omega;\omega')=\frac{1}{(2\pi)^4}\int_V d^3R\int_{-\infty}^{\infty}G(\mathbf{R},T;\omega')\times e^{-i(\mathbf{K}\cdot\mathbf{R}-\Omega T)}dT \ , \tag{5.11a}$$

or, more explicitly, using Eq. (4.1)

$$\mathcal{S}(\mathbf{K},\Omega;\omega')=\frac{1}{(2\pi)^4}\int_V d^3R \times\int_{-\infty}^{\infty}\langle\hat{\eta}^*(\mathbf{r},t;\omega')\hat{\eta}(\mathbf{r}+\mathbf{R},t+T;\omega')\rangle \times e^{-i(\mathbf{K}\cdot\mathbf{R}-\Omega T)}dT \ , \tag{5.11b}$$

i.e., it is the space-time Fourier transform of the two-point correlation function of the dielectric susceptibility of the scattering medium. We may thus regard the function $\mathcal{S}(\mathbf{K},\Omega;\omega')$ as the *generalized structure function* of the medium.[16] Hence we see from Eq. (5.10) that when the incident field is a linearly polarized polychromatic plane wave, the spectral density of the scattered field is equal, apart from simple geometrical factors, to a "weighted integral" taken over the spectrum of the incident field, the weighting factor being the generalized structure function of the medium.

B. Plane, monochromatic, linearly polarized incident wave

Suppose that the incident field is again a linearly polarized plane wave, but that it is monochromatic. Then the spectrum $S^{(i)}(\omega)$ has the form

$$S^{(i)}(\omega)=\tfrac{1}{2}I_0[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)] \ , \tag{5.12}$$

where ω_0 and I_0 are positive constants. The formula (5.10) now reduces to

$$S^{(s)}(r\mathbf{u},\omega)=\frac{(2\pi)^3I_0Vk^4\sin^2\psi}{2r^2}\times[\mathcal{S}(\mathbf{k}-\mathbf{k}_0,\omega-\omega_0;\omega_0)+\mathcal{S}(\mathbf{k}+\mathbf{k}_0,\omega+\omega_0;-\omega_0)] \ , \tag{5.13}$$

where

$$\mathbf{k}_0=\frac{\omega_0}{c}\mathbf{u}_0 \ . \tag{5.14}$$

If the fluctuations of the dielectric susceptibility are slow compared to the optical period $2\pi/\omega_0$, then for $\omega>0$,

$$\mathcal{S}(\mathbf{k}+\mathbf{k}_0,\omega+\omega_0;-\omega_0)\cong 0 \ , \tag{5.15}$$

and Eq. (5.13) reduces to

$$S^{(s)}(r\mathbf{u},\omega)\approx\frac{(2\pi)^3 I_0 V k^4 \sin^2\psi}{2r^2}\mathcal{S}(\mathbf{k}-\mathbf{k}_0,\omega-\omega_0;\omega_0) \ . \tag{5.16}$$

This formula is essentially a well-known expression of classical scattering theory for the intensity of the scattered light.[17–19]

C. Static limit

Finally, let us consider the special case when the physical properties of the scattering medium do not change in the course of time but they still change randomly with position. $\hat{\eta}(\mathbf{r},t;\omega')$ will then be independent of t [in which case we will write $\tilde{\eta}(\mathbf{r},\omega')$ in place of $\hat{\eta}(\mathbf{r},t;\omega')$ as we did before—cf. Eq. (2.2)]. The correlation function $G(\mathbf{R},T;\omega')$ defined by Eq. (4.1), will then be independent of the temporal argument and we will denote it by $g(\mathbf{R},\omega')$. Instead of Eq. (4.1) we now have[20]

$$\langle \tilde{\eta}^*(\mathbf{r}_1,\omega')\tilde{\eta}(\mathbf{r}_2,\omega')\rangle = g(\mathbf{r}_2-\mathbf{r}_1,\omega') \ , \tag{5.17}$$

and Eq. (4.10) gives

$$\bar{G}(\mathbf{R},\Omega;\omega')=g(\mathbf{R},\omega')\delta(\Omega) \ . \tag{5.18}$$

On substituting from this equation into the general formula (4.12) we obtain the following expression for the spectral density of the scattered field in the static limit:

$$S^{(s)}(r\mathbf{u},\omega)=\frac{Vk^4}{r^2}(\delta_{lm}-u_l u_m) \times \int_V g(\mathbf{R},\omega)W_{lm}^{(i)}(\mathbf{R},\omega)e^{-i\mathbf{k}\cdot\mathbf{R}}d^3R \ . \tag{5.19}$$

Two special cases of this formula are of particular interest. When the incident field is a linearly polarized polychromatic plane wave [Eq. (5.1)], the cross-spectral density tensor of the incident field is given by Eq. (5.5), and, if we also make use of the identity (5.8), Eq. (5.19) reduces to

$$S^{(s)}(r\mathbf{u},\omega)=\frac{(2\pi)^3 V k^4 \sin^2\psi}{r^2}\tilde{g}(\mathbf{k}-\mathbf{k}_0,\omega)S^{(i)}(\omega) \ , \tag{5.20}$$

where, as before, $\mathbf{k}=(\omega/c)\mathbf{u}$, $\mathbf{k}_0=(\omega/c)\mathbf{u}_0$, and $\tilde{g}(\mathbf{k},\omega)$ is the three-dimensional Fourier transform of the function $g(\mathbf{R},\omega)$, i.e.,

$$\tilde{g}(\mathbf{K},\omega)=\frac{1}{(2\pi)^3}\int_V g(\mathbf{R},\omega)e^{-i\mathbf{K}\cdot\mathbf{R}}d^3R \ , \tag{5.21}$$

or, more explicitly, using Eq. (5.17),

$$\tilde{g}(\mathbf{K},\omega)=\frac{1}{(2\pi)^3}\int_V \langle \tilde{\eta}^*(\mathbf{r},\omega')\tilde{\eta}(\mathbf{r}+\mathbf{R},\omega')\rangle e^{-i\mathbf{K}\cdot\mathbf{R}}d^3R \ . \tag{5.22}$$

The formula (5.20) is the electromagnetic analogue of a formula for scalar scattering, derived not long ago[4] in a different manner.

Finally, let us suppose that the field incident on the scatterer is monochromatic. Then $S^{(i)}(\omega)$ is given by Eq. (5.12) and Eq. (5.20) reduces, when $\omega>0$, to

$$S^{(s)}(r\mathbf{u},\omega)=\frac{(2\pi)^3 I_0 V k^4 \sin^2\psi}{2r^2}\tilde{g}(\mathbf{k}-\mathbf{k}_0,\omega)\delta(\omega-\omega_0) \ . \tag{5.23}$$

A formula of this kind was first derived by Einstein[21] in a well-known investigation that was the starting point of the statistical theory of light scattering.

VI. CONCLUDING REMARKS

We have developed, in this paper, a statistical continuum theory of scattering of electromagnetic fields, valid within the accuracy of the first-order Born approximation. The theory has a much wider range of validity than those that are currently available. In particular the incident field may be of any state of coherence and polarization and have arbitrary spectrum, provided only that it is statistically stationary and homogeneous. The medium is assumed to be linear, statistically homogeneous, isotropic, and nonmagnetic, and of linear dimensions that are large compared with the spatial correlation lengths over which the random variation of its physical properties are correlated at each effective frequency contained in the spectrum of the incident field. No assumption is made regarding the thermodynamic state of the scattering medium.

The response of the scattering medium is described by a generalized "two-time" dielectric susceptibility function, which takes into account the effects of both its spatial and its temporal variations. Our analysis elucidates, as a by-product, the physical significance and the approximate nature of the "one-time" response function that is employed in the usual theories.

Our main formula [Eq. (4.12)] expresses the spectrum of the scattered field as a linear transform of the cross-spectral density tensor of the fluctuating incident field. The kernel of the transform is, apart from a simple geometrical factor, a Fourier transform of the two-point correlation function of the generalized dielectric susceptibility of the scattering medium. We show that many of the well-known formulas of the usual scattering theories readily follow from it as special or limiting cases. In particular, Eq. (4.12) yields a well-known expression derived by Einstein in a classic paper that was the starting point of the statistical theory of light scattering, as well as various formulas that are frequently in the analysis of modern scattering experiments with laser light.

In general, the interaction of an electromagnetic field with a random medium produces changes in the spectrum of the field. In an accompanying paper[7] we show on the basis of the present theory that the modification may be such as to produce frequency shifts of spectral lines.

ACKNOWLEDGMENTS

This research was supported by the National Science Foundation and by the U.S. Army Research Office.

APPENDIX A: A VECTOR IDENTITY USED IN THE DERIVATION OF EQ. (4.7)

We start with the vector identity

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}) \ . \tag{A1}$$

It follows at once from this identity that

$$\begin{aligned}\{\mathbf{u}\times[\mathbf{u}\times\mathcal{P}_1^*(\mathbf{k},\omega)]\}\cdot\{\mathbf{u}\times[\mathbf{u}\times\mathcal{P}_1(\mathbf{k}',\omega')]\}&=(\mathbf{u}\cdot\mathbf{u})[\mathbf{u}\times\mathcal{P}_1^*(\mathbf{k},\omega)]\cdot[\mathbf{u}\times\mathcal{P}_1(\mathbf{k}',\omega')]\\&\quad-\mathbf{u}\cdot[\mathbf{u}\times\mathcal{P}_1(\mathbf{k}',\omega')]\{[\mathbf{u}\times\mathcal{P}_1^*(\mathbf{k},\omega)]\cdot\mathbf{u}\}\\&=[\mathbf{u}\times\mathcal{P}_1^*(\mathbf{k},\omega)]\cdot[\mathbf{u}\times\mathcal{P}_1(\mathbf{k}',\omega')] \ .\end{aligned} \tag{A2}$$

On using the identity (A1) once again we obtain from Eq. (A2) the identities

$$\begin{aligned}\{\mathbf{u}\times[\mathbf{u}\times\mathcal{P}_1^*(\mathbf{k},\omega)]\}\cdot\{\mathbf{u}\times[\mathbf{u}\times\mathcal{P}_1(\mathbf{k}',\omega')]\}&=\mathcal{P}_1^*(\mathbf{k},\omega)\cdot\mathcal{P}_1(\mathbf{k}',\omega')-[\mathbf{u}\cdot\mathcal{P}_1^*(\mathbf{k},\omega)][\mathbf{u}\cdot\mathcal{P}_1(\mathbf{k}',\omega')]\\&=(\delta_{lm}-u_lu_m)\mathcal{P}_{1l}^*(\mathbf{k},\omega)\mathcal{P}_{1m}(\mathbf{k}',\omega') \ ,\end{aligned} \tag{A3}$$

where $\mathcal{P}_{1l}$ and $\mathcal{P}_{1m}$ denote the lth and the mth component, respectively, of $\mathcal{P}_1$, δ_{lm} denotes the Kronecker symbol, and summation over repeated suffixes is implied.

APPENDIX B: DERIVATION OF FORMULA (4.9)

It follows from Eq. (3.32) that

$$\langle\langle\mathcal{P}_{1l}^*(\mathbf{k},\omega)\mathcal{P}_{1m}(\mathbf{k}',\omega')\rangle\rangle=\frac{1}{(2\pi)^6}\int_V d^3r_1\int_V d^3r_2\int_{-\infty}^{\infty}d\omega_1\int_{-\infty}^{\infty}d\omega_2\, e^{i(\mathbf{k}\cdot\mathbf{r}_1-\mathbf{k}'\cdot\mathbf{r}_2)}\times\langle\tilde{\eta}^*(\mathbf{r}_1,\omega-\omega_1;\omega_1)\tilde{\eta}(\mathbf{r}_2,\omega'-\omega_2;\omega_2)\rangle\times\langle[\tilde{E}_l^{(i)}(\mathbf{r}_1,\omega_1)]^*\tilde{E}_m^{(i)}(\mathbf{r}_2,\omega_2)\rangle \ , \tag{B1}$$

where we have made use of the assumption that the fluctuations of the medium and of the incident field are statistically independent. Now according to Eq. (4.3) the second expectation value that occurs on the right-hand side of Eq. (B1) is given by

$$\langle[\tilde{E}_l^{(i)}(\mathbf{r}_1,\omega_1)]^*\tilde{E}_m^{(i)}(\mathbf{r}_2,\omega_2)\rangle=W_{lm}^{(i)}(\mathbf{r}_2-\mathbf{r}_1,\omega_1)\delta(\omega_2-\omega_1) \ . \tag{B2}$$

On substituting from Eq. (B2) into Eq. (B1) and on carrying out the trivial integration with respect to ω_2, we find that

$$\langle\langle\mathcal{P}_{1l}^*(\mathbf{k},\omega)\mathcal{P}_{1m}(\mathbf{k}',\omega')\rangle\rangle=\frac{1}{(2\pi)^6}\int_V d^3r_1\int_V d^3r_2\int_{-\infty}^{\infty}d\omega_1\, e^{i(\mathbf{k}\cdot\mathbf{r}_1-\mathbf{k}'\cdot\mathbf{r}_2)}\times\langle\tilde{\eta}^*(\mathbf{r}_1,\omega-\omega_1;\omega_1)\tilde{\eta}(\mathbf{r}_2,\omega'-\omega_1;\omega_1\rangle\times W_{lm}^{(i)}(\mathbf{r}_2-\mathbf{r}_1,\omega_1) \ . \tag{B3}$$

Now the expectation value on the right-hand side of Eq. (B3), which involves the dielectric susceptibility, may be expressed in a simpler form. We find from Eqs. (3.31), (4.1), and the Wiener-Khintchine theorem that

$$\langle\tilde{\eta}^*(\mathbf{r}_1,\Omega;\omega')\tilde{\eta}(\mathbf{r}_2,\Omega';\omega')\rangle=\overline{G}(\mathbf{r}_2-\mathbf{r}_1,\Omega;\omega')\delta(\Omega-\Omega') \ , \tag{B4}$$

where

$$\overline{G}(\mathbf{R},\Omega,\omega')=\frac{1}{2\pi}\int_{-\infty}^{\infty}G(\mathbf{R},T;\omega')e^{i\Omega T}dT \ , \tag{B5}$$

$G(\mathbf{R},T;\omega')$ being the correlation function defined by Eq. (4.1), viz.,

$$G(\mathbf{R},T;\omega')=\langle\hat{\eta}^*(\mathbf{r},t;\omega')\hat{\eta}(\mathbf{r}+\mathbf{R},t+T;\omega')\rangle \ . \tag{B6}$$

On substituting from Eq. (B4) into Eq. (B3) we obtain the formula

$$\langle\langle\mathcal{P}_{1l}^*(\mathbf{k},\omega)\mathcal{P}_{1m}(\mathbf{k}',\omega')\rangle\rangle=\frac{\delta(\omega-\omega')}{(2\pi)^6}\int_V d^3r_1\int_V d^3r_2\, e^{-i\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)}\times\int_{-\infty}^{\infty}d\omega_1\,\overline{G}(\mathbf{r}_2-\mathbf{r}_1,\omega-\omega_1;\omega_1)W_{lm}^{(i)}(\mathbf{r}_2-\mathbf{r}_1,\omega_1) \ . \tag{B7}$$

Because of our assumption that the spatial correlation length of the dielectric susceptibility fluctuations is small compared with the linear dimensions of the scattering volume, the above expression can be readily shown to reduce to

$$\langle\langle\mathcal{P}_{1l}^*(\mathbf{k},\omega)\mathcal{P}_{1m}(\mathbf{k}',\omega')\rangle\rangle=\frac{V\delta(\omega-\omega')}{(2\pi)^6}\int_V d^3R\, e^{-i\mathbf{k}\cdot\mathbf{R}}\int_{-\infty}^{\infty}d\omega_1\,\overline{G}(\mathbf{R},\omega-\omega_1;\omega_1)W_{lm}^{(i)}(\mathbf{R},\omega_1) \ , \tag{B8}$$

where V is the volume occupied by the scattering medium.

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[1]For reviews of some of these contributions, see, for example, L. Fabelinskii, *Molecular Scattering of Light* (Plenum, New York, 1968); G. B. Benedek, in *Brandeis University Summer Institute on Theoretical Physics, Boston, 1966,* edited by M. Chretien, E. P. Gross, and S. Desser (Gordon and Breach, New York, 1968), Vol. 2, pp. 5–98; B. Crosignani, P. Di Porto, and M. Bertolotti, *Statistical Properties of Scattered Light* (Academic, New York, 1975); B. J. Berne and R. Pecora, *Dynamic Light Scattering* (Wiley, New York, 1976).

[2]See, for example, B. Chu, *Laser Light Scattering* (Academic, New York, 1974); or B. J. Berne and R. Pecora, cited in Ref. 1.

[3]Notable exceptions are (a) L. Mandel, Phys. Rev. **181**, 75 (1969); (b) P. N. Pusey, in *Photon Correlation Spectroscopy and Velocimetry,* edited by H. A. Cummins and E. R. Pike (Plenum, New York, 1977), pp. 45–141.

[4]E. Wolf, J. T. Foley, and F. Gori, J. Opt. Soc. Am. A (to be published).

[5]See, for example, J. Schroder, in *Treatise on Material Science and Technology* (Academic, New York, 1977), Vol. 12, p. 158, and references given therein.

[6]This theory will, however, not apply when the effective frequencies of the incident field are very close to any of the resonance frequencies of the medium because the first-order Born approximation will then no longer adequately describe the scattering process.

[7]J. T. Foley and E. Wolf, following paper, Phys. Rev. A **40**, 588 (1989).

[8]It is customary in optical coherence theory to represent the real fields (such as the electric field **E** or the polarization **P**) by so-called analytic signals (see, for example, Ref. 9, Sec. 10.2). We do *not* adopt this procedure here.

[9]M. Born and W. Wolf, *Principles of Optics,* 6th ed. (Pergamon, Oxford, England, 1980).

[10]Cf. L. Mandel and E. Wolf, Opt. Commun. **8**, 95 (1973).

[11]A. T. Friberg and E. Wolf, J. Opt. Soc. Am. **73**, 26 (1983), Eq. (4.5).

[12]W. B. Davenport and W. L. Root, *Random Signals and Noise* (McGraw-Hill, New York, 1958), p. 60.

[13]These quantities must be interpreted in the sense of the theory of generalized functions because, as is well known, the sample functions of a stationary random process do not possess Fourier transforms within the framework of ordinary function theory.

[14]See, for example, C. Kittel, *Elementary Statistical Physics* (Wiley, New York, 1958), Sec. 28.

[15]L. Van Hove, Phys. Rev. **95**, 249 (1954); Physica, **24**, 404 (1958); see also R. J. Glauber, in *Lectures on Theoretical Physics,* edited by W. E. Brittin, B. W. Downs, and J. Downs (Interscience, New York, 1962), Vol IV, Sec. V, p. 571.

[16]When $\mathcal{S}(\mathbf{K},\Omega;\omega')$ is independent of ω' it becomes the analogue (for fluctuations of the dielectric susceptibility) of the usual dynamical structure factor for particle-density fluctuations.

[17]L. I. Komarov and I. Z. Fisher, Zh. Eksp. Teor. Fiz. **43**, 1927 (1962) [Sov. Phys.—JETP **16**, 1358 (1963)], Eq. (24).

[18]N. G. Van Kampen, in *Quantum Optics,* Proceedings of the International School of Physics "Enrico Fermi," Course XLII, Varenna, 1967, edited by R. J. Glauber (Academic, New York, 1969), p. 235, Eq. (7a).

[19]R. Pecora, J. Chem. Phys. **40**, 1604 (1964), Eq. (30).

[20]The function $g(\mathbf{R},\omega')$ is analogous to the pair distribution function introduced by F. Zernike and J. A. Prins [Z. Phys. **41**, 184 (1927)], in their well-known study of x-ray scattering from liquids.

[21]A. Einstein, Ann. Phys. **33**, 1275 (1910). An English translation of this paper was published in *Colloid Chemistry,* edited by J. Alexander (Reinhold, New York, 1926), Vol. 1, pp. 323–339.