Lasers with sub-Poissonian pump

M. A. M. Marte and P. Zoller
Institut für Theoretische Physik, Universität Innsbruck, Innsbruck, Austria
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It is shown that, introducing quantum-mechanical degrees of freedom for the pump field, one can model optical-pumping processes of a laser with sub-Poissonian statistics. For a class of pump models containing regular pumping at one end of the range and Poissonian pumping at the other, the exact stationary moments for the pump-field-averaged laser field are calculated in the strong-saturation limit of the lasing transition. It is demonstrated that, in a photodetection experiment of the laser output, complete noise suppression in the photocurrent fluctuation spectrum is, at least in principle, achievable. Finally, an approximate Fokker-Planck equation for photon distribution of the laser field with explicit appearance of the pump-light intensity correlation function in the diffusion term is derived and solved in the stationary limit.

I. INTRODUCTION

In the wake of successful generation of squeezed light and particularly of light exhibiting antibunching,[1] recently interest has also turned to questions on the possibility of sub-Poissonian statistics in laser systems: For example sub-Poissonian laser statistics have been predicted[2] and found[3] in the micromaser. Marte *et al.*[4] have studied the influence of a squeezed environment on laser atoms.

Especially reduced laser noise due to pump fluctuation suppression has been the topic of recent investigations; Yamamoto and co-workers[5] have demonstrated that controlling the pump fluctuations a sub-Poissonian output from a semiconductor laser is achievable. Golubev and Sokolev[6] have shown in their analysis based on the Scully-Lamb laser theory[7] that deterministic regular pumping by short laser pulses of a laser results in a photocurrent fluctuation spectrum of the laser output which drops below the shot-noise level. Recently Haake *et al.*[8] have calculated and simulated the effect of pump fluctuations, including fluctuations due to spontaneous emission from the upper lasing level in their treatment. Kennedy and Walls[9] compare atomic and semiconductor lasers with regular pumping and inhibited spontaneous emission.

In this paper we present a novel stochastic treatment of pump fluctuations: Instead of characterizing the pump mechanism by a simple pumping rate R on one hand or by averaging the laser field over Poissonian pump field statistics (as, e.g., in the Scully-Lamb laser theory), on the other hand, we explicitly introduce quantum-mechanical degrees of freedom for the pump field. This has the great advantage that the theory can be made sufficiently general to contain the results of both regular pumping and Poissonian pump light as special cases.

The paper is organized as follows. Section II briefly outlines the Scully-Lamb laser theory for later reference. In Sec. III we introduce a quantum-mechanical model for the pump-light field so that pump field averages of the laser field are found by tracing over the pump degrees of freedom. Following a recent analysis by Carmichael,[10] a connection to the experimentally relevant photocurrent fluctuation spectrum is established. Section IV specifies these general considerations to a class of pump-field models ranging from regular pumping to Poissonian pumping; in the strong-saturation limit the stationary moments of the corresponding laser equations can be calculated exactly. Finally in Sec. V an approximate Fokker-Planck equation is derived and used to calculate the stationary photon distribution. It is shown that in the ideal case of regular pumping with subsequent ideal photodetection complete noise suppression in some frequency components of the photocurrent is theoretically achievable.

II. LASER MODEL

In the following we will briefly outline the laser theory of Scully and Lamb.[7] Besides the laser problem their approach has been, for example, particularly useful when studying the micromaser,[2,3] as it deals with the contribution of single incoherently pumped atoms, entering a high-Q cavity, to the laser field inside. Recent work[6,8] on pump fluctuations in lasers have been based on this laser treatment and it has also proven convenient as a basis for our pump-noise model.

Figure 1 schematically depicts a model atom entering the cavity: The incoherent pumping process prepares the atom in an excited level $|2\rangle$ at a rate R; laser action takes place in the inverted transition $|2\rangle \rightarrow |1\rangle$, which is assumed resonant with the single-mode cavity. The levels $|1\rangle$ and $|2\rangle$ decay according to the atomic decay rates γ_1 and γ_2 to some lower levels and subsequently the atom becomes "invisible."

The cavity damping of the resulting laser field is modeled by a zero-temperature bath, i.e., the equation of motion of the reduced-field density matrix contains the following damping terms:

$$\frac{d\rho_F}{dt} = L\rho_F \quad \text{with } L\rho_F = \frac{\kappa}{2}(2a\rho_F a^\dagger - a^\dagger a \rho_F - \rho_F a^\dagger a) , \tag{2.1}$$

where κ denotes the cavity decay rate.

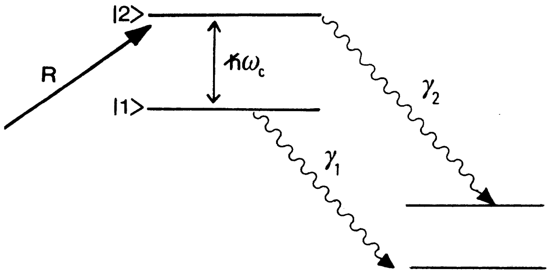

FIG. 1. Schematic diagram of effective levels in the laser atoms: The transition $|2\rangle \rightarrow |1\rangle$, which is assumed resonant with the laser frequency ω_c in the cavity, is pumped at a rate R. γ_2 and γ_1 are the atomic decay rates.

Let some operator F ($\equiv 1+\hat{u}$) denote the change of the cavity field that an excited atom (entering the cavity at time t_i) brings about after having interacted for a time $t_{\rm int}$ with the cavity:

$$\rho_F(t_i+t_{\rm int})=F\rho_F(t_i) \ . \tag{2.2}$$

We assume the interaction time to be very short compared to the cavity lifetime, i.e., $t_{\rm int} \ll 1/\kappa$. This means that each atom sees an effectively undamped field during its interaction time with the cavity. Obviously, F has to satisfy $\mathrm{Tr}_F F\rho_F = \mathrm{Tr}_F \rho_F = 1$ for any density operator ρ_F.

Combining n atomic contributions and the cavity damping yields the following density matrix for the field:

$$\rho_F(t|t_n,\ldots,t_1)=e^{L(t-t_n)}Fe^{L(t_n-t_{n-1})} \times F \cdots Fe^{Lt_1}F\rho_F(0) \ . \tag{2.3}$$

Usually at this stage one carries out an averaging procedure over the distribution of excited atoms entering the cavity and assumes this distribution to be Poissonian.[1-3] The Poisson distribution can be characterized by the fact that the conditional probability density $\tilde{c}(\tau)$ that, given an atom was excited at $t=0$ in state $|2\rangle$ or entered the cavity at time $t=0$, the *next* one will appear at time τ, is given by

$$\tilde{c}(\tau)=R\,e^{-R\tau} \ . \tag{2.4}$$

Thus the averaged-field density matrix is given by

$$\langle\langle \rho_F(t)\rangle\rangle = \sum_{n=0}^{\infty} \int_0^t dt_n \cdots \int_0^{t_2} dt_1 p_{[0,t)}(t_n,\ldots,t_1) \times \rho_F(t|t_n,\ldots,t_1) \ , \tag{2.5}$$

with

$$p_{[0,t)}(t_n,\ldots,t_1)=e^{-R(t-t_n)}Re^{-R(t_n-t_{n-1})}R \cdots Re^{-Rt_1} \ . \tag{2.6}$$

Differentiating Eq. (2.5) with respect to time, this averaged density operator is seen to obey the following master equation (which is also encountered in the micromaser theory[2]):

$$\frac{d}{dt}\langle\langle \rho_F(t)\rangle\rangle = (-R+L)\langle\langle \rho_F(t)\rangle\rangle + RF\langle\langle \rho_F(t)\rangle\rangle \ . \tag{2.7}$$

Since we are only interested in fluctuations of the intensity, in the following we can restrict ourselves to the diagonal elements of the field density operator $p_n(t) \equiv \langle\langle \rho_F(t)\rangle\rangle_{n,n}$ in the number state representation. In the Scully-Lamb laser theory the corresponding equations of motion explicitly read

$$\frac{dp_n}{dt} = -\kappa[np_n-(n+1)p_{n+1}] + R(\beta_n p_{n-1}-\beta_{n+1}p_n) \ , \tag{2.8}$$

where

$$\beta_n = \beta_c \frac{n/n_s}{1+n/n_s} \quad \text{with } \beta_c = \frac{\gamma_1}{\gamma_1+\gamma_2} \ . \tag{2.9}$$

Here n_s denotes the saturation photon number $n_s = \gamma_1\gamma_2/4g^2$ with g being the atomic dipole coupling constant.

The stationary solution of Eq. (2.8) is found to be

$$p_n = N_0 \prod_{k=1}^{n} \left[\frac{\beta_k}{k}\frac{R}{\kappa} \right] \tag{2.10}$$

(with N_0 being a normalization constant), and for β_k approximately constant it agrees with a coherent state.

III. PUMP STATISTICS AND AVERAGES

In the above laser theory pump fluctuations have been incorporated in form of fluctuations in the beam of excited atoms passing through the cavity. Alternatively, instead of dealing with the statistics of the atomic beam, one can study the influence of the photon statistics of the pump light field on the cavity field. In this section we will develop a formalism for incorporating pump-field statistics and calculating the pump-field averaged laser field.

A. Characterization of the pump-light field

As we are dealing with incoherent pump light, the relevant quantities in order to characterize the pump process are the mean intensity as well as the intensity correlation function. Adopting a notation which resembles the photodetection theory of Srinivas and Davies,[11] we write the mean intensity (in units of a rate) as

$$\langle I \rangle = \mathrm{Tr}_P(J\rho_P) \ , \tag{3.1}$$

where ρ_P is the pump-field density matrix in the Hilbert space $\mathcal{H}_P$ of the pump degrees of freedom and Tr_P stands for the trace in $\mathcal{H}_P$. We assume the pump light to be stationary, i.e., $(d/dt)\rho_p = \Lambda\rho_p = 0$, with Λ being the Liouville operator governing the time evolution of the pump field. The operator J acts on ρ_P in the following way:

$$J\rho_P = \xi E^+ \rho_P E^- \tag{3.2}$$

[with $E^{\pm}$ being the positive (negative) frequency part of the electric field Schrödinger operator and ξ a constant conversion factor to units of photon flux].

A characterization of the intensity fluctuations of the pump light is achieved by specifying the two time intensity correlations (with : : denoting normal and time ordering)

$$c(\tau)=\mathrm{Tr}_P(Je^{\Lambda\tau}J\rho_P)/\mathrm{Tr}_P(J\rho_P)$$
$$=\langle :I(\tau)I(0):\rangle/\langle I(0)\rangle \ , \tag{3.3a}$$
$$\tilde{c}(\tau)=\mathrm{Tr}_P(Je^{(\Lambda-J)\tau}J\rho_P)/\mathrm{Tr}_P(J\rho_P)$$
$$=\left\langle :I(\tau)\exp\left[-\int_0^\tau I(t)dt\right]I(0):\right\rangle \Big/ \langle I(0)\rangle \ . \tag{3.3b}$$

(The last equalities may be derived from the Kelly-Kleiner photodetection formula, analogous to the procedure in an Appendix of Ref. 10.) Here again the operator Λ is the not-yet-specified Liouville operator belonging to the dynamics in $\mathcal{H}_P$. Equations (3.1)–(3.3) throughout reflect a strong parallel to the Srinivas-Davies theory: The stochastic pump process is modeled in analogy to the photodetection of pump photons. Given a pump photon excited an atom at time $t=0$, $c(\tau)$ represents the probability that another (not necessarily the next one) will arrive at time $t=\tau$, whereas $\tilde{c}(\tau)$ is the probability density for subsequent photons at times $t=0$ and $t=\tau$.

With Eqs. (2.5) and (2.6) in mind we interpret

$$p_{[0,t)}(t_n,\ldots,t_1)=\mathrm{Tr}_P(e^{(\Lambda-J)(t-t_n)}Je^{(\Lambda-J)(t_n-t_{n-1})}$$
$$\times J\cdots Je^{(\Lambda-J)t_1}\rho_P) \tag{3.4}$$

as a conditional probability density for pumping the laser field (or equivalently preparing the laser atoms in the level $|2\rangle$) at times $t_1\le\cdots\le t_n$ in a time interval $[0,t)$.

B. Pump-field averages

By means of Eq. (2.5) we will now define stochastic averages $\langle\!\langle\ \rangle\!\rangle$ over the pump statistics of the field density operator $\rho_F(t)$, given a hierarchy of distributions $p_{[0,t)}(t_n,\ldots,t_1)$ [compare Eq. (3.4)]. To this end it is useful to construct a density operator $W(t)$ in an extended product Hilbert space $\mathcal{H}_F\otimes\mathcal{H}_P$ (with $\mathcal{H}_F$ the Hilbert space of the laser field), defined by

$$W(t)=\sum_{n=0}^{\infty}\int_0^t dt_n\ldots\int_0^{t_2}dt_1\, e^{(L+\Lambda-J)(t-t_n)}JF$$
$$\times e^{(L+\Lambda-J)(t_n-t_{n-1})}\cdots$$
$$\times JFe^{(L+\Lambda-J)t_1}\rho_F(0)\otimes\rho_P(0) \ . \tag{3.5}$$

The motivation for the construction (3.5) is that the averaged-field density matrix Eq. (2.5) is found by tracing $W(t)$ over the pump degrees of freedom,

$$\langle\!\langle\rho_F(t)\rangle\!\rangle=\mathrm{Tr}_P W(t) \ . \tag{3.6}$$

(Note that the operators Λ and J operate in $\mathcal{H}_P$, whereas L operates in $\mathcal{H}_F$.) Furthermore, differentiating Eq. (3.5) with respect to time, it is straightforward to prove that $W(t)$ obeys the equation of motion

$$\frac{d}{dt}W(t)=(\Lambda-J+L)W(t)+JFW(t) \tag{3.7}$$

with initial condition $W(0)=\rho_F(0)\otimes\rho_P(0)$. Here the terms involving Λ and L describe the evolution of the pump field and cavity damping, respectively, whereas the terms proportional to J and F constitute the pumping. Tracing Eq. (3.7) over the cavity field, the pump-field density matrix can be recovered according to $\rho_P(t)=\mathrm{Tr}_F W(t)$ [which follows from Eq. (3.7) in view of $\mathrm{Tr}_F LW(t)=0$ and $\mathrm{Tr}_F JFW(t)=J\rho_p(t)$]. To summarize, given a hierarchy of distributions (3.4) for the pump statistics, the average-field density operator can be found by solving Eq. (3.7) and tracing over the pump degrees of freedom.

The Scully-Lamb laser theory with Poissonian pump is recovered, if one specializes Eq. (3.7) assuming $\mathcal{H}_p$ to be one-dimensional: the operators Λ and J then become c numbers, $J=R$ and $\Lambda=0$, where R is the pump rate and the last equality follows from the requirement $\mathrm{Tr}_p\Lambda W=0$ [compare Eqs. (2.5) or (2.6)]. In view of this, one can regard the present formulation as a generalization of the prevailing laser theory to include a quantum-mechanical pump process: Instead of modeling the pump statistics by stochastic c-number pump rates, one treats the pump field as additional quantum-mechanical degrees of freedom. This difference is of great importance, as it opens up the possibility for quantum effects such as sub-Poissonian statistics, as we shall see later.

Inserting Eq. (2.1) for the cavity damping L and (2.8) for the operator F describing the interaction of the laser field with the atoms, the diagonal elements $\langle n|W(t)|n\rangle=W_n(t)$ satisfy the equations of motion

$$\frac{d}{dt}W_n(t)=(\Lambda-\kappa n)W_n+\kappa(n+1)W_{n+1}$$
$$+J(\beta_n W_{n-1}-\beta_{n+1}W_n) \ . \tag{3.8}$$

[Note that $W_n(t)$ is still an operator in $\mathcal{H}_p$.] With the help of the solutions $W_n(t)$ the intracavity moments such as the mean photon number can be calculated:

$$\langle n\rangle=\sum_{n=0}^{\infty}np_n(t) \quad \text{with } p_n=\mathrm{Tr}_P W_n \ . \tag{3.9}$$

In a similar fashion, one can obtain the Mandel Q parameter

$$Q=[\langle n(n-1)\rangle-\langle n\rangle^2]/\langle n\rangle \ , \tag{3.10}$$

which will be of interest later on, when discussing the measurement of the laser output intensity fluctuation spectrum. Q negative/positive indicates sub/super-Poissonian deviations in the photon statistics from the Poissonian distribution of a coherent state which has $Q=0$.

C. Photocurrent fluctuation spectrum

Following a recent analysis by Carmichael,[10] we shall briefly explain the relation of the theoretically derivable intracavity moments of the preceding section with measurable quantities.

In an experiment the laser output falls on a photodetector converting incident photons into photoelectrons. A single detector, positioned at a distance z_d from the (single) laser output port, with quantum efficiency η and amplification G measures a mean photocurrent given by[10]

$$\overline{i(t)}=\eta Ge\kappa\langle a^\dagger(t-z_d/c)a(t-z_d/c)\rangle$$
$$\equiv\eta Ge\xi\langle I(z_d,t)\rangle \; , \tag{3.11}$$

(with $\xi=2\epsilon_0 cA/\hbar\omega$ a conversion factor to photon flux units, e the electronic charge, and A the transverse area).

However, in this context, where one is interested in noise properties, one rather looks at the spectrum of the photocurrent fluctuations, which is related to the two time correlation functions by Fourier transform. Recently Carmichael has discussed in detail the photocurrent spectrum of squeezed light in a homodyne detection scheme. His general results are applicable to the present case (with the modification that the amplitude of the local oscillator is zero here) and yield

$$\overline{i(0)i(\tau)}=\mathrm{DC}+\mathrm{SH}+\eta^2(Ge)^2\xi^2\langle :I(z_d,0),I(z_d,\tau):\rangle \; . \tag{3.12}$$

Here again : : denotes normal and time ordering, i.e.,

$$\xi^2\langle :I(z_d,0),I(z_d,\tau):\rangle=\kappa^2[\langle a^\dagger(0)a^\dagger(\tau)a(\tau)a(0)\rangle-\langle a^\dagger(0)a(0)\rangle^2] \; ,$$

and stationarity of the field is assumed. DC and SH stand for a dc contribution and a δ-correlated "shot-noise" contribution, given by

$$\mathrm{DC}=\eta^2(Ge)^2\xi^2\langle I(z_d,0)\rangle^2 \tag{3.13}$$

and

$$\mathrm{SH}=\eta(Ge)^2\xi\langle I(z_d,0)\rangle\delta(\tau) \; , \tag{3.14}$$

respectively. The observable power spectrum (which is symmetric in ω) is the Fourier cosine transform

$$P(\omega)=\frac{1}{\pi}\int_0^\infty d\tau\cos(\omega\tau)\overline{i(0)i(\tau)}$$
$$=P_{\mathrm{SH}}(\omega)[1+\eta S(\omega)] \; . \tag{3.15}$$

In the last expression $P(\omega)$ has been factorized into the flat shot-noise background

$$P_{\mathrm{SH}}(\omega)=\eta(Ge)^2\frac{1}{2\pi}\kappa\langle a^\dagger(0)a(0)\rangle \tag{3.16}$$

and the positive variance $V=1+\eta S(\omega)$ with

$$S(\omega)=2\,\mathrm{Re}\left[\int_0^\infty d\tau\, e^{i\omega\tau}[\langle a^\dagger(0)a^\dagger(\tau)a(\tau)a(0)\rangle-\langle a^\dagger(0)a(0)\rangle^2]\right]\kappa\Big/\langle a^\dagger(0)a(0)\rangle \; . \tag{3.17}$$

$V=1$ is the coherent (vacuum) level which is often referred to as "standard quantum limit." For a classical field the function $S(\omega)$ can be shown to be positive semidefinite; for quantum fields $S(\omega)$ can become negative and thus subtracts from the vacuum level $V=1$. In the ideal case one reaches $S(\omega)=-1$ over some frequency interval; with $\eta=1$ this means perfect noise suppression in the photocurrent.

The stationary two-time correlation function $\langle a^\dagger(0)a^\dagger(\tau)a(\tau)a(0)\rangle$ may be calculated for $\tau\geq 0$ with the help of a quantum-regression theorem generalized to include the pump degrees of freedom,

$$\langle a^\dagger(0)a^\dagger(\tau)a(\tau)a(0)\rangle=\mathrm{Tr}_{P+F}(a\{e^{(L+\Lambda+J\hat{u})\tau}[aW(0)a^\dagger]\}a^\dagger) \tag{3.18}$$

with $W(0)$ the stationary density matrix in $\mathcal{H}_F\otimes\mathcal{H}_P$.

IV. EXACTLY SOLUBLE MODEL

A. A class of sub-Poissonian pump fields

In this section we will introduce a whole class of pump-field models, containing Poissonian pump light at one end of the spectrum and deterministic (regular) pumping at the other end as special cases. As explained in Sec. III A, the photon statistics of the pump-light field can be characterized by the conditional probability density $\tilde{c}(\tau)$ [cf. Eq. (3.3)]. Thus we introduce a class of probability densities $\tilde{c}_N(\tau)$ parametrically dependent on an integer $N\in[1,\infty)$,

$$\tilde{c}_N(\tau)=r\frac{(r\tau)^{N-1}}{(N-1)!}e^{-r\tau} \tag{4.1}$$

with $r=N/T$ and $T=$const. The above choice for the probability density is motivated by the fact that this set contains the Poissonian and the regular pump limit and furthermore leads to simple closed form expressions: obviously, setting $N=1$ yields Poissonian statistics; on the other hand, in the limit $N\to\infty$ $\tilde{c}_N(\tau)$ approaches the δ-function $\delta(\tau-T)$ corresponding to regular pumping.[6,8] Other values $N>1$, but finite, specify a whole range of sub-Poissonian pump-light fields which exhibit antibunching, i.e., $\tilde{c}_N(0)=0$.

The constant T implicit in Eq. (4.1) can be interpreted as the main time interval between two subsequent pump photons, since

$$\langle\tau\rangle\equiv\int_0^\infty d\tau\,\tau\tilde{c}_N(\tau)=N/r=T \; . \tag{4.2}$$

The corresponding variance is found to be

$$(\Delta\tau)^2\equiv\langle\tau^2\rangle-\langle\tau\rangle^2=T^2/N \; , \tag{4.3}$$

and thus the relative error $\Delta\tau/\langle\tau\rangle=1\sqrt{N}$ clearly goes to zero for large N, i.e., when approaching the limit of regular pumping. Figures 2 and 3 illustrate the situation.

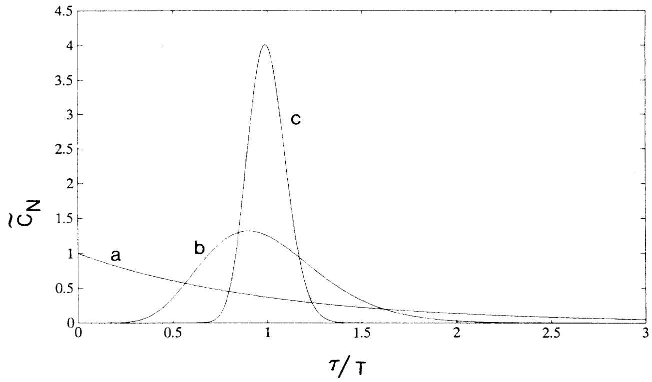

FIG. 2. The conditional probability density $\tilde{c}_N(\tau)$ for $N=1$ (curve *a*), $N=10$ (curve *b*), and $N=100$ (curve *c*).

In Fig. 2 the functions $\tilde{c}_N(\tau)$ and in Fig. 3 simulated photon emissions according to the statistics determined by $\tilde{c}_N(\tau)$ are plotted for some values of N.

Note that a link to the framework of Sec. III A can be made. There exists a matrix representation for the operators J and Λ, satisfying Eq. (3.3) with $\tilde{c}_N(\tau)$. The explicit form of these matrices is given in Appendix A. One can show that the parameter $1/T$ is the pump rate,

$$\langle I \rangle = \mathrm{Tr}_P(J\rho_P) = 1/T \; . \tag{4.4}$$

B. The strong saturation limit

In the limit of strong saturation, that is $\langle n \rangle \gg n_s$ the laser model with pump-field statistics specified by Eq. (4.1) becomes exactly soluble. According to Eq. (2.9), in the strong-saturation limit one can ignore the dependence of β_n on the subscript $n\,(\beta_n \approx \beta_c = \mathrm{const})$. For constant β one could speak of a "kicked cavity" in analogy to the kicked rotator: The field in the cavity is pumped or "kicked" by a constant amount, the kicks being distributed in time according to the probability density $\tilde{c}_N(\tau)$. (In Sec. V, when deriving a Fokker-Planck equation for the laser photon distribution, we will return to the general case keeping the n dependence.) In the present case it is possible to compute the stationary moments of the photon distribution of the laser by means of a generating "function" $G(z,t)$ (which is actually a density matrix in $\mathcal{H}_P$),

$$G(z,t) = \sum_{n=0}^{\infty} z^n W_n(t) \tag{4.5}$$

[with $G(z=1,t) \equiv \rho_P(t)$], satisfying the differential equation

$$\frac{\partial G}{\partial t} = [\Lambda + \beta_c J(z-1)]G - \kappa(z-1)\frac{\partial G}{\partial z} \; . \tag{4.6}$$

Obviously the following relations hold:

$$\langle n \rangle = \mathrm{Tr}_P \left[\left. \frac{\partial G}{\partial z} \right|_{z=1} \right] , \tag{4.7}$$

$$\langle n(n-1) \rangle = \mathrm{Tr}_P \left[\left. \frac{\partial^2 G}{\partial z^2} \right|_{z=1} \right] . \tag{4.8}$$

Differentiating Eq. (4.6) we derive

$$\frac{d}{dt}\langle n \rangle = -\kappa \langle n \rangle + \beta_c \mathrm{Tr}_P(JG|_{z=1}) , \tag{4.9}$$

$$\frac{d}{dt}\langle n(n-1) \rangle = -2\kappa \langle n(n-1) \rangle + 2\beta_c \mathrm{Tr}_P \left[J \left. \frac{\partial G}{\partial z} \right|_{z=1} \right] . \tag{4.10}$$

Identifying the last term in Eq. (4.9) with the stationary pump rate $\langle I \rangle = 1/T$ according to Eq. (4.4), i.e.,

$$\langle I \rangle = \mathrm{Tr}_P(J\rho_P) \equiv \mathrm{Tr}_P(JG|_{z=1}) , \tag{4.11}$$

and inserting the stationary solution

$$\left. \frac{\partial G}{\partial z} \right|_{z=1} = (\kappa - \Lambda)^{-1} \beta_c J \rho_p \tag{4.12}$$

into Eq. (4.10), we get the stationary moments

$$\langle n \rangle = \beta_c \langle I \rangle / \kappa , \tag{4.13}$$

$$Q = -\beta_c \{ \langle I \rangle / \kappa - \mathrm{Tr}_P[J(\kappa - \Lambda)^{-1} J \rho_p] / \langle I \rangle \} \; . \tag{4.14}$$

By means of the Laplace transform

$$\hat{c}(s) = \mathrm{Tr}_P[J(s-\Lambda)^{-1} J \rho_p] / \mathrm{Tr}_P(J\rho_p)$$

of $c(\tau)$, defined by Eq. (3.3a), it is possible to rewrite Eq. (4.14) in the following way:

$$Q = [\hat{c}(s=\kappa) - \langle I \rangle / \kappa]\beta_c \equiv Q_P(\langle n \rangle)\beta_c \; . \tag{4.15}$$

As shown in Appendix A, by means of the explicit operator representation (J,Λ) compatible with Eqs. (3.3a) and (3.3b) one derives

$$Q_P(\langle n \rangle) = \left[\left[1 + \frac{\beta_c}{\langle n \rangle N} \right]^N - 1 \right]^{-1} - \langle n \rangle / \beta_c \; . \tag{4.16}$$

In Fig. 4 the Mandel Q parameter is plotted as a function of $\Delta\tau / \langle \tau \rangle = 1/\sqrt{N}$ assuming $\beta_c = 1$, that is, spon-

FIG. 3. Simulated photon emission; each line corresponds to an emitted photon. (a) $N=1$ Poissonian statistics; (b) $N=10$ antibunching.

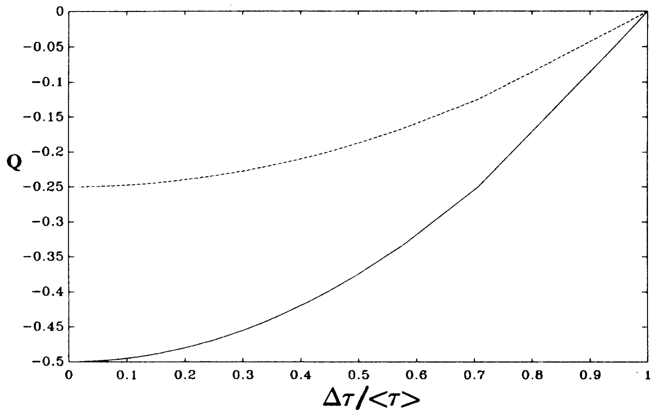

FIG. 4. The Mandel Q parameter as a function of $\Delta\tau/\langle t\rangle$ for $\langle I\rangle/\kappa=10^3$; $\beta_c=1$ (solid line) and $\beta_c=0.5$ (dashed line).

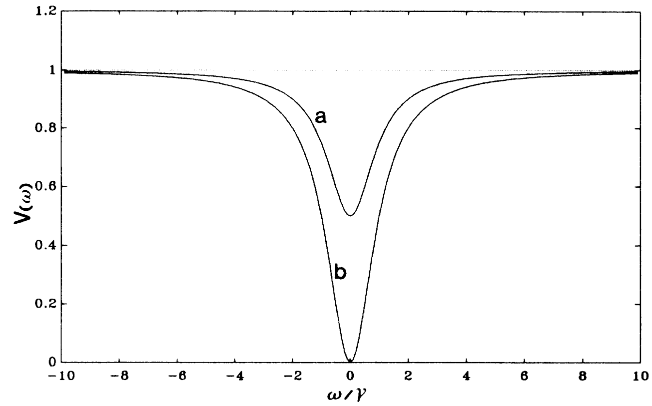

FIG. 5. The spectral variance $V(\omega)$ for $\langle I\rangle/\kappa=100$; $\eta=0.5$ (curve a) and $\eta=1$ (curve b).

taneous emission from the upper compared to the lower level is ignored. There the lowest achievable $Q=Q_P$ value is seen to be -0.5 for $N\rightarrow\infty$. This can be made obvious in an expansion in the limit $\langle n\rangle=\langle I\rangle/\kappa\gg 1$—which is the case of interest—yielding the following leading terms for Q_P:

$$Q_P(\langle n\rangle)=-\frac{1}{2}\frac{N-1}{N}+\frac{1}{\langle n\rangle}\frac{N^2-1}{12N^2}+O\left[\left[\frac{1}{\langle n\rangle}\right]^2\right]$$
$$=-\frac{1}{2}\left[1-\left[\frac{\Delta\tau}{\langle\tau\rangle}\right]^2\right]$$
$$+\frac{1}{\langle n\rangle}\frac{1}{12}\left[1-\left[\frac{\Delta\tau}{\langle\tau\rangle}\right]^4\right]$$
$$+O\left[\left[\frac{1}{\langle n\rangle}\right]^2\right]. \tag{4.17}$$

Note that for a number state one has $Q=-1$ compared to the optimum value of $Q=-\frac{1}{2}$ in the present case; this is analogous to the 50% squeezing attainable in the intracavity mode.[10] However, in the subsequent section it will be demonstrated that this lower limit on Q leads (for a single port laser cavity and an ideal photodetector) to perfect noise suppression in the photocurrent fluctuations outside. In this sense Eq. (4.17) really is the optimum theoretically achievable value. To lowest order in $1/\langle n\rangle$ and for $N\rightarrow\infty$ our result Eq. (4.17) agrees with the theories by Sokolev and Golubev[6] and Haake *et al.*[8] in spite of the fact that there one deals with a nonstationary pump for which only stroboscopic mean values at times $0,T,2T,\ldots$ are calculated.

We now turn to the stationary pump field averaged normally and time ordered correlation function

$$g(\tau)=\langle\langle a^\dagger(0)a^\dagger(\tau)a(\tau)a(0)\rangle\rangle .$$

The technique of calculation is similar to the method of Sec. IV B. Details on the calculation of $g(\tau)$ and its Laplace transform $\hat{g}(s)$ by means of a quantum-fluctuation regression theorem are given in Appendix B. The spectrum $S(\omega)$ Eq. (3.17) expressed in terms of the Laplace transforms $\hat{g}(s)$ [with $\hat{c}(s)$ given in Appendix A] is found to be

$$S(\omega)=\kappa\{2\,\mathrm{Re}[\hat{g}(-i\omega)-\langle n\rangle^2/(-i\omega)]\}/\langle n\rangle \tag{4.18}$$

with

$$\hat{g}(s)=\frac{\langle n\rangle}{s^2-\kappa^2}[s\hat{c}(s=\kappa)-\kappa\hat{c}(s)] .$$

The results are graphically shown in Fig. 5. The variance $V(\omega)=1+\eta S(\omega)$ is plotted for some N values: $N=1$ gives a straight reference line (normalized to one) corresponding to a Poisson distribution, such as for a field in a coherent state: $N=1000$ displays almost perfect noise suppression—provided the quantum efficiency η of the detector is close enough to unity and the spontaneous emission is small, i.e., $\beta_c=\gamma_1/(\gamma_1+\gamma_2)\approx 1$. Smaller N, spontaneous emission, a nonideal detector or a second laser output port lead to an accordingly diminished effect. Again for very large $\langle n\rangle$ and $\Delta\tau/T\rightarrow 0$ this agrees with the result by Sokolev and Golubev.[6]

V. FOKKER-PLANCK EQUATION

In the limit of large mean photon number $\langle n\rangle=\beta_c/(\kappa T)$, that is, in the case that the pumping rate $1/T$ largely exceeds the cavity decay rate, one can derive an approximate Fokker-Planck equation. To this end we recall Eq. (3.7)

$$\frac{dW}{dt}=(\Lambda+L+J\hat{u})W(t) , \tag{5.1}$$

where $\hat{u}\equiv(F-1)$ operates in the following way on the pump-field-averaged density matrix in the number state representation [compare Eq. (2.8)]:

$$(\hat{u}\langle\langle\rho_F\rangle\rangle)_{n,n}=-\beta_{n+1}p_n+\beta_n p_{n-1} . \tag{5.2}$$

We realize that this may be rewritten by means of a difference operator $\mathcal{D}$ between terms with subscript n shifted by 1, i.e., $\beta_n p_{n-1}-\beta_{n+1}p_n\equiv\mathcal{D}(\beta_{n+1}p_n)$.

Approximating the discrete index n by a continuous variable n, we rewrite $\mathcal{D}$ in terms of the continuous displacement operator as $\mathcal{D}=(e^{-\partial/\partial n}-1)$. Thus increasing powers of $\hat{u}$ roughly correspond to increasing powers of $\partial/\partial n$; in the Fokker-Planck approximation one retains terms up to second order in $\partial/\partial n$. This motivates a second-order perturbation treatment of $\hat{u}$ in Eq. (5.1).

Perturbation theory to lowest order in the cavity damping L and second order in $\hat{u}$ leads to the following second-order equation for $\langle\langle\rho_F(t)\rangle\rangle=\mathrm{Tr}_P W(t)$:

$$\frac{d}{dt}\langle\langle\rho_F(t)\rangle\rangle=\left[L+\mathrm{Tr}_P[J\rho_p(t)]\hat{u}+\left[\int_0^\infty dt'\,\mathrm{Tr}_P[Je^{\Lambda(t-t')}J\rho_p(t')]-\int_0^\infty dt'\,\mathrm{Tr}_P[J\rho_p(t)]\mathrm{Tr}_P[J\rho_p(t')]\right]\hat{u}^2\right]\langle\langle\rho_F(t)\rangle\rangle\ . \tag{5.3}$$

In view of previous results we recognize

$$\mathrm{Tr}_P[J\rho_p(t)]=\langle I(t)\rangle$$

as the mean pump rate and, from Eq. (3.3a),

$$\begin{aligned}\mathrm{Tr}_P[Je^{\Lambda(t-t')}J\rho_p(t')]-\mathrm{Tr}_P[J\rho_p(t)]\mathrm{Tr}_P[J\rho_p(t')]&=[c(t-t')-\langle I(t)\rangle]\langle I(t')\rangle\\&=\langle :I(t),I(t'):\rangle\end{aligned} \tag{5.4}$$

as the intensity correlation function. Assuming a stationary pump field $\mathrm{Tr}_P[J\rho_p(t)]=\langle I\rangle$ and using Eq. (4.15) in the relation

$$\begin{aligned}\int_0^\infty dt'\langle :I(t),I(t'):\rangle&=\lim_{s\to 0}\left[\hat{c}(s)-\frac{\langle I\rangle}{s}\right]\langle I\rangle\\&\equiv Q_p(\infty)\langle I\rangle\ ,\end{aligned} \tag{5.5}$$

one arrives at the approximate equation

$$\frac{d}{dt}\langle\langle\rho_F(t)\rangle\rangle=[L+\langle I\rangle\hat{u}+\langle I\rangle Q_P(\infty)\hat{u}^2]\langle\langle\rho_F(t)\rangle\rangle\ . \tag{5.6}$$

Note that in view of Eq. (4.17) $Q_P(\infty)$ is equal to $-\frac{1}{2}$ for regular pumping; this follows in analogy to the discussion of Eqs. (4.15)–(4.17) concerning the Mandel Q of the *output* field in the strong-saturation limit. The case $Q_P=0$ corresponds to the usual laser theory with Poissonian pump. An equation of the form of Eq. (5.6) has also been derived by Sokolev and Golubev[6] for the special case of regular pumping.

Finally, inserting a continuous variable version of Eq. (5.2), we get a Fokker-Planck equation for p_n, valid for $1/\langle n\rangle=\kappa/\langle I\rangle\beta_c\ll 1$:

$$\frac{\partial}{\partial t}p(n,t)=\left[-\frac{\partial}{\partial n}A(n)+\frac{1}{2}\frac{\partial^2}{\partial n^2}D(n)\right]p(n,t) \tag{5.7a}$$

with drift term

$$A(n)=-\kappa n+\langle I\rangle\beta(n) \tag{5.7b}$$

and diffusion

$$D(n)=\langle I\rangle\beta(n)+2Q_P(\infty)\langle I\rangle\beta^2(n)+\kappa n\ . \tag{5.7c}$$

We emphasize that this equation explicitly depends on the parameter Q_P defined in Eq. (5.5). For sub-Poissonian pump light Q_P leads to a negative contribution to the diffusion resulting in reduced fluctuations. Again the regular pump case has the same structure as the heuristic Fokker-Planck model for the micromaser.[2]

Since the problem is one-dimensional, a stationary potential solution

$$p(n)=Ne^{-V(n)} \tag{5.8a}$$

can be found[12] with N a normalization constant and

$$V(n)=\int_{n_0}^n dx\left[\frac{1}{D(x)}\left[\frac{d}{dx}D(x)-2A(x)\right]\right]\ . \tag{5.8b}$$

It is straightforward to carry out this integration; the results are displayed in Fig. 6, where the normalized distribution is plotted as a function of $n/\bar{n}_{sc}$ with $\bar{n}_{sc}$ denoting the semiclassical stationary solution

$$\bar{n}_{sc}=\frac{1}{\kappa}\langle I\rangle\beta_c-n_s\ , \tag{5.9}$$

found by setting the diffusion D in Eq. (5.7) to zero. The case $Q_P=0$ has been compared with a Poisson distribution of the same mean value and excellent agreement was found. The curves clearly demonstrate that $Q_P<0$ and $Q_P>0$ lead to a narrowed or broadened distribution, respectively, compared to the corresponding Poissonian distribution. The effect of subtracting Q_P is strongest for

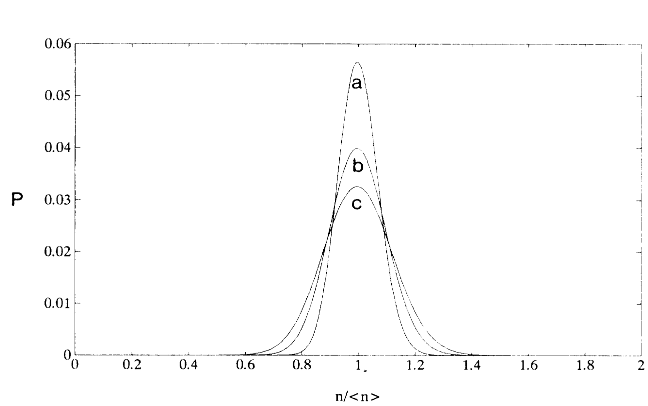

FIG. 6. Stationary distribution $p(n)$ of the Fokker-Planck equation in arbitrary units in the strong-saturation limit $\langle n\rangle/n_s=10^5$; $\beta_c=1$, $Q_P=-0.5$ (curve *a*), $Q_P=0$ (curve *b*), and $Q_P=0.5$ (curve *c*).

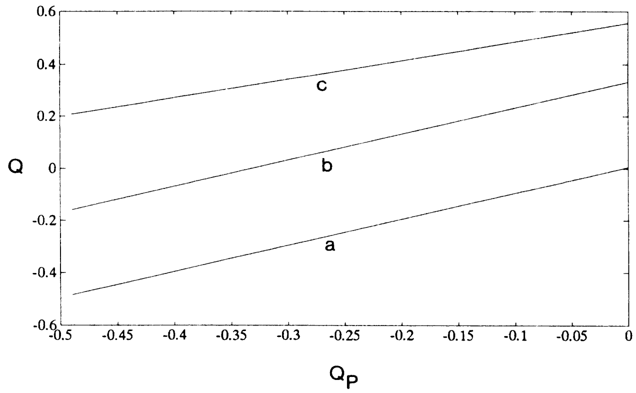

FIG. 7. The Mandel Q parameter for the laser output as a function of Q_P for $\langle n \rangle = 200$; $n_s \rightarrow 0$, $\beta_c = 1$ (curve *a*), $n_s = 50$, $\beta_c = 1$ (curve *b*), $n_s = 50$, $\beta_c = 0.7$ (curve *c*).

$\beta_c = \gamma_1/(\gamma_1 + \gamma_2) \approx 1$; thus large spontaneous emission from the upper level tends to destroy the sub-Poissonian statistics. Figure 7 shows the Mandel Q parameter of the laser field as a function of Q_P for some values of β_c and n_s.

VI. CONCLUSIONS

We have developed a laser theory with sub-Poissonian pump statistics. The central equation for the treatment of pump statistics is a master equation explicitly containing the pump degrees of freedom besides the laser field. In the present paper we have calculated the stationary moments and the photon number distribution of the pump-field-averaged laser field for a class of pump models which contain both the Poissonian and regular laser pump as limiting cases. Complete suppression of photocurrent fluctuations is found for regular pumping, negligible spontaneous decay, and ideal photodetection. The present theory has the advantage that it applies to a large number of pump-field inputs. A particularly relevant example is a squeezed input; corresponding results will be published elsewhere.

Note added in proof. Recently, a series of preprints by Scully and co-workers has been brought to our attention which address somewhat related questions, dealing with pump-noise quenching in correlated spontaneous emission lasers.

APPENDIX A

The purpose of this Appendix is to identify the operators Λ and J for the pump model defined in Sec. III A [Eq. (4.1) for $\tilde{c}_N(\tau)$]. In general, the density operator of the pump field projected on an n-photon subspace, $\rho_p^{(n)}(t)$, obeys the equation

$$\frac{d}{dt}\rho_p^{(n)}(t) = (\Lambda - J)\rho_p^{(n)}(t) + J\rho_p^{(n-1)}(t)(1-\delta_{n0}) , \quad (n = 0, 1, 2, \ldots) . \tag{A1}$$

Denoting by $p^{(n)}$ the diagonal elements of $\rho_p^{(n)}$ our model is defined by the set of rate equations

$$\begin{aligned} \frac{d}{dt}p_1^{(n)} &= -rp_1^{(n)} + rp_N^{(n-1)}(1-\delta_{n0}) , \\ \frac{d}{dt}p_i^{(n)} &= -rp_i^{(n)} + rp_{i-1}^{(n)}, \quad i = 2, \ldots, N \end{aligned} \tag{A2}$$

from which by comparison with Eq. (A1) the explicit form of the operators Λ and J is easily read off. In particular we find that Eqs. (A2) give rise to the 'next photon' intensity correlation function $\tilde{c}_N(\tau)$ given in Eq. (4.1). Finally, it is easy to prove that in the present model we have the following integral equation for the intensity correlation function $c_N(t)$,

$$c_N(t) = \tilde{c}_N(t) + \int_0^t d\tau\, \tilde{c}_N(t-\tau) c_N(\tau) . \tag{A3}$$

The Laplace transform of Eq. (A3) used in Eq. (4.16) is then found to be

$$\hat{c}(s) = \frac{1}{(1 + sT/N)^N - 1} . \tag{A4}$$

APPENDIX B

Here we wish to calculate the pump-field-averaged two-time intensity correlation function, defined as [cf. Eq. (3.18)]

$$g(t,t') = \mathrm{Tr}_{P+F}[\hat{I} e^{(\Lambda + L + J\hat{u})(t-t')} \hat{I} e^{(\Lambda + L + J\hat{u})t'} \times \rho_F(0) \otimes \rho_P(0)] , \tag{B1}$$

where $\hat{I}W = aWa^\dagger$ is a Liouville operator in $\mathcal{H}_F \otimes \mathcal{H}_P$. Let us introduce a pseudodensity matrix $\overline{W}$

$$\overline{W}(t) = e^{(\Lambda + L + J\hat{u})(t-t')} \hat{I} W(t') \quad \text{for } t \geq t' \tag{B2}$$

satisfying Eq. (3.7) for $t \geq t'$ with the "initial condition" $\overline{W}(t') = \hat{I} W(t')$. In particular, for Eq. (2.8) considered master equation in the strong-saturation limit with $\beta = \mathrm{const} = \beta_c$, we get Eq. (3.8) for $\overline{W}_n(t)$ with the initial condition $\overline{W}_n(t') = (n+1) W_{n+1}(t')$. These equations can again (analogous to Sec. IV B) be solved by means of a generating function

$$\overline{G}(z,t) = \sum_{n=0}^{\infty} z^n \overline{W}_n(t) ,$$

satisfying the differential equation (4.6) with initial condition $\overline{G}(z,t') = (\partial/\partial z) G(z,t')$. Thus we can write

$$g(t,t') = \mathrm{Tr}_{F+P}[\hat{I} \overline{W}(t)] = \mathrm{Tr}_P \left[\left. \frac{\partial \overline{G}}{\partial z} \right|_{z=1} \right] . \tag{B3}$$

From the differential equation for $\overline{G}$ one derives in the stationary limit $t' \rightarrow \infty$

$$\left[\frac{d}{d\tau}+\kappa\right]g(\tau)=\mathrm{Tr}_P[Je^{\Lambda\tau}(\kappa-\Lambda)^{-1}J\rho_P] \quad \text{(B4)}$$

with $g(0)=\langle n(n-1)\rangle$. Taking the Laplace transform of this equation and inserting the explicit expressions of Appendix A, one finds

$$\hat{g}(s)=\frac{1}{s+\kappa}\{\langle n(n-1)\rangle + \mathrm{Tr}_P[J(s-\Lambda)^{-1}(\kappa-\Lambda)^{-1}J\rho_P]\} = \frac{\langle n\rangle}{s^2-\kappa^2}[s\hat{c}(\kappa)-\kappa\hat{c}(s)] \; . \quad \text{(B5)}$$

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