Quantum theory of two-photon correlated-spontaneous-emission lasers: Exact atom-field interaction Hamiltonian approach

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A quantum theory of two-photon correlated-spontaneous-emission lasers (CEL's) is developed, starting from the exact atom-field interaction Hamiltonian for cascade three-level atoms interacting with a single-mode radiation field. We consider the situation where the active atoms are prepared initially in a coherent superposition of three atomic levels and derive a master equation for the field-density operator by using a quantum theory for coherently pumped lasers. The master equation is transformed into a Fokker-Planck equation for the antinormal-ordering Q function. The drift coefficients of the Fokker-Planck equation enable us to study the steady-state operation of the two-photon CEL's analytically. We have studied both resonant two-photon CEL for which there is no threshold, and off-resonant two-photon CEL for which there exists a threshold. In both cases the initial atomic coherences provide phase locking, and squeezing in the phase quadrature of the field is found. The off-resonant two-photon CEL can build up from a vacuum when its linear gain is larger than the cavity loss (even without population inversion). Maximum squeezing is found in the no-population-inversion region with the laser intensities far below saturation in both cases, which are more than 90% for the resonant two-photon CEL and nearly 50% for the off-resonant one. Approximate steady-state Q functions are obtained for the resonant two-photon CEL and, in certain circumstances, for the off-resonant one.

I. INTRODUCTION

The studies of the generation and properties of the squeezed states of light have attacked great interest in the last few years following successful experimental realization in several laboratories.[1] It is very appealing to find new optical devices that can generate bright squeezed light, since the coherent amplitudes of the squeezed light produced by the typical parametric down-conversion and four-wave-mixing processes, etc. are small. Lasers (operated above threshold) may be such candidates. In fact, a two-photon laser was proposed to produce squeezed light during the early stage of studying the squeezed states.[2] The possibility of squeezing in an ordinary two-photon laser, however, was later carefully examined and ruled out[3,4] due to phase-insensitive spontaneous emission. On the other hand, Scully *et al.* have shown,[5] in a linear theory of two-photon correlated-spontaneous-emission lasers (CEL's), that such new two-photon lasers can generate bright light with phase squeezing.

The two-photon CEL's consist of cascade "three-level" atoms interacting with a single field mode with the atoms initially prepared in a coherent superposition of the three levels. With both one- and two-photon resonances the intracavity field of the (resonant) two-photon CEL can be near perfectly squeezed in the phase quadrature.[5] With (1) two-photon resonance but large detunings for the one-photon transitions, and (2) initial atomic coherence between the top and bottom levels only, the (off-resonant) two-photon CEL can exhibit net linear gain and phase squeezing simultaneously, in which a maximum of 50% squeezing can be approached in the two-photon transition limit.[5] In contrast to an ordinary two-photon laser in which atoms are incoherently pumped to the top level, the spontaneous emission fluctuations in the two-photon CEL's are phase sensitive owing to the presence of the initial atomic coherences. Compared with the ordinary two-photon laser, the linear theory of the two-photon CEL's shows that the quantum fluctuations in the two-photon CEL's are decreased in one quadrature of the field but increased by the same percentage in the other quadrature. The phase locking in the two-photon CEL's, also due to the initial atomic coherences, determines whose noise is reduced in the steady state.

In this paper we develop an all-order nonlinear (quantum) theory of the two-photon CEL's starting from an exact atom-field interaction Hamiltonian. The nonlinear theory of an ordinary (single-mode) two-photon laser has been studied theoretically for many years,[2–4,6–13] but with few experimental realizations.[14,15] Recently, Brune and co-workers studied the problem of a two-photon micromaser theoretically,[16,17] with a successful experimental demonstration.[18] Most of these theoretical treatments assumed large one-photon detunings and ended up, sooner or later, with the use of an effective atom-field interaction Hamiltonian for a two-level system coupled by a two-photon transition. Zhu and Li[13] treated the problem of the ordinary two-photon laser by starting from an exact atom-field interaction Hamiltonian. As to the two-photon CEL, an all-order nonlinear theory beginning with the effective atom-field interaction Hamiltonian for the two-photon transition has been developed recently,[19] which is simple and predicts the behavior and noise properties of the off-resonant two-photon CEL, from below threshold to near threshold and to far above threshold, in the two-photon transition limit. Another fourth-order nonlinear theory starting from the exact interaction Hamiltonian has also been formulated,[20] which, valid near the threshold, focuses on the properties of the off-resonant two-photon CEL in the two-photon transition limit too.

In this work we study the steady-state operation and noise properties of both resonant and off-resonant two-photon CEL's by using the exact interaction Hamiltonian, as in Ref. 5. This enables us to formulate the quantum theory of the two-photon CEL's in a unified approach and to study the effects of one-photon detunings on the off-resonant two-photon CEL. Compared to the linear theory of Ref. 5, our nonlinear theory developed in this paper is capable of studying the squeezing in the phase quadrature in terms of the field's Hermitian-quadrature operators directly. We show that the laser field will build up from a vacuum without triggering and become stable if the linear gain of the off-resonant two-photon CEL is larger than the cavity loss. The largest squeezings in the phase quadrature are found in the no-population-inversion region when the laser intensities are far below saturation. Approximate Q (antinormal-ordering) functions[2,21,22] are obtained for the off-resonant two-photon CEL either near threshold or in a two-photon transition limit, and for the resonant two-photon CEL.

The organization of the paper is as follows. In Sec. II, by making use of a recently developed quantum theory of coherently pumped lasers,[23] we derive the master equation for the reduced field-density operator. In Sec. III we convert the master equation into a Fokker-Planck equation for the Q function. We study the resonant and off-resonant two-photon CEL's in Secs. IV and V, respectively. We discuss the steady-state operation of the lasers and give general expressions for the laser intensities, mode pulling, the quadrature variances, and the steady-state solutions of the Q functions. In Sec. VI we compare in detail the results for the off-resonant two-photon CEL with those in Ref. 19 and discuss the validity of the effective interaction Hamiltonian. Finally, we summarize our results in Sec. VII.

II. MASTER EQUATION

We consider cascade three-level atoms interacting with a single mode of radiation field in a laser cavity (see Fig. 1). The top level a and the bottom level c are of the same parity, which is opposite to that of the middle level b. The energy of level A ($A=a,b,c$) is $\hbar\omega_A$. The upper atomic transition a-b and the lower one b-c interact with the same mode of the cavity field. We consider the situation where the active three-level atoms are prepared initially in a coherent superposition of the three atomic levels a, b, and c. We will use the exact atom-field interaction Hamiltonian to study such a coherently pumped (single-mode) two-photon laser.

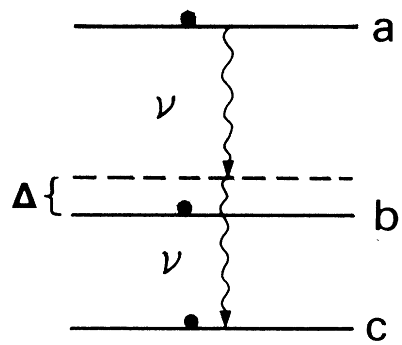

FIG. 1. Energy-level diagram for two-photon CEL's. Atoms are prepared initially in a coherent superposition of levels a, b, and c.

The quantum theory of a coherently pumped laser[23] has been developed recently by properly generalizing the Scully-Lamb theory of lasers.[24] It can be summarized into two basic equations of motion. One governs the reduced field-density operator ρ for the (single-mode) laser field in the interaction picture,

$$\dot\rho = -i(\Omega-\nu)[a^\dagger a,\rho] - i\sum_j \Theta(t-t_j)\mathrm{Tr}_{A_j}[\tilde V_j,\tilde\rho_j^f] + \tfrac{1}{2}\gamma(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a)\ . \tag{2.1}$$

The other treats the reduced density operator $\tilde\rho_j^f$ for the jth atom and the field in the interaction picture,

$$\dot{\tilde\rho}_j^f = -i\Theta(t-t_j)[\tilde V_j,\tilde\rho_j^f] - \tfrac{1}{2}(\Gamma^j\tilde\rho_j^f + \tilde\rho_j^f\Gamma^j)\ , \tag{2.2}$$

as in the Scully-Lamb theory of lasers. In Eqs. (2.1) and (2.2), Ω is the cavity-mode frequency, ν the actual laser frequency, a ($a^\dagger$) the field annihilation (creation) operator, γ the cavity-loss rate, t_j the injection time of the jth atom (assumed to be random), $\Theta(t-t_j)$ the unit step function: $\Theta(t-t_j)=1$ for $t\ge t_j$ and $\Theta(t-t_j)=0$ for $t<t_j$, and

$$\Gamma^j = \sum_{A=a,b,c}\Gamma_A |A^j\rangle\langle A^j| \tag{2.3}$$

the decay operator for the jth atom, where Γ_A is the decay rate of the atomic level A. Also $\hbar\tilde V_j$ is the interaction Hamiltonian of the jth atom with the laser field in the interaction picture, which is obtained from that in the Schrödinger picture $\hbar V_j$ through the unitary transformation

$$\tilde V_j = \exp[i\nu a^\dagger a t + iH_j^{\mathrm{at}}(t-t_j)]V_j \times \exp[-i\nu a^\dagger a t - iH_j^{\mathrm{at}}(t-t_j)]\ , \tag{2.4}$$

with $\hbar H_j^{\mathrm{at}} = \sum_A \hbar\omega_A |A^j\rangle\langle A^j|$ being the free Hamiltonian of the jth atom. Summation over the randomly injected atoms in Eq. (2.1) can be replaced by an integral over the injection time t_j, $\sum_j \Theta(t-t_j) \to r_a\int_{-\infty}^{t} dt_j$, where r_a is the atomic injection rate.

For a (single-mode) two-photon laser (see Fig. 1) the interaction Hamiltonian $\hbar V_j$ in the Schrödinger picture is

(under the rotating-wave approximation)

$$V_j=(g_1|a^j\rangle\langle b^j|+g_2|b^j\rangle\langle c^j|)a+\text{H.c.}\ , \tag{2.5}$$

where g_1 and g_2 are the atom-field coupling constants for the a-b and b-c transitions, respectively, and are chosen to be real. Applying the transformation (2.4) to the two-photon laser, the interaction Hamiltonian $\hbar\tilde{V}_j$ in the interaction picture is found to be

$$\tilde{V}_j=[g_1|a^j\rangle\langle b^j|e^{i(\Delta_1 t-\omega_{ab}t_j)}+g_2|b^j\rangle\langle c^j|e^{i(\Delta_2 t-\omega_{bc}t_j)}]a+\text{H.c.}\ , \tag{2.6}$$

where

$$\Delta_1=\omega_{ab}-\nu=\omega_a-\omega_b-\nu\ , \tag{2.7a}$$

$$\Delta_2=\omega_{bc}-\nu=\omega_b-\omega_c-\nu \tag{2.7b}$$

are the one-photon detunings for the upper and lower transitions, respectively. Substituting Eq. (2.6) into Eq. (2.1) one finds the equations of motion for the matrix elements of the reduced field-density operator,

$$\begin{aligned}\dot{\rho}_{nm}=&-i(\Omega-\nu)(n-m)\rho_{nm}-ir_a\int_{-\infty}^{t}dt_j[g_1e^{i(\Delta_1 t-\omega_{ab}t_j)}(\sqrt{n+1}\tilde{\rho}^{\,j}_{bn+1,am}-\sqrt{m}\,\tilde{\rho}^{\,j}_{bn,am-1})\\&+g_1e^{-i(\Delta_1 t-\omega_{ab}t_j)}(\sqrt{n}\,\tilde{\rho}^{\,j}_{an-1,bm}-\sqrt{m+1}\tilde{\rho}^{\,j}_{an,bm+1})\\&+g_2e^{i(\Delta_2 t-\omega_{bc}t_j)}(\sqrt{n+1}\tilde{\rho}^{\,j}_{cn+1,bm}-\sqrt{m}\,\tilde{\rho}^{\,j}_{cn,bm-1})\\&+g_2e^{-i(\Delta_2 t-\omega_{bc}t_j)}(\sqrt{n}\,\tilde{\rho}^{\,j}_{bn-1,cm}-\sqrt{m+1}\tilde{\rho}^{\,j}_{bn,cm+1})]\\&+\gamma(n+1)^{1/2}(m+1)^{1/2}\rho_{n+1,m+1}-\tfrac{1}{2}\gamma(n+m)\rho_{nm}\ ,\end{aligned} \tag{2.8}$$

where $\tilde{\rho}^{\,j}_{An,A'm}$ $(A,A'=a,b,c)$ are the matrix elements of the density operator $\tilde{\rho}^{\,f}_j$ to be calculated from Eq. (2.2). A simple way to accomplish this is to introduce A^j_n such that

$$\tilde{\rho}^{\,j}_{An,A'm}=A^j_n(A'^j_m)^*,\quad A,A'=a,b,c\ . \tag{2.9}$$

The equations of motion for A^j_n are found by substituting Eqs. (2.3), (2.6), and (2.9) into Eq. (2.2),

$$\dot{a}^{\,j}_n=-\Gamma_a a^j_n-ig_1\sqrt{n+1}e^{i(\Delta_1 t-\omega_{ab}t_j)}b^j_{n+1}\ , \tag{2.10a}$$

$$\dot{b}^{\,j}_{n+1}=-\Gamma_b b^j_{n+1}-ig_1\sqrt{n+1}e^{-i(\Delta_1 t-\omega_{ab}t_j)}a^j_n-ig_2\sqrt{n+2}e^{i(\Delta_2 t-\omega_{bc}t_j)}c^j_{n+2}\ , \tag{2.10b}$$

$$\dot{c}^{\,j}_{n+2}=-\Gamma_c c^j_{n+2}-ig_2\sqrt{n+2}e^{-i(\Delta_2 t-\omega_{bc}t_j)}b^j_{n+1}\ , \tag{2.10c}$$

for $t\geq t_j$.

For simplicity, we consider the actual two-photon resonance, $\omega_a-\omega_c=2\nu$ (i.e., $\Delta_1=-\Delta_2\equiv\Delta$), and equal atomic decay rates, $\Gamma_a=\Gamma_b=\Gamma_c\equiv\Gamma$, in the following. The solutions of Eqs. (2.10), in terms of initial conditions $A^j_n(t_j)$, are then found to be

$$\begin{aligned}a^j_n(t)=e^{-\Gamma\tau/2}\{&(\chi^2_{1n}+\chi^2_{2n})^{-1}[\chi^2_{1n}e^{i\Delta\tau/2}(\cos y_n-i\Delta\Omega_n^{-1}\sin y_n)+\chi^2_{2n}]a^j_n(t_j)-i\chi_{1n}\Omega_n^{-1}e^{i\Delta\tau/2}\sin y_n b^j_{n+1}(t_j)\\&+\chi_{1n}\chi_{2n}(\chi^2_{1n}+\chi^2_{2n})^{-1}[e^{i\Delta\tau/2}(\cos y_n-i\Delta\Omega_n^{-1}\sin y_n)-1]c^j_{n+2}(t_j)\}\ ,\end{aligned} \tag{2.11a}$$

$$\begin{aligned}b^j_{n+1}(t)=e^{-\Gamma\tau/2}[&-i\chi_{1n}\Omega_n^{-1}e^{-i\Delta\tau/2}\sin y_n a^j_n(t_j)+(\cos y_n+i\Delta\Omega_n^{-1}\sin y_n)b^j_{n+1}(t_j)\\&-i\chi_{2n}\Omega_n^{-1}e^{-i\Delta\tau/2}\sin y_n c^j_{n+2}(t_j)]\ ,\end{aligned} \tag{2.11b}$$

$$\begin{aligned}c^j_{n+2}(t)=e^{-\Gamma\tau/2}\{&\chi_{1n}\chi_{2n}(\chi^2_{1n}+\chi^2_{2n})^{-1}[e^{i\Delta\tau/2}(\cos y_n-i\Delta\Omega_n^{-1}\sin y_n)-1]a^j_n(t_j)-i\chi_{2n}\Omega_n^{-1}e^{i\Delta\tau/2}\sin y_n b^j_{n+1}(t_j)\\&+(\chi^2_{1n}+\chi^2_{2n})^{-1}[\chi^2_{1n}+\chi^2_{2n}e^{i\Delta\tau/2}(\cos y_n-i\Delta\Omega_n^{-1}\sin y_n)]c^j_{n+2}(t_j)\}\ ,\end{aligned} \tag{2.11c}$$

where

$$\begin{aligned}&\chi_{1n}=2g_1\sqrt{n+1}\ ,\\&\chi_{2n}=2g_2\sqrt{n+2}\ ,\\&\Omega_n=(\chi^2_{1n}+\chi^2_{2n}+\Delta^2)^{1/2}\ ,\\&\tau=t-t_j,\quad y_n=\tfrac{1}{2}\Omega_n\tau\ .\end{aligned} \tag{2.12}$$

Substitution of Eqs. (2.11) into Eqs. (2.9) leads to $\tilde{\rho}^{\,j}_{An,A'm}(t)$ expressed in terms of the matrix elements of the initial density operator

$$\tilde{\rho}^{\,f}_j(t_j)=\rho(t_j)\otimes\rho^j(t_j)\ , \tag{2.13}$$

since the jth atom is injected at time t_j. Here $\rho^j(t_j)$ is the initial density operator for the jth atom in the Schrödinger picture. For the two-photon CEL we are interested in the following general form:

$$\rho^j(t_j)=\begin{bmatrix}\rho_{aa} & \bar{\rho}_{ab}e^{-i\nu t_j} & \bar{\rho}_{ac}e^{-i2\nu t_j}\\ \bar{\rho}_{ba}e^{i\nu t_j} & \rho_{bb} & \bar{\rho}_{bc}e^{-i\nu t_j}\\ \bar{\rho}_{ca}e^{i2\nu t_j} & \bar{\rho}_{cb}e^{i\nu t_j} & \rho_{cc}\end{bmatrix}, \quad j=1,2,\ldots, \tag{2.14}$$

where ρ_{aa}, ρ_{bb}, ρ_{cc}, $\bar{\rho}_{ab}=\bar{\rho}^*_{ba}$, $\bar{\rho}_{bc}=\bar{\rho}^*_{cb}$, and $\bar{\rho}_{ac}=\bar{\rho}^*_{ca}$ are the same for all atoms.

In the good-cavity limit $\gamma \ll \Gamma$, for which the laser field does not change appreciably on a time scale of atomic lifetime, one can obtain the coarse-grained time rate of change for the field operator ρ from Eq. (2.8) by using an approximation $\rho(t_j)\approx\rho(t)$. Such an approximation is all right when $t_j \geq t-\Gamma^{-1}$. It is also allowed when $t_j < t-\Gamma^{-1}$, since $\tilde{\rho}^f_j$ decays with an overall factor $e^{-\Gamma(t-t_j)} \ll 1$. Substituting Eqs. (2.9) and (2.11)–(2.14) into Eqs. (2.8) and accomplishing the integral over t_j, we find the master equation for the laser field (assuming $g_1=g_2\equiv g$),

$$\begin{aligned}\dot{\rho}_{nm}=&\alpha\rho_{aa}\{-\tfrac{1}{2}\rho_{nm}[(n+2)(m+1)\mu^{-1}_{nm}+(n+1)(m+2)\epsilon^{-1}_{nm}+2(n+1)(m+1)\kappa_{nm}\sigma^{-1}_{nm}]+\rho_{n-1,m-1}\sqrt{nm}\,\xi^{-1}_{n-1,m-1}\\
&\quad+\tfrac{1}{2}\rho_{n-2,m-2}[n(n-1)m(m-1)]^{1/2}(\mu^{-1}_{n-2,m-2}+\epsilon^{-1}_{n-2,m-2}-2\kappa_{n-2,m-2}\sigma^{-1}_{n-2,m-2})\}\\
&+\alpha\rho_{bb}[\rho_{n+1,m+1}\sqrt{(n+1)(m+1)}\xi^{-1}_{nm}-\rho_{nm}\kappa_{m-1,n-1}\xi^{-1}_{n-1,m-1}+\rho_{n-1,m-1}\sqrt{nm}\,\xi^{-1}_{n-2,m-2}]\\
&+\alpha\rho_{cc}\{\tfrac{1}{2}\rho_{n+2,m+2}[(n+1)(n+2)(m+1)(m+2)]^{1/2}(\mu^{-1}_{nm}+\epsilon^{-1}_{nm}-2\kappa_{nm}\sigma^{-1}_{nm})\\
&\quad+\rho_{n+1,m+1}(n+1)^{1/2}(m+1)^{1/2}\xi^{-1}_{n-1,m-1}\\
&\quad-\tfrac{1}{2}\rho_{nm}[(n-1)m\mu^{-1}_{n-2,m-2}+n(m-1)\epsilon^{-1}_{n-2,m-2}+2nm\kappa_{n-2,m-2}\sigma^{-1}_{n-2,m-2}]\}\\
&+(iS\bar{\rho}_{ab}\{\rho_{n,m+1}\sqrt{m+1}[(n+1)(2n+3)^{-1}\eta_{mn}\xi^{-1}_{nm}+(n+2)\mu^{-1}_{nm}]-\rho_{n-1,m}\sqrt{n}\,\eta_{nm}\xi^{-1}_{n-1,m-1}\\
&\quad+\rho_{n-2,m-1}\sqrt{n(n-1)m}\,[\eta_{mn}(2n-1)^{-1}\xi^{-1}_{n-2,m-2}-\mu^{-1}_{n-2,m-2}]\}\\
&-iS\bar{\rho}_{bc}\{\rho_{n+1,m+2}[(n+1)(m+1)(m+2)]^{1/2}[\eta^*_{nm}(2m+3)^{-1}\xi^{-1}_{nm}-\epsilon^{-1}_{nm}]-\rho_{n,m+1}\sqrt{m+1}\eta^*_{mn}\xi^{-1}_{n-1,m-1}\\
&\quad+\rho_{n-1,m}\sqrt{n}\,[(m-1)\epsilon^{-1}_{n-2,m-2}+m(2m-1)^{-1}\eta^*_{nm}\xi^{-1}_{n-2,m-2}]\}\\
&+\alpha\bar{\rho}_{ac}\{\tfrac{1}{2}\rho_{n,m+2}(m+1)^{1/2}(m+2)^{1/2}[(n+1)\epsilon^{-1}_{nm}-(n+2)\mu^{-1}_{nm}-2(n+1)\kappa_{nm}\sigma^{-1}_{nm}]\\
&\quad+\rho_{n-1,m+1}\sqrt{n(m+1)}\xi^{-1}_{n-1,m-1}\\
&\quad+\tfrac{1}{2}\rho_{n-2,m}\sqrt{n(n-1)}[m\mu^{-1}_{n-2,m-2}-(m-1)\epsilon^{-1}_{n-2,m-2}-m\kappa_{n-2,m-2}\sigma^{-1}_{n-2,m-2}]\}+(\text{c.c.})_{n\leftrightarrow m})\\
&-i(\Omega-\nu)(n-m)\rho_{nm}+\gamma\sqrt{(n+1)(m+1)}\rho_{n+1,m+1}-\tfrac{1}{2}\gamma(n+m)\rho_{nm}\ ,\end{aligned} \tag{2.15}$$

with

$$\xi_{nm}=1+\delta^2+(n+m+3)\beta/\alpha+(n-m)^2\beta^2/4\alpha^2\ , \tag{2.16a}$$

$$\sigma_{nm}=(2n+3)(2m+3)\xi_{nm}\ , \tag{2.16b}$$

$$\mu_{nm}=(2n+3)[1+i\delta+(2m+3)\beta/4\alpha]\ , \tag{2.16c}$$

$$\epsilon_{nm}=(2m+3)[1-i\delta+(2n+3)\beta/4\alpha]\ , \tag{2.16d}$$

$$\kappa_{nm}=n+m+3+(n-m)^2\beta/2\alpha+i\delta(n-m)\ , \tag{2.16e}$$

$$\eta_{nm}=1-i\delta+(n-m)\beta/2\alpha\ , \tag{2.16f}$$

where

$$\alpha=\frac{2r_a g^2}{\Gamma^2},\quad \beta=\frac{8r_a g^4}{\Gamma^4},\quad S=\frac{r_a g}{\Gamma},\quad \delta=\frac{\Delta}{\Gamma} \tag{2.17}$$

are a linear-gain coefficient, saturation parameter, driving-force parameter, and normalized one-photon detuning, respectively. Also, $(\text{c.c.})_{n\leftrightarrow m}$ denotes complex-conjugate terms with n and m interchanged. Alternatively, S can be written as

$$S=\alpha\sqrt{\alpha/\beta} \tag{2.18}$$

according to Eqs. (2.17). The master equation for an ordinary two-photon laser, in which atoms are incoherently pumped to the top level a, is a special case of Eq. (2.15) with $\rho_{aa}=1$ and all other $\rho_{AA'}=0$.

The photon statistics of the two-photon CEL may be studied from the equations of motion for the diagonal elements ρ_{nn} of the field-density operator, which can be obtained from Eq. (2.15) by setting $m=n$. Besides the usual diagonal couplings among ρ_{nn}, $\rho_{n+1,n+1}$, and $\rho_{n+2,n+2}$ as in an ordinary two-photon laser, there are additional couplings to off-diagonal density-matrix elements $\rho_{n,n+1}=\rho^*_{n+1,n}$ and $\rho_{n,n+2}=\rho^*_{n+2,n}$. The appearance of such couplings makes a direct investigation [i.e., using the master equation (2.15)] for the steady-state operation, photon statistics, and phase noise, etc. very difficult. An easy way to accomplish these is to transform the master equation (2.15) into a Fokker-Planck equation for a c-number representation of the field operator ρ. We do this in Sec. III.

III. FOKKER-PLANCK EQUATION FOR THE Q FUNCTION

For squeezed states of light the normal-ordering Glauber-Sudarshan P function[25,26] can no longer be interpreted as a quasiprobability distribution, since it is no longer positive definite. The positive P function,[27] which is a four-dimensional normal-ordering function, is suitable for representing squeezed states of light.[28] However, it is not convenient to convert nonlinear master equations for laser fields into Fokker-Planck equations for the positive P functions. On the other hand, the Q function,[2,21,22] which is a two-dimensional antinormal-ordering function, is a quasiprobability distribution even when squeezing occurs, and it is easy to transform a nonlinear master equation into a Fokker-Planck equation for the Q function.

To study squeezing in the two-photon CEL we will use the Q function to represent the field-density operator ρ in this paper, as in Ref. 19. Since the derivation of the Fokker-Planck equation for the Q function in this paper is very similar to that in Ref. 19, we will only outline the derivation here. First, we write the Q function in terms of the field-density matrix elements $\rho_{nm}=\langle n|\rho|m\rangle$,

$$Q(\mathcal{E},\mathcal{E}^*)=\pi^{-1}\sum_{n,m=0}^{\infty} e^{-|\mathcal{E}|^2}\frac{(\mathcal{E}^*)^n\mathcal{E}^m}{\sqrt{n!m!}}\rho_{nm} . \tag{3.1}$$

Then by taking the time derivative on both sides of Eq. (3.1) and substituting the master equation (2.15) into Eq. (3.1), we obtain an equation of motion for the Q function which contains derivatives with respect to $\mathcal{E}$ and $\mathcal{E}^*$ in both numerators and denominators of its various terms.[19,23,29] The Fokker-Planck equation for the Q function is obtained by expanding the equation in terms of the derivatives and keeping terms up to second order in the derivatives. Note that the terms containing third- and higher-order derivatives do not affect the first and second moments of the field, which are the quantities we are going to calculate in this work. Assuming that the average photon number of the two-photon CEL is much larger than 1, we can safely neglect 1 compared to $|\mathcal{E}|^2$ in the Q's equation of motion. After some lengthy calculations the Fokker-Planck equation for the Q function is finally found to be

$$\frac{\partial}{\partial t}Q(\mathcal{E},\mathcal{E}^*,t)=\left[-\frac{\partial}{\partial\mathcal{E}}d_{\mathcal{E}}+\frac{\partial^2}{\partial\mathcal{E}\,\partial\mathcal{E}^*}D_{\mathcal{E}\mathcal{E}^*}+\frac{\partial^2}{\partial\mathcal{E}^2}D_{\mathcal{E}\mathcal{E}}+\text{c.c.}\right]Q(\mathcal{E},\mathcal{E}^*,t) , \tag{3.2}$$

with the drift coefficient

$$\begin{aligned} d_{\mathcal{E}}=\frac{\alpha\mathcal{E}}{2}&\left[\frac{(\rho_{aa}-\rho_{cc})(1+|\mathcal{E}|^2\beta/2\alpha)+(i\delta\bar{\rho}_{ca}\mathcal{E}/\mathcal{E}^*+\text{c.c.})}{(1+|\mathcal{E}|^2\beta/2\alpha)^2+\delta^2}+i\delta\frac{2\rho_{bb}-\rho_{aa}-\rho_{cc}-(\bar{\rho}_{ca}\mathcal{E}/\mathcal{E}^*+\text{c.c.})}{1+\delta^2+2|\mathcal{E}|^2\beta/\alpha}\right]\\ &-\frac{iS\mathcal{E}}{2}\left[\frac{(\bar{\rho}_{ab}/\mathcal{E}+\bar{\rho}_{cb}/\mathcal{E}^*)(1-i\delta+2|\mathcal{E}|^2\beta/\alpha)}{1+\delta^2+2|\mathcal{E}|^2\beta/\alpha}+\text{c.c.}\right]\\ &-\frac{S\mathcal{E}}{2}\left[\frac{i(\bar{\rho}_{ab}/\mathcal{E}-\bar{\rho}_{cb}/\mathcal{E}^*)(1-i\delta+|\mathcal{E}|^2\beta/2\alpha)}{(1+|\mathcal{E}|^2\beta/2\alpha)^2+\delta^2}+\text{c.c.}\right]+[i(\nu-\Omega)-\tfrac{1}{2}\gamma]\mathcal{E} , \end{aligned} \tag{3.3}$$

and the diffusion coefficients

$$\begin{aligned} D_{\mathcal{E}\mathcal{E}^*}=\frac{\alpha}{4}&\left[\frac{2\rho_{cc}(1+|\mathcal{E}|^2\beta/2\alpha)+(i\delta\bar{\rho}_{ac}\mathcal{E}^*/\mathcal{E}+\text{c.c.})}{(1+|\mathcal{E}|^2\beta/2\alpha)^2+\delta^2}\right.\\ &+\frac{|\mathcal{E}|^2\beta}{2\alpha}\frac{(\rho_{aa}-\rho_{cc})[(1+|\mathcal{E}|^2\beta/2\alpha)^2-\delta^2]+2(i\delta\bar{\rho}_{ca}\mathcal{E}/\mathcal{E}^*+\text{c.c.})(1+|\mathcal{E}|^2\beta/2\alpha)}{[(1+|\mathcal{E}|^2\beta/2\alpha)^2+\delta^2]^2}\\ &\left.+\frac{2\rho_{bb}+(\rho_{aa}+2\rho_{bb}+\rho_{cc})|\mathcal{E}|^2\beta/\alpha+[(i\delta+|\mathcal{E}|^2\beta/\alpha)\bar{\rho}_{ca}\mathcal{E}/\mathcal{E}^*+\text{c.c.}]}{1+\delta^2+2|\mathcal{E}|^2\beta/\alpha}\right]\\ &+\left[\frac{iS}{4}\left[\frac{\bar{\rho}_{ba}\mathcal{E}\beta/\alpha+(\delta^2-i\delta+|\mathcal{E}|^2\beta/\alpha)\bar{\rho}_{bc}/\mathcal{E}}{1+\delta^2+2|\mathcal{E}|^2\beta/\alpha}\right.\right.\\ &\left.\left.+\frac{\bar{\rho}_{ba}\mathcal{E}\beta/2\alpha+[\delta^2+i\delta+(i\delta-1)|\mathcal{E}|^2\beta/2\alpha-|\mathcal{E}|^4\beta^2/4\alpha^2]\bar{\rho}_{bc}/\mathcal{E}}{(1-i\delta+|\mathcal{E}|^2\beta/2\alpha)^2}\right]+\text{c.c.}\right]+\tfrac{1}{2}\gamma , \end{aligned} \tag{3.4a}$$

$$
\begin{aligned}
D_{\mathcal{E}\mathcal{E}}=\frac{\alpha\mathcal{E}^2}{4}\Bigg[& \frac{(\rho_{aa}+\rho_{cc})(1+|\mathcal{E}|^2\beta/2\alpha)+i\delta(\rho_{aa}-\rho_{cc})|\mathcal{E}|^{-2}+[(1+i\delta)\bar{\rho}_{ca}(\mathcal{E}^*)^{-2}-\text{c.c.}]}{(1+|\mathcal{E}|^2\beta/2\alpha)^2+\delta^2} \\
& +\frac{(\rho_{aa}-\rho_{cc})\beta/2\alpha}{(1+i\delta+|\mathcal{E}|^2\beta/2\alpha)^2}+\frac{i\delta(1-i\delta+|\mathcal{E}|^2\beta/2\alpha)[\bar{\rho}_{ca}(\mathcal{E}^*)^{-2}-\text{c.c.}]|\mathcal{E}|^2\beta/\alpha}{[(1+|\mathcal{E}|^2\beta/2\alpha)^2+\delta^2]^2} \\
& +\frac{(1-i\delta+|\mathcal{E}|^2\beta/\alpha)(\bar{\rho}_{ac}\mathcal{E}^{-2}+\rho_{cc}|\mathcal{E}|^{-2})-2\rho_{bb}\beta/\alpha+(1+i\delta+|\mathcal{E}|^2\beta/\alpha)[\rho_{aa}|\mathcal{E}|^{-2}+\bar{\rho}_{ca}(\mathcal{E}^*)^{-2}]}{1+\delta^2+2|\mathcal{E}|^2\beta/\alpha} \\
& -\frac{2i\beta\delta}{\alpha}\frac{\rho_{aa}+\rho_{cc}-2\rho_{bb}+(\bar{\rho}_{ac}\mathcal{E}^*/\mathcal{E}+\text{c.c.})}{(1+\delta^2+2|\mathcal{E}|^2\beta/\alpha)^2}\Bigg] \\
+\frac{iS\mathcal{E}\beta}{4\alpha}\Bigg[& \frac{\bar{\rho}_{bc}(1-2i\delta+3\delta^2+2|\mathcal{E}|^2\beta/\alpha)-\bar{\rho}_{ab}(1-2i\delta-\delta^2+2|\mathcal{E}|^2\beta/\alpha)}{(1+\delta^2+2|\mathcal{E}|^2\beta/\alpha)^2} \\
& -\frac{\mathcal{E}}{\mathcal{E}^*}\frac{\bar{\rho}_{cb}(1-2i\delta-\delta^2+2|\mathcal{E}|^2\beta/\alpha)+2\bar{\rho}_{ba}(1+i\delta+2|\mathcal{E}|^2\beta/\alpha)}{(1+\delta^2+2|\mathcal{E}|^2\beta/\alpha)^2}+\frac{\bar{\rho}_{cb}\mathcal{E}/\mathcal{E}^*-\bar{\rho}_{ab}}{(1+i\delta+|\mathcal{E}|^2\beta/2\alpha)^2} \\
& +\frac{\mathcal{E}\bar{\rho}_{ba}}{\mathcal{E}^*(1-i\delta)}\left[\frac{3+i\delta}{1+\delta^2+2|\mathcal{E}|^2\beta/\alpha}-\frac{1}{1-i\delta+|\mathcal{E}|^2\beta/2\alpha}\right]\Bigg] .
\end{aligned}
\tag{3.4b}
$$

Note that the cavity-loss rate enters into the diffusion coefficient $D_{\mathcal{E}\mathcal{E}^*}$ when the antinormal-ordering Q function is used. When $\rho_{aa}=1$ and all other initial atomic variables vanish, the above Fokker-Planck equation reduces to the Q's Fokker-Planck equation for an ordinary (single-mode) two-photon laser.

The phase of the laser field in the two-photon CEL is locked (say, to ϕ_0), in contrast to the situation in an ordinary two-photon laser. To calculate the moments for the amplitude and phase quadratures of the laser field, we introduce Hermitian amplitude- and phase-quadrature operators a_1 and a_2 through the relation

$$a=(a_1+ia_2)e^{i\phi_0} . \tag{3.5}$$

The quadrature operators a_1 and a_2 satisfy the commutation relation $[a_1,a_2]=\frac{1}{2}i$. Corresponding to Eq. (3.5) we also introduce (real) c-number quadrature variables $\mathcal{E}_1$ and $\mathcal{E}_2$ via

$$\mathcal{E}=(\mathcal{E}_1+i\mathcal{E}_2)e^{i\phi_0} . \tag{3.6}$$

Rewriting the Fokker-Planck equation (3.2) in terms of $\mathcal{E}_1$ and $\mathcal{E}_2$, we arrive at

$$
\begin{aligned}
\frac{\partial}{\partial t}Q(\mathcal{E}_1,\mathcal{E}_2,t)=\Bigg[& -\frac{\partial}{\partial\mathcal{E}_1}d_1-\frac{\partial}{\partial\mathcal{E}_2}d_2 \\
& +\frac{\partial^2}{\partial\mathcal{E}_1^2}D_{11}+\frac{\partial^2}{\partial\mathcal{E}_2^2}D_{22} \\
& +2\frac{\partial^2}{\partial\mathcal{E}_1\partial\mathcal{E}_2}D_{12}\Bigg]Q(\mathcal{E}_1,\mathcal{E}_2,t) ,
\end{aligned}
\tag{3.7}
$$

where the new drift and diffusion coefficients are related to the old ones by

$$d_1=\text{Re}(d_{\mathcal{E}}e^{-i\phi_0}) , \tag{3.8a}$$

$$d_2=\text{Im}(d_{\mathcal{E}}e^{-i\phi_0}) , \tag{3.8b}$$

$$D_{jj}=\tfrac{1}{2}[D_{\mathcal{E}\mathcal{E}^*}-(-1)^j\text{Re}(D_{\mathcal{E}\mathcal{E}}e^{-i2\phi_0})], \quad j=1,2 \tag{3.9a}$$

$$D_{12}=\tfrac{1}{2}\text{Im}(D_{\mathcal{E}\mathcal{E}}e^{-i2\phi_0}) . \tag{3.9b}$$

Since the Q function is an antinormal-ordering function, we find, by using Eqs. (3.5) and (3.6), the expectation values of the quadrature operators, $\langle a_j\rangle=\langle\mathcal{E}_j\rangle$ $(j=1,2)$, and quadrature variances,

$$\langle(\Delta a_j)^2\rangle=\langle\vdots(\Delta a_j)^2\vdots\rangle-\tfrac{1}{4}=\langle(\delta\mathcal{E}_j)^2\rangle-\tfrac{1}{4}, \quad j=1,2 \tag{3.10}$$

where $\Delta a_j=a_j-\langle a_j\rangle$, $\delta\mathcal{E}_j=\mathcal{E}_j-\langle\mathcal{E}_j\rangle$, and $\vdots\;\vdots$ denotes antinormal ordering of a and $a^\dagger$. Making use of the Fokker-Planck equation (3.7) we find the equations of motion for the first moments $\langle\mathcal{E}_j\rangle$ of the field,

$$\frac{d}{dt}\langle\mathcal{E}_j\rangle=\langle d_j\rangle, \quad j=1,2 \tag{3.11}$$

and those for antinormally ordered quadrature variances $\langle(\delta\mathcal{E}_j)^2\rangle$ and covariance $\langle\delta\mathcal{E}_1\delta\mathcal{E}_2\rangle$,

$$\frac{d}{dt}\langle(\delta\mathcal{E}_j)^2\rangle=2\langle d_j\delta\mathcal{E}_j\rangle+2\langle D_{jj}\rangle, \quad j=1,2 \tag{3.12a}$$

$$\frac{d}{dt}\langle\delta\mathcal{E}_1\delta\mathcal{E}_2\rangle=\langle d_1\delta\mathcal{E}_2\rangle+\langle d_2\delta\mathcal{E}_1\rangle+2\langle D_{12}\rangle . \tag{3.12b}$$

In the steady state, we have $d/dt=0$. Because of Eqs. (3.11) the steady-state locking point(s) $(\mathcal{E}_{10},\mathcal{E}_{20})$ of the laser field, which represents the maximum point(s) in the steady-state quasiprobability Q function, satisfies the following deterministic equations:

$$d_j(\mathcal{E}_{10},\mathcal{E}_{20})=0, \quad j=1,2 . \tag{3.13}$$

To ensure ϕ_0 being the laser phase, one should have $\mathcal{E}_{20}=0$. Denoting $A_{jk}\equiv\partial d_j(\mathcal{E}_{10},\mathcal{E}_{20})/\partial\mathcal{E}_k$ $(j,k=1,2)$, the condition for a stable locking in the $\mathcal{E}_j$ direction $(j=1,2)$ is

$$A_{jj} < 0 \ , \tag{3.14a}$$

provided

$$A_{12} = 0 \ \text{ or } \ A_{21} = 0 \tag{3.14b}$$

or both. A stable locking point corresponds to a peak (maximum point) in the Q function. Expanding d_j and D_{jk} in Eqs. (3.12) around a stable locking point $(\mathcal{E}_{10}, \mathcal{E}_{20})$ up to first order in $\delta\mathcal{E}_1$ and $\delta\mathcal{E}_2$, one finds that in the steady state $(d/dt=0)$

$$A_{jj}\langle(\delta\mathcal{E}_j)^2\rangle + A_{jk}\langle\delta\mathcal{E}_1\delta\mathcal{E}_2\rangle + D^0_{jj} = 0, \quad k \neq j \tag{3.15a}$$

$$(A_{11}+A_{22})\langle\delta\mathcal{E}_1\delta\mathcal{E}_2\rangle + A_{12}\langle(\delta\mathcal{E}_2)^2\rangle + A_{21}\langle(\delta\mathcal{E}_1)^2\rangle + 2D^0_{12} = 0 \ , \tag{3.15b}$$

where $D^0_{jk} \equiv D_{jk}(\mathcal{E}_{10}, \mathcal{E}_{20})$ $(j,k=1,2)$ are the diffusion coefficients in the steady state. Solving Eqs. (3.15) one can obtain $\langle(\delta\mathcal{E}_j)^2\rangle$ and $\langle\delta\mathcal{E}_1\delta\mathcal{E}_2\rangle$, implying that we only need to know the steady-state diffusion coefficients D^0_{jk} for calculating variances.

In Secs. IV and V we study, separately, two cases for the two-photon CEL: (1) $\delta=0$, $\bar\rho_{ab}\neq 0$, $\bar\rho_{bc}\neq 0$, $\theta_{ab}=\theta_{bc}$ $(\theta_{jk}=\arg\bar\rho_{jk}, \ j,k=a,b,c)$ and (2) $\delta\neq 0$, $\bar\rho_{ac}\neq 0$, $\rho_{bb}=\bar\rho_{ab}=\bar\rho_{bc}=0$. A preliminary discussion for two such cases based on a linear theory of the two-photon CEL has been given in Ref. 5.

IV. RESONANT TWO-PHOTON CEL

When the three atomic levels a, b, and c are equally spaced, $\omega_{ab}=\omega_{bc}$, and $\Omega=\omega_{ab}=\omega_{bc}$, we have both one- and two-photon resonances. From symmetry considerations there is no mode pulling, $\nu=\Omega$, similar to the case of one-photon lasers.[23] Consequently, we have simply $\Delta=\omega_{ab}-\nu=0$. We consider the case where the two initial atomic coherences $\bar\rho_{ab}$ and $\bar\rho_{bc}$ involving the middle level b are nonzero and have the same phase, $\theta_{ab}=\theta_{bc}$ $(=\frac{1}{2}\theta_{ac})$, and let[5]

$$\phi_0 = \theta_{ab} - \tfrac{1}{2}\pi \ . \tag{4.1}$$

Substituting Eq. (3.3) into Eqs. (3.8) one finds the drift coefficients in the amplitude and phase quadratures of the field,

$$d_1 = \tfrac{1}{2}\mathcal{E}_1\{[\alpha(\rho_{aa}-\rho_{cc}) + 2S\mathcal{E}_1^{-1}(|\bar\rho_{ab}|+|\bar\rho_{bc}|)(1+\mathcal{E}_2^2\beta/2\alpha)][1+(\mathcal{E}_1^2+\mathcal{E}_2^2)\beta/2\alpha]^{-1} - \gamma\} \ , \tag{4.2a}$$

$$d_2 = \tfrac{1}{2}\mathcal{E}_2\{[\alpha(\rho_{aa}-\rho_{cc}) - S(|\bar\rho_{ab}|+|\bar\rho_{bc}|)\mathcal{E}_1\beta/\alpha][1+(\mathcal{E}_1^2+\mathcal{E}_2^2)\beta/2\alpha]^{-1} - \gamma\} \ . \tag{4.2b}$$

The initial atomic variables $\bar\rho_{ac}$ and ρ_{bb} do not enter into the drift coefficients due to the resonant atom-field interaction, $\delta=0$. The population difference $\rho_{aa}-\rho_{cc}$ plays the usual role of laser gain. The initial atomic coherences $\bar\rho_{ab}$ and $\bar\rho_{bc}$ involving the middle level b act as a "driving force" to the laser amplitude $\mathcal{E}_1$ so that $\mathcal{E}_{10}>0$. Due to the existence of such a driving force there is no threshold in the resonant two-photon CEL and population inversion is not necessary,[5] in contrast to ordinary lasers. Solving Eqs. (3.13) with d_j given in Eqs. (4.2), we find that in the steady state

$$\frac{\alpha(\rho_{aa}-\rho_{cc}) + 2S(|\bar\rho_{ab}|+|\bar\rho_{bc}|)/\mathcal{E}_{10}}{1+\mathcal{E}_{10}^2\beta/2\alpha} = \gamma, \quad \mathcal{E}_{10} > 0 \tag{4.3a}$$

$$\mathcal{E}_{20} = 0 \ . \tag{4.3b}$$

We now define

$$N_0 = \mathcal{E}_{10}^2\beta/\alpha \ , \tag{4.4}$$

which is a normalized mean photon number and represents the degree of saturation. Using Eqs. (2.18) and (4.4), Eq. (4.3a) can be rearranged to the usual form of a cubic algebraic equation,

$$N_0^{3/2} + 2[1+(\alpha/\gamma)(\rho_{cc}-\rho_{aa})]N_0^{1/2} - 4(\alpha/\gamma)(|\bar\rho_{ab}|+|\bar\rho_{bc}|) = 0 \ , \tag{4.5}$$

which can have either one positive root or one positive and two negative roots. However, only the positive solution can be realized, and is also stable as we are going to show in the following.

It is easy to see from Eqs. (4.2) that both cross derivatives $\partial d_1/\partial\mathcal{E}_2$ and $\partial d_2/\partial\mathcal{E}_1$ vanish at $\mathcal{E}_{20}$, since d_1 is an even function of $\mathcal{E}_2$ and d_2 is proportional to $\mathcal{E}_2$. Namely,

$$A_{12} = A_{21} = 0 \ , \tag{4.6}$$

satisfying Eqs. (3.14b). Making use of Eqs. (4.3), (2.18), and (4.4), we find from Eqs. (4.2) that both self-derivatives $\partial d_j/\partial\mathcal{E}_j$ $(j=1,2)$ are negative at the locking point $\mathcal{E}_{10}>0$, $\mathcal{E}_{20}=0$,

$$A_{11} = -\frac{\gamma N_0 + 2\alpha(|\bar\rho_{ab}|+|\bar\rho_{bc}|)/\sqrt{N_0}}{2+N_0} < 0 \ , \tag{4.7a}$$

$$A_{22} = -\alpha(|\bar\rho_{ab}|+|\bar\rho_{bc}|)/\sqrt{N_0} < 0 \ . \tag{4.7b}$$

Consequently, the locking point $\mathcal{E}_{10}>0$ [from Eq. (4.3a) or (4.5)], $\mathcal{E}_{20}=0$ is stable.

The steady-state diffusion coefficients can be obtained by substituting Eqs. (3.4) into Eqs. (3.9) and using Eqs. (4.3b), (2.18), and (4.4). For example,

$$D^0_{22} = \frac{\gamma}{4} + \frac{\alpha}{4}\left[\frac{\rho_{cc}}{1+N_0/2} + \frac{\rho_{bb}}{1+2N_0} - |\bar\rho_{ac}|\right] + \frac{\alpha N_0}{16}\left[\frac{8\rho_{bb}}{1+2N_0} + \frac{\rho_{aa}+\rho_{cc}}{1+N_0/2}\right] + \frac{\alpha\sqrt{N_0}}{8}\left[\frac{|\bar\rho_{ab}|}{1+N_0/2} + \frac{|\bar\rho_{bc}|(1+N_0)}{(1+N_0/2)^2}\right] , \tag{4.8a}$$

$$D^0_{12}=0 \ . \tag{4.8b}$$

Using Eqs. (4.6), (4.7), and (4.8b) in Eq. (3.15b) we have the vanishing covariance

$$\langle \delta\mathcal{E}_1\delta\mathcal{E}_2\rangle=0 \ . \tag{4.9}$$

Similarly it follows from Eqs. (3.10), (3.15a), (4.6), and (4.7) that the quadrature variances are

$$\langle(\Delta a_j)^2\rangle=\frac{D^0_{jj}}{|A_{jj}|}-\tfrac{1}{4}, \quad j=1,2 \ . \tag{4.10}$$

Substituting Eqs. (4.7b) and (4.8a) into Eq. (4.10) and using Eq. (4.5), we obtain the variance in the phase quadrature

$$\langle(\Delta a_2)^2\rangle=\frac{\rho_{aa}+2\rho_{bb}+\rho_{cc}-2|\bar\rho_{ac}|+\gamma/\alpha+\frac{1}{2}|\bar\rho_{ac}|N_0^{3/2}(1+N_0/2)^{-2}}{4[\rho_{cc}-\rho_{aa}+(1+N_0/2)\gamma/\alpha]} \ . \tag{4.11}$$

When $N_0 \ll 1$, Eq. (4.11) reduces to the result obtained from the linear theory

$$\langle(\Delta a_2)^2\rangle=\frac{\rho_{aa}+2\rho_{bb}+\rho_{cc}-2|\bar\rho_{ac}|+\gamma/\alpha}{4(\rho_{cc}-\rho_{aa}+\gamma/\alpha)} , \tag{4.12}$$

which can be much smaller than $\frac{1}{4}$.[5]

As an example, we consider the case in which the initial atomic populations are

$$\begin{aligned}\rho_{aa}&=\tfrac{1}{2}\{1-[2(2\lambda+1)\gamma/\alpha]^{1/2}-(\lambda-1)\gamma/\alpha\} ,\\ \rho_{bb}&=\lambda\gamma/\alpha ,\\ \rho_{cc}&=\tfrac{1}{2}\{1+[2(2\lambda+1)\gamma/\alpha]^{1/2}-(\lambda+1)\gamma/\alpha\} ,\end{aligned} \tag{4.13a}$$

with coherences

$$|\bar\rho_{jk}|=(\rho_{jj}\rho_{kk})^{1/2}, \quad j,k=a,b,c \ . \tag{4.13b}$$

When $(2\gamma/\alpha)^{1/2} \ll 1$, Eqs. (4.12) gives a linear-theory result (requiring $N_0 \ll 1$ thus $\lambda \ll 1$),

$$\langle(\Delta a_2)^2\rangle\approx\tfrac{1}{4}[2(2\lambda+1)\gamma/\alpha]^{1/2}\ll\tfrac{1}{4} , \tag{4.14}$$

which represents more than 90% squeezing in the a_2 quadrature. In general, one should use expression (4.11) with N_0 determined by substituting the initial conditions (4.13) into Eq. (4.5). The variance $\langle(\Delta a_2)^2\rangle$ is plotted in Fig. 2 as a function of the parameter λ for different values of γ/α. The degree of squeezing decreases with increasing λ.

An approximate steady-state solution for $Q(\mathcal{E}_1,\mathcal{E}_2)$ can be found after linearizing the Fokker-Planck equation (3.7). Expanding the drift coefficients d_j and diffusion coefficients D_{jk} around the locking point $(\mathcal{E}_{10},0)$ up to first and zeroth order in $\delta\mathcal{E}_1$ and $\delta\mathcal{E}_2(=\mathcal{E}_2)$, respectively, and making use of Eqs. (4.6) and (4.8b), one has the linearized Fokker-Planck equation

$$\frac{\partial}{\partial t}Q(\mathcal{E}_1,\mathcal{E}_2,t)=-\sum_{j=1}^{2}\frac{\partial}{\partial\mathcal{E}_j}\left[A_{jj}\delta\mathcal{E}_j-D^0_{jj}\frac{\partial}{\partial\mathcal{E}_j}\right]\times Q(\mathcal{E}_1,\mathcal{E}_2,t) \ . \tag{4.15}$$

Using Eq. (4.7) the steady-state solution of Eq. (4.15) is found to be

$$Q(\mathcal{E}_1,\mathcal{E}_2)=\prod_{j=1}^{2}\left[\frac{|A_{jj}|}{2\pi D^0_{jj}}\right]^{1/2}\times\exp\left[-\frac{|A_{jj}|(\mathcal{E}_j-\mathcal{E}_{j0})^2}{2D^0_{jj}}\right] , \tag{4.16}$$

which is a two-dimensional Gaussian distribution centered at the locking point $(\mathcal{E}_{10},0)$. The quadrature variances and covariance may also be calculated by direct integrations over the Q function. The results are the same as Eqs. (4.9)–(4.11), as expected. In fact, $D^0_{jj}/|A_{jj}|$ is the width (at $e^{-1/2}$ peak height) square of the quasiprobability distribution $Q(\mathcal{E}_1,\mathcal{E}_2)$ in the $\mathcal{E}_j$ quadrature and represents antinormally ordered quadrature variance $\langle(\delta\mathcal{E}_j)^2\rangle$ $(j=1,2)$.

V. OFF-RESONANT TWO-PHOTON CEL

In this section we examine the case $\delta\neq 0$, $\bar\rho_{ac}\neq 0$ for the two-photon CEL. For simplicity, we restrict our discussion to the situation where the intermediate level b is not populated initially, i.e., $\rho_{bb}=\bar\rho_{ab}=\bar\rho_{bc}=0$.

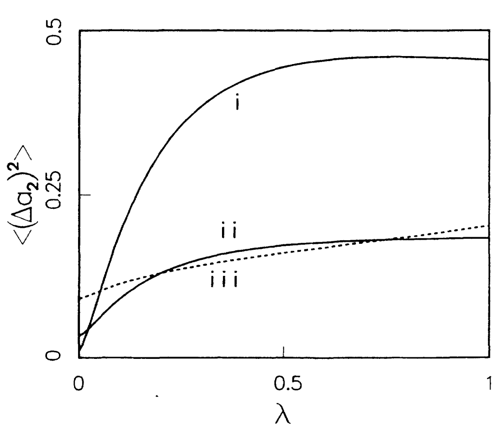

FIG. 2. Quadrature variance $\langle(\Delta a_2)^2\rangle$ as a function of λ for the initial atomic conditions given in Eqs. (4.13). The solid curves *i* and *ii* and the dashed one *iii* correspond to $\gamma/\alpha=0.001$, 0.01, and 0.1, respectively. The vacuum noise level is $\langle(\Delta a_2)^2\rangle=0.25$.

A. Steady-state operation

For such an off-resonant two-photon CEL the drift and diffusion coefficients in Eqs. (3.3) and (3.4) become much simpler, since all terms proportional to the parameter S vanish. According to the steady-state laser phase,[5] here we choose ϕ_0 in Eq. (3.5) to be

$$\phi_0=\tfrac{1}{2}\theta_{ac}-\tfrac{1}{4}\pi\,\mathrm{sgn}\delta \; . \qquad (5.1)$$

Substituting Eq. (3.3) into Eqs. (3.8) we obtain the drift coefficients for the amplitude and phase quadratures,

$$d_1=\tfrac{1}{2}(G-\gamma)\mathcal{E}_1-A_\phi\mathcal{E}_2 \; , \qquad (5.2a)$$

$$d_2=\tfrac{1}{2}(G-\gamma)\mathcal{E}_2+A_\phi\mathcal{E}_1 \; , \qquad (5.2b)$$

with

$$G(\mathcal{E}_1,\mathcal{E}_2)=\alpha\{(\rho_{aa}-\rho_{cc})[1+(\mathcal{E}_1^2+\mathcal{E}_2^2)\beta/2\alpha]+2|\bar{\rho}_{ac}\delta|(\mathcal{E}_1^2-\mathcal{E}_2^2)(\mathcal{E}_1^2+\mathcal{E}_2^2)^{-1}\}\{[1+(\mathcal{E}_1^2+\mathcal{E}_2^2)\beta/2\alpha]^2+\delta^2\}^{-1} \; , \qquad (5.3a)$$

$$A_\phi(\mathcal{E}_1,\mathcal{E}_2)=\nu-\Omega-\tfrac{1}{2}\alpha[(\rho_{aa}+\rho_{cc})\delta+4|\bar{\rho}_{ac}\delta|\mathcal{E}_1\mathcal{E}_2(\mathcal{E}_1^2+\mathcal{E}_2^2)^{-1}][1+\delta^2+2(\mathcal{E}_1^2+\mathcal{E}_2^2)\beta/\alpha]^{-1} \; . \qquad (5.3b)$$

In contrast to the resonant two-photon CEL discussed in Sec. IV, the initial atomic coherence $\bar{\rho}_{ac}$ enters into the drift coefficients in the off-resonant two-photon CEL. Since ϕ_0 (or $\phi_0+\pi$) is the steady-state laser phase, we have

$$\mathcal{E}_{20}=0 \; . \qquad (5.4)$$

The substitution of Eqs. (5.2) and (5.4) into Eqs. (3.13) leads to (1)

$$G(\mathcal{E}_{10},0)=\gamma \; , \qquad (5.5a)$$

$$A_\phi(\mathcal{E}_{10},0)=0 \; , \qquad (5.5b)$$

and/or (2)

$$\mathcal{E}_{10}=0 \; . \qquad (5.6)$$

Here $G(\mathcal{E}_1,0)$ is a saturated gain of the off-resonant two-photon CEL.

Equation (5.5a) determines the laser amplitude. If $\mathcal{E}_{10}$ is a stable (unstable) solution of Eq. (5.5a), then $-\mathcal{E}_{10}$ is also a stable (unstable) one, since $G(\mathcal{E}_1,0)$ is an even function of $\mathcal{E}_1$,

$$G(\mathcal{E}_1,0)=\alpha\frac{(\rho_{aa}-\rho_{cc})(1+\mathcal{E}_1^2\beta/2\alpha)+2|\bar{\rho}_{ac}\delta|}{(1+\mathcal{E}_1^2\beta/2\alpha)^2+\delta^2} \; . \qquad (5.7)$$

This reflects the fact that the off-resonant two-photon CEL can be locked to either $\mathcal{E}_{10}>0$ branch or the opposite $\mathcal{E}_{10}<0$ branch. Besides the usual (saturated) two-photon laser gain (proportional to $\rho_{aa}-\rho_{cc}$), there exists an extra two-photon-CEL gain proportional to $\alpha|\bar{\rho}_{ac}\delta|$, which is due to both the initial atomic coherence $\bar{\rho}_{ac}$ and the two-photon transition (indicated by $\delta\neq 0$). For a small normalized mean photon number, $N_0\ll 1$, we recover the linear gain

$$G(0,0)=[\alpha(\rho_{aa}-\rho_{cc})+2|\bar{\rho}_{ac}\delta|](1+\delta^2)^{-1} \qquad (5.8)$$

found in Ref. 5; $G(0,0)=\gamma$ is the threshold of the off-resonant two-photon CEL.

We plot, in Fig. 3, the gain $G(\mathcal{E}_1,0)$ as a function of $\mathcal{E}_1(\beta/\alpha)^{1/2}$ for $\rho_{aa}>\rho_{cc}$, $\rho_{aa}=\rho_{cc}$, and $\rho_{aa}<\rho_{cc}$, respectively. When $\delta^2\gg 1$, the gain G behaves differently, depending on whether there is a population inversion ($\rho_{aa}>\rho_{cc}$) or not. With population inversion $\rho_{aa}>\rho_{cc}$, there is (i) zero, (ii) two, and (iii) one positive solution of N_0 [see Eq. (4.4)] to Eq. (5.5a) for (i) $G_{\max}(\mathcal{E}_1,0)<\gamma$, (ii) $G(0,0)<\gamma<G_{\max}(\mathcal{E}_1,0)$, and (iii) $G(0,0)>\gamma$, respectively. In order to start laser operation one needs triggering in situation (ii) since the linear gain $G(0,0)<\gamma$, whereas triggering is no longer needed in situation (iii) since $G(0,0)>\gamma$. Without population inversion $\rho_{aa}\leq\rho_{cc}$, there is one (zero) positive solution of N_0 to Eq. (5.5a) if $G(0,0)=G_{\max}(\mathcal{E}_1,0)>\gamma\;(<\gamma)$. For $G(0,0)>\gamma$, the laser field can build up from a vacuum via spontaneous emission (i.e., without triggering). Overall, it follows from Eqs. (5.5b) and (5.7) that the solutions for N_0 are

$$N_0=(\rho_{aa}-\rho_{cc})\alpha/\gamma-2\pm\{[(\rho_{aa}-\rho_{cc})\alpha/\gamma]^2+8|\bar{\rho}_{ac}\delta|\alpha/\gamma-4\delta^2\}^{1/2} \; , \qquad (5.9)$$

provided that the value of { } is non-negative. Note that the smaller N_0 (taking minus sign), and even the larger N_0 (plus sign) may be negative. The negative N_0 should be dropped.

After knowing N_0, Eq. (5.5b) gives the actual laser frequency [by Eqs. (5.3b) and (5.4)],

$$\nu=\frac{\Gamma\Omega+\tilde{\gamma}\omega_{ab}}{\Gamma+\tilde{\gamma}} \; , \qquad (5.10a)$$

where

$$\tilde{\gamma}=\frac{\alpha(\rho_{aa}+\rho_{cc})}{2(1+\delta^2+2N_0)} \qquad (5.10b)$$

resembles the role of the cavity-loss rate γ in determining the actual laser frequency in an ordinary two-level one-photon laser. Notice that $\tilde{\gamma}$ depends on the laser intensity (N_0), which in turn depends on $\bar{\rho}_{ac}$ and γ, etc. Equation (5.10a) shows that, even in the case of the actual two-photon resonance $\omega_{ac}=2\nu$, there still exists mode pulling in the off-resonant two-photon CEL (as well as in an ordinary two-photon maser[17]). The direction of the mode pulling follows that of the one-photon detuning for the upper transition a-b. [When $\Omega=\omega_{ab}$ Eq. (5.10a) leads to $\nu=\Omega=\omega_{ab}=\omega_{bc}$, which is just the resonant case dis-

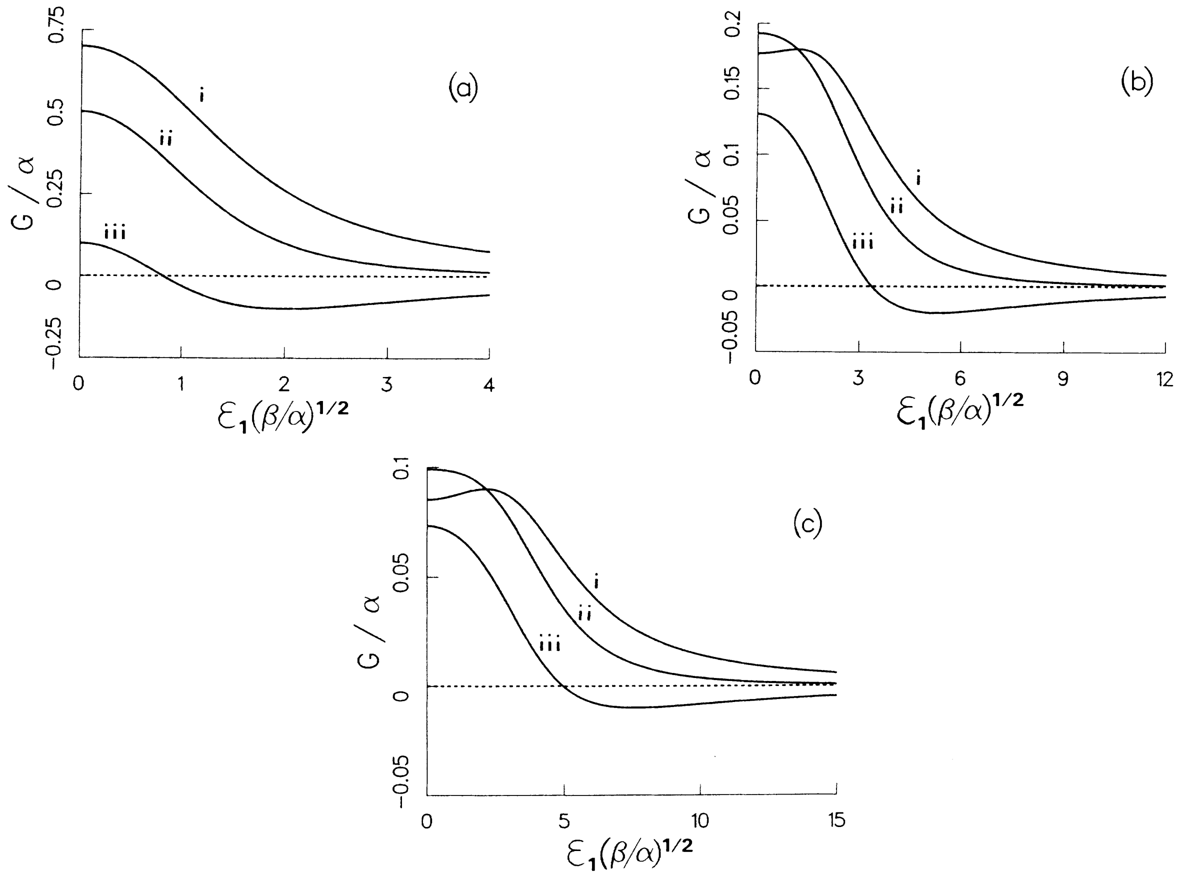

FIG. 3. Gain $G(\mathcal{E}_1,0)$ of the off-resonant two-photon CEL as a function of a normalized laser amplitude $\mathcal{E}_1\sqrt{\beta/\alpha}$ for different $\rho_{aa}=1-\rho_{cc}$ with $|\bar{\rho}_{ac}|=(\rho_{aa}\rho_{cc})^{1/2}$. The curves *i*, *ii*, and *iii* correspond to $\rho_{aa}=0.8$, 0.5, and 0.2, respectively. (a), (b), and (c) represent $|\delta|=1$, 5, and 10, respectively.

cussed in Sec. IV.] It is of practical importance to know how to set the cavity-mode frequency according to the atomic transition frequencies in order to achieve the actual two-photon resonance in a two-photon laser. The rule follows from Eq. (5.10a) as

$$2\Omega=\omega_{ac}+(\omega_{bc}-\omega_{ab})\bar{\gamma}/\Gamma \; . \tag{5.11}$$

$2\Omega-\omega_{ac}$ has the same sign as $\omega_{bc}-\omega_{ab}$.

B. Stability analysis

In the following we examine the stability of the locking point $(\mathcal{E}_{10},\mathcal{E}_{20})$ found above. When $\mathcal{E}_{10}\neq 0$ (i.e., $N_0>0$), we find from Eqs. (5.2) that

$$A_{1k}=\frac{\mathcal{E}_{10}}{2}\frac{\partial G(\mathcal{E}_{10},0)}{\partial \mathcal{E}_k} \; , \tag{5.12a}$$

$$A_{2k}=\mathcal{E}_{10}\frac{\partial A_{\phi}(\mathcal{E}_{10},0)}{\partial \mathcal{E}_k} \tag{5.12b}$$

$(k=1,2)$ by using Eqs. (5.4) and (5.5). Since G [see Eq. (5.3a)] is an even function of $\mathcal{E}_2$, we have

$$A_{12}=0 \; , \tag{5.13}$$

which means that (3.14b) is satisfied. Other A_{jk}'s are found, by using Eqs. (5.3) and (4.4), to be

$$A_{11}=\mathcal{E}_{10}^2\frac{\partial G(\mathcal{E}_{10},0)}{\partial \mathcal{E}_1^2}=-\frac{\alpha N_0}{2}\frac{(\rho_{aa}-\rho_{cc})[(1+N_0/2)^2-\delta^2]+4|\bar{\rho}_{ac}\delta|(1+N_0/2)}{[(1+N_0/2)^2+\delta^2]^2}$$
$$=-\frac{\alpha N_0}{(1+N_0/2)^2+\delta^2}\left[\frac{\rho_{cc}-\rho_{aa}}{2}+\frac{\gamma}{\alpha}\left[1+\frac{N_0}{2}\right]\right] , \tag{5.14a}$$

$$A_{21}=\frac{2\alpha N_0(\rho_{aa}+\rho_{cc})\delta}{(1+\delta^2+2N_0)^2}\neq 0 \; , \tag{5.14b}$$

$$A_{22}=-\frac{2\alpha|\bar{\rho}_{ac}\delta|}{1+\delta^2+2N_0}<0 \; . \tag{5.14c}$$

In arriving at the third equality of Eq. (5.14a), Eq. (5.8) has been used. Inequality (5.14c) shows that the locking in the $\mathcal{E}_2$ quadrature (i.e., the laser-phase locking) is always stable. In order for the locking in the $\mathcal{E}_2$ quadrature not to be too weak, neither $|\bar{\rho}_{ac}|$ nor $|\delta|$ can be too small. Namely, we should avoid $|\bar{\rho}_{ac}| \ll 1$ or $|\delta| \ll 1$ in the following treatments. Equation (5.14) demonstrates that the locking in the $\mathcal{E}_1$ quadrature (i.e., the laser-amplitude locking) is stable (1) if

$$\rho_{aa} \le \rho_{cc} , \tag{5.15}$$

or (2) if $\rho_{aa} > \rho_{cc}$ but

$$N_0 > (\rho_{aa} - \rho_{cc})\alpha/\gamma - 2 . \tag{5.16}$$

Of course, when inequality (5.15) holds, inequality (5.16) also holds. Alternatively, we may discuss the stability of the laser-amplitude locking by referring to Fig. 3. According to Eq. (3.14a) and the first equality in Eq. (5.14a), a stable amplitude locking requires

$$\frac{\partial G(\mathcal{E}_{10},0)}{\partial \mathcal{E}_1^2} < 0 , \tag{5.17}$$

which means a decreasing gain with increasing laser intensity around the locking point $(\mathcal{E}_{10},0)$. Physically this can be understood easily. For situation (ii), i.e., $\rho_{aa} > \rho_{cc}$ and $G(0,0) < \gamma < G_{\max}(\mathcal{E}_1,0)$, the larger N_0 is stable whereas the smaller N_0 is unstable as is evident in Fig. 3. This conclusion agrees with Eq. (5.16). For the situation $G(0,0) > \gamma$, in which there exists only one N_0 (see Fig. 3), the locking is always stable (for both $\rho_{aa} > \rho_{cc}$ and $\rho_{aa} \le \rho_{cc}$), again in agreement with (5.15) and (5.16).

When $\mathcal{E}_{10} = 0$ [see Eq. (5.6)], we find from Eqs. (5.2) and (5.3) that[30]

$$A_{11} = \tfrac{1}{2}[G(0,0) - \gamma] , \tag{5.18a}$$

$$A_{22} = \tfrac{1}{2}[G(0,0) - \gamma] - 2\alpha|\bar{\rho}_{ac}\delta|(1+\delta^2)^{-1} , \tag{5.18b}$$

and, if Eq. (5.9) holds, that

$$A_{12} = A_{21} = 0 . \tag{5.19}$$

Consequently, the locking at the origin (0,0) is stable only if the linear gain $G(0,0) < \gamma$, as expected. For situation (ii) discussed previously, $\mathcal{E}_{10} = \mathcal{E}_{20} = 0$ is another stable locking point besides the one with $\mathcal{E}_{10}^2 \gg 1$. With a proper triggering and under certain conditions, it may be possible that the two stable locking points ($\mathcal{E}_{10} = 0$ and $\mathcal{E}_{10}^2 \gg 1$) are equally probable so that one can find two peaks in the photon-number distribution ρ_{nn} with one peaked at $n = 0$. The peak of ρ_{nn} appearing at $n = 0$ is just a thermal field, arising from the below-threshold operation of the off-resonant two-photon CEL. In the following discussion for the field quadrature variances with stable laser operation we assume no triggering so that the coincidence does not happen.

C. Quadrature variances

Because of the laser locking the diffusion coefficients in the steady state take their values at the stable locking point. Substituting Eqs. (3.4) into Eqs. (3.9) and making use of Eqs. (5.4) and (4.4), we obtain the steady-state diffusion coefficients as

$$D_{11}^0 = \frac{\gamma}{4} + \frac{\alpha}{8}\left[\frac{\rho_{aa}(1+3N_0/2) + \rho_{cc}(3+N_0/2) - 2|\bar{\rho}_{ac}\delta|}{(1+N_0/2)^2+\delta^2} - \frac{\rho_{aa}+\rho_{cc}}{1+\delta^2+2N_0} + N_0 \frac{4|\bar{\rho}_{ac}\delta|(1+N_0/2) + 2(\rho_{cc}-\rho_{aa})\delta^2}{[(1+N_0/2)^2+\delta^2]^2} \right] , \tag{5.20a}$$

$$D_{22}^0 = \frac{\gamma}{4} + \frac{\alpha}{8}\left[\frac{(\rho_{cc}-\rho_{aa})(1+N_0/2) - 2|\bar{\rho}_{ac}\delta|}{(1+N_0/2)^2+\delta^2} + \frac{4|\bar{\rho}_{ac}\delta| + (\rho_{aa}+\rho_{cc})(1+N_0/2)}{1+\delta^2+2N_0} \right]$$
$$= \frac{\gamma}{8} + \frac{\alpha}{8}\,\frac{4|\bar{\rho}_{ac}\delta| + (\rho_{aa}+\rho_{cc})(1+2N_0)}{1+\delta^2+2N_0} , \tag{5.20b}$$

$$D_{12}^0 = \frac{\alpha}{8}\left[\frac{(\rho_{aa}-\rho_{cc})\delta - 2|\bar{\rho}_{ac}|\operatorname{sgn}\delta}{(1+N_0/2)^2+\delta^2} + \frac{(\rho_{cc}-\rho_{aa})\delta}{1+\delta^2+2N_0} - \frac{2(\rho_{aa}+\rho_{cc})N_0\delta}{(1+\delta^2+2N_0)^2} - \frac{[2|\bar{\rho}_{ac}\delta| + (\rho_{aa}-\rho_{cc})(1+N_0/2)]N_0\delta}{[(1+N_0/2)^2+\delta^2]^2} \right] . \tag{5.20c}$$

The second equality in Eq. (5.20b) is found after using Eq. (5.9). Because of Eq. (5.13), the variance in the amplitude quadrature is simply given by Eqs. (3.10) and (3.15a) as

$$\langle(\Delta a_1)^2\rangle=\frac{D_{11}^0}{|A_{11}|}-\frac{1}{4}\ . \tag{5.21}$$

On the other hand, since A_{21} does not vanish, $\langle(\delta\mathcal{E}_2)^2\rangle$ is coupled to $\langle\delta\mathcal{E}_1\delta\mathcal{E}_2\rangle$, which in turn relates to $\langle(\delta\mathcal{E}_1)^2\rangle$. Solving Eqs. (3.15) with Eq. (5.13) we find [by Eqs. (3.10)]

$$\langle(\Delta a_2)^2\rangle=\frac{D_{22}^0}{|A_{22}|}-\tfrac{1}{4}+\frac{A_{21}}{|A_{22}|}\frac{2D_{12}^0+A_{21}|A_{11}|^{-1}D_{11}^0}{|A_{11}|+|A_{22}|}\ , \tag{5.22}$$

which shows that the fluctuations in the phase quadrature a_2 are coupled to those in the amplitude quadrature a_1 in general. The explicit expressions for the quadrature variances $\langle(\Delta a_j)^2\rangle$ ($j=1,2$) are now readily obtained by substituting Eqs. (5.14) and (5.20) into Eqs. (5.21) and (5.22), which are quite lengthy. Relatively simple expressions can be found in various special cases. We discuss two such cases in Secs. V D and V E, respectively, and plot $\langle(\Delta a_2)^2\rangle$ for the general case in Sec. V F.

D. Near threshold $N_0 \ll 1$

Near the threshold, characterized by $G(0,0)\gtrsim\gamma$ or $N_0\ll 1$ (i.e., far below saturation), we can expand various quantities in terms of N_0 up to first order in N_0. From Eqs. (5.7) and (5.14a) we find

$$G(\mathcal{E}_{10},0)=G(0,0)+A_{11}\ , \tag{5.23a}$$

$$A_{11}=-\tfrac{1}{2}\alpha N_0|\mathcal{L}|^4[4|\bar{\rho}_{ac}\delta|+(\rho_{cc}-\rho_{aa})(\delta^2-1)]\ , \tag{5.23b}$$

where $|\mathcal{L}|^2=(1+\delta^2)^{-1}$. Equation (5.23b) gives the stability condition

$$4|\bar{\rho}_{ac}\delta|+(\rho_{cc}-\rho_{aa})(\delta^2-1)>0 \tag{5.24}$$

near the threshold. Equations (5.23) and (5.5a) give the normalized mean photon number

$$N_0=\frac{G(0,0)-\gamma}{\alpha|\mathcal{L}|^4[2|\bar{\rho}_{ac}\delta|+(\rho_{cc}-\rho_{aa})(\delta^2-1)/2]}\ . \tag{5.25}$$

Using Eqs. (5.14), (5.20), and (5.25) we find the quadrature variances from Eqs. (5.21) and (5.22),

$$\langle(\Delta a_1)^2\rangle=\tfrac{1}{4}+\frac{\rho_{aa}+|\bar{\rho}_{ac}\delta|+\tfrac{1}{4}N_0|\mathcal{L}|^2[\rho_{aa}(5\delta^2-1)+\rho_{cc}(5-\delta^2)-12|\bar{\rho}_{ac}\delta|]}{2N_0|\mathcal{L}|^2[4|\bar{\rho}_{ac}\delta|+(\rho_{cc}-\rho_{aa})(\delta^2-1)]}\ , \tag{5.26a}$$

$$\begin{aligned}\langle(\Delta a_2)^2\rangle=&(8|\bar{\rho}_{ac}\delta|)^{-1}\{\rho_{aa}+|\bar{\rho}_{ac}\delta|+\tfrac{1}{4}N_0|\mathcal{L}|^2[\rho_{aa}(5\delta^2+3)+\rho_{cc}(3\delta^2+1)+4|\bar{\rho}_{ac}\delta|]\}\\&+\frac{N_0|\mathcal{L}|^2(\rho_{aa}+\rho_{cc})}{4|\bar{\rho}_{ac}|^2|\delta|}\frac{|\bar{\rho}_{ac}|[\rho_{aa}(3\delta^2-1)+\rho_{cc}(\delta^2+1)+2|\delta|(\rho_{aa}^2+\rho_{aa}\rho_{cc}-2|\bar{\rho}_{ac}|^2)}{4|\bar{\rho}_{ac}\delta|+(\delta^2-1)(\rho_{cc}-\rho_{aa})}\ ,\end{aligned} \tag{5.26b}$$

which show that the variance in the amplitude quadrature a_1 is always much larger than $\frac{1}{4}$, and that in the phase quadrature a_2 can be less than the vacuum noise level $\frac{1}{4}$ for large enough $|\delta|$. Thus we find phase squeezing. With increasing N_0, $\langle(\Delta a_1)^2\rangle$ decreases, whereas $\langle(\Delta a_2)^2\rangle$ increases, which means that the larger degree of squeezing is found with smaller N_0. Alternatively, Eqs. (5.23)–(5.26) can be derived by treating the off-resonant two-photon CEL in a perturbation manner up to g^4. We examine more closely two limiting situations (1) $|\delta|\gg 1$ and (2) $N_0\simeq 0$ in the following.

(1) In the two-photon transitions limit $|\delta|\gg 1$, the degree of coupling to $\langle\delta\mathcal{E}_1\delta\mathcal{E}_2\rangle$ for $\langle(\delta\mathcal{E}_2)^2\rangle$ is small. Consequently, it follows from Eqs. (5.22) and (5.26b) that

$$\langle(\Delta a_2)^2\rangle=\frac{D_{22}^0}{|A_{22}|}-\tfrac{1}{4}=\tfrac{1}{8}+\frac{\rho_{aa}+\tfrac{1}{4}N_0(5\rho_{aa}+3\rho_{cc})}{8|\bar{\rho}_{ac}\delta|}\ . \tag{5.27}$$

An approximate steady-state solution for $Q(\mathcal{E}_2,\mathcal{E}_2)$ can be found in this limit. Similar to Sec. IV we first linearize the Fokker-Planck equation (3.7) around the *two possible* locking points $(\pm\mathcal{E}_{10},0)$ and then find the steady-state solution for the linearized Fokker-Planck equation,

$$Q(\mathcal{E}_1,\mathcal{E}_2)=\frac{1}{4\pi}\left[\frac{A_{11}A_{22}}{D_{11}^0D_{22}^0}\right]^{1/2}\sum_{m=1}^{2}\exp\left[-\frac{|A_{11}|}{2D_{11}^0}[\mathcal{E}_1-(-1)^m|\mathcal{E}_{10}|]^2-\frac{|A_{22}|}{2D_{22}^0}\mathcal{E}_2^2\right]\ , \tag{5.28}$$

which consists of two identical Gaussian peaks located at the two possible locking points. Similar to Sec. IV one can calculate the quadrature variances directly from the steady-state solution of $Q(\mathcal{E}_1,\mathcal{E}_2)$. The results are the same as those given in Eqs. (5.26a) and (5.27), since the field is locked to either $(\mathcal{E}_{10},0)$ or $(-\mathcal{E}_{10},0)$ depending on initial field fluctuations. If we further assume $|(\rho_{cc}-\rho_{aa})\delta|\gg|\bar{\rho}_{ac}|$, then Eq. (5.26a) reduces to

$$\langle(\Delta a_1)^2\rangle \approx \tfrac{1}{4} + \frac{\rho_{aa}+|\bar\rho_{ac}\delta| + \tfrac{1}{4}N_0(5\rho_{aa}-\rho_{cc})}{2N_0(\rho_{cc}-\rho_{aa})} . \quad (5.29)$$

Equations (5.27) and (5.29) are identical to the quadrature variances obtained by us in Ref. 20 from a fourth-order (in g) field master equation under the (almost same) conditions $|\delta| \gg 1$ and $|(\rho_{cc}-\rho_{aa})\delta| \gg |\bar\rho_{ac}|$. For a given laser intensity (i.e., N_0) requirement, a minimum value for $\langle(\Delta a_2)^2\rangle$ can be reached by properly choosing initial atomic variables. Assuming a full atomic coherence $|\bar\rho_{ac}| = (\rho_{aa}\rho_{cc})^{1/2} = [\rho_{aa}(1-\rho_{aa})]^{1/2}$, we find from Eq. (5.27)

$$\langle(\Delta a_2)^2\rangle_{\min} \approx \tfrac{1}{8} + \frac{\sqrt{3N_0}}{8|\delta|} , \quad (5.30)$$

when

$$\rho_{aa} = 1-\rho_{cc} = \frac{3N_0}{4(1+2N_0)} . \quad (5.31)$$

Meanwhile, Eq. (5.29) gives

$$\langle(\Delta a_1)^2\rangle \approx |\bar\rho_{ac}\delta|[2N_0(\rho_{cc}-\rho_{aa})]^{-1} = \sqrt{3}|\delta|/4\sqrt{N_0} \gg 1 .$$

[The cavity-loss rate γ should be chosen according to Eqs. (5.25) and (5.31).] The stability condition (5.24) is satisfied due to $|\delta| \gg 1$ and $\rho_{aa} \ll \rho_{cc}$.

(2) In the zero intensity limit $N_0 \simeq 0$, we find from Eq. (5.26b)

$$\langle(\Delta a_2)^2\rangle = \tfrac{1}{8} + \frac{\rho_{aa}}{8|\bar\rho_{ac}\delta|} , \quad (5.32)$$

which agrees with the phase variance found in Ref. 5 derived from a linear master equation (to g^2). Squeezing in the a_2 quadrature occurs when $\rho_{aa} < |\bar\rho_{ac}\delta|$. The degree of squeezing is $\frac{1}{2}(1-\rho_{aa}|\bar\rho_{ac}\delta|^{-1})$, which approaches its maximum value[5] 50% in the two-photon transition limit $|\delta| \gg 1$.

E. $\delta^2 \gg N_0 \gg 1$

We consider the special case

$$\delta^2 \gg N_0 \gg 1 , \quad (5.33)$$

i.e., $\Delta^2 \gg (2g\mathcal{E}_{10})^2 \gg \Gamma^2$ in Sec. V E, where $2g\mathcal{E}_{10}$ is the Rabi frequency for the one-photon transitions. Under conditions (5.33), we find from Eq. (5.7) the steady-state gain

$$G(\mathcal{E}_{10},0) = \frac{\alpha}{|\delta|}\frac{N_0'(\rho_{aa}-\rho_{cc})+2|\bar\rho_{ac}|}{1+N_0'^2} , \quad (5.34)$$

where

$$N_0' = N_0/2|\delta| \quad (5.35)$$

is a new normalized mean photon number. Equations (5.34), (5.12a), and (5.5a) leads to the stability condition [cf. Eq. (5.17)],

$$A_{11} = -\frac{2\alpha|\delta^{-1}\bar\rho_{ac}| + \gamma(N_0'^2-1)}{1+N_0'^2} < 0 . \quad (5.36)$$

Using Eqs. (5.5a) and (5.34) we obtain the variance in the amplitude quadrature a_1 from Eqs. (5.21), (5.20a), and (5.36)

$$\langle(\Delta a_1)^2\rangle = \frac{2(1+N_0'^2)|\bar\rho_{ac}| + N_0'[(3N_0'^2-1)\rho_{aa}+(5+N_0'^2)\rho_{cc}]}{8N_0'[4N_0'|\bar\rho_{ac}| + (1-N_0'^2)(\rho_{cc}-\rho_{aa})]} , \quad (5.37)$$

which is much larger than $\frac{1}{4}$ when $N_0' \ll 1$. Due to (5.33) the three steady-state diffusion coefficients D_{jk}^0 in Eqs. (5.20) are of the same order and, according to Eqs. (5.14), $|A_{21}| \ll |A_{11}|, |A_{22}|$. Thus it follows from Eqs. (5.22), (5.20b), and (5.14c) that

$$\begin{aligned}\langle(\Delta a_2)\rangle &= \frac{D_{22}^0}{|A_{22}|} - \tfrac{1}{4} \\ &= \frac{\gamma + 4N_0'(\rho_{aa}+\rho_{cc})\alpha|\delta|^{-1}}{16|\bar\rho_{ac}|\alpha|\delta|^{-1}} \\ &= \frac{1+N_0'|\bar\rho_{ac}|^{-1}[\rho_{aa}+(\frac{3}{2}+2N_0'^2)(\rho_{aa}+\rho_{cc})]}{8(1+N_0'^2)} .\end{aligned} \quad (5.38)$$

For given N_0', the variance $\langle(\Delta a_2)^2\rangle$ in Eq. (5.38) is a function of ρ_{aa} [let $\rho_{cc} = 1-\rho_{aa}$ and $|\bar\rho_{ac}| = (\rho_{aa}\rho_{cc})^{1/2}$ again]. Its minimum value

$$\langle(\Delta a_2)^2\rangle_{\min} = \frac{1+N_0'(3+4N_0'^2)^{1/2}(5+4N_0'^2)^{1/2}}{8(1+N_0'^2)} \quad (5.39)$$

is found at

$$\rho_{aa} = 1-\rho_{cc} = \frac{\frac{3}{2}+2N_0'^2}{4(1+N_0'^2)} . \quad (5.40)$$

This is in the $\rho_{aa} < \rho_{cc}$ region and the stability condition is always satisfied. [The cavity-loss rate γ needs to be chosen according to Eqs. (5.5a), (5.34), and (5.40).] It is straightforward to show from Eq. (5.39) that squeezing is possible only when $N_0' < 0.275$. We plot Eq. (5.39) in Fig. 4. Quite accurately (and somewhat surprisingly), $\langle(\Delta a_2)^2\rangle_{\min}$ increases linearly with N_0'.

In this case we can also find the steady-state solution for the linearized Fokker-Planck equation, which is of the exactly same form as that in Eq. (5.28). Of course the explicit expressions for A_{jj} and D_{jj}^0 $(j=1,2)$ are different, since they belong to two different special cases. Once

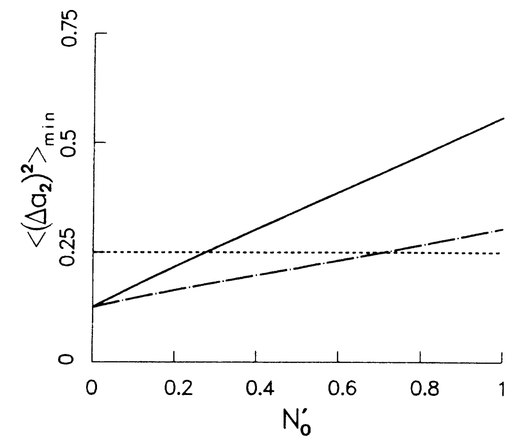

FIG. 4. Minimum variance $\langle(\Delta a_2)^2\rangle_{\min}$ [Eq. (5.39)] vs the normalized mean photon number N_0' (solid line), as given in Eq. (5.35). The dashed line indicates the variance in the vacuum state. The dash-dotted line is the minimum variance $\langle(\Delta a_2)^2\rangle_{\min}$ found in Eq. (5.22a′) (of Ref. 19) from an effective interaction Hamiltonian.

again such a solution for $Q(\mathcal{E}_1,\mathcal{E}_2)$ leads to the same quadrature variances as given in Eqs. (5.37) and (5.38).

F. Graphic illustrations

In the last part of Sec. V, we plot the phase-quadrature variance $\langle(\Delta a_2)^2\rangle$ by using Eq. (5.22). We use $\rho_{aa}+\rho_{cc}=1$ and $|\bar{\rho}_{ac}|=(\rho_{aa}\rho_{cc})^{1/2}$. In Fig. 5, the a_2's variance is drawn as a function of the normalized photon number N_0' with ρ_{aa} taking values given in Eq. (5.40). One sees that $\langle(\Delta a_2)^2\rangle$ increases with increasing N_0' for a given $|\delta|$ value, but decreases with increasing $|\delta|$ at a given N_0. A comparison of Fig. 5 (e.g., $|\delta|=10$) with Fig. 4 shows that the expression (5.39) is valid even beyond the region $\delta^2 \gg N_0 \gg 1$. In Fig. 6, the variance

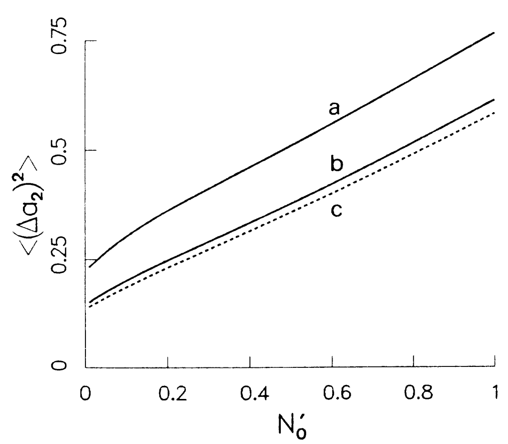

FIG. 5. Phase-quadrature variance as a function of the normalized photon number N_0' when the initial atomic variables take the values given in Eq. (5.40). The solid curves a and b and the dashed one c correspond to $|\delta|=1$, 5, and 10, respectively.

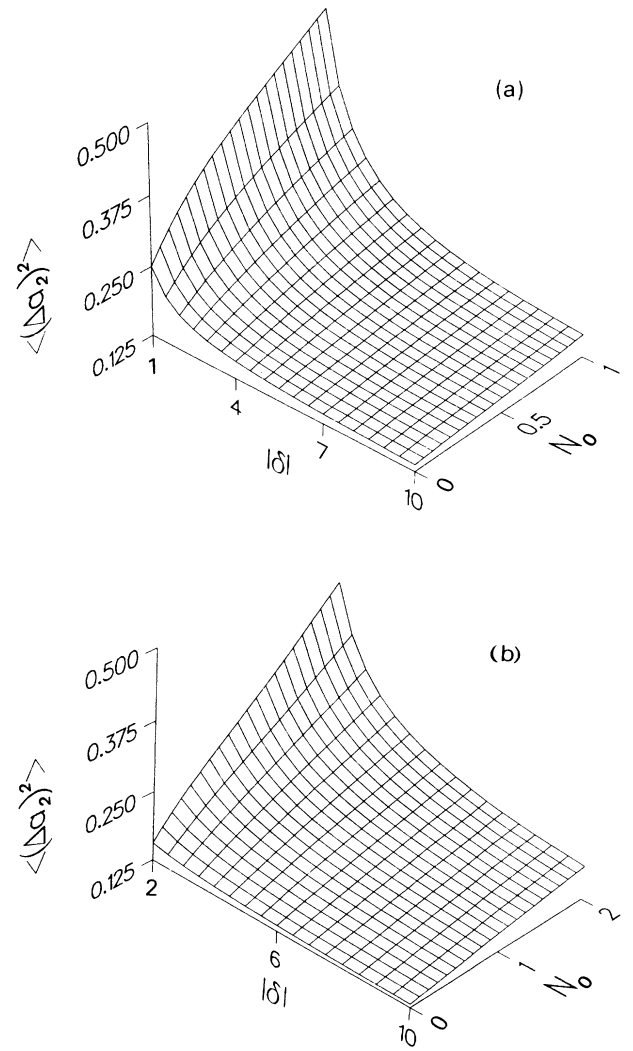

FIG. 6. Variance in the a_2 quadrature as a function of both one-photon detuning $|\delta|$ and the normalized photon number N_0 with initial atomic condition $\rho_{cc}=1-\rho_{aa}$, $|\bar{\rho}_{ac}|=(\rho_{aa}\rho_{cc})^{1/2}$, and (a) $\rho_{aa}=0.5$, (b) $\rho_{aa}=0.2$. The vacuum noise level is 0.25.

$\langle(\Delta a_2)^2\rangle$ is plotted as a function of both one-photon detuning $|\delta|$ and the normalized photon number N_0 for $\rho_{aa}=0.5$ and 0.2. Common features are that the variance increases with increasing N_0 and/or decreasing $|\delta|$ and squeezing occurs for large ranges of $|\delta|$ and N_0. A difference is that the smaller ρ_{aa} leads to smaller a_2's variance. In Fig. 7 we plot $\langle(\Delta a_2)^2\rangle$ as a function of both $|\delta|$ and N_0 when ρ_{aa} takes the expression (5.31), which shows once again the common features illustrated in Fig. 6.

VI. COMPARISON WITH THE EFFECTIVE HAMILTONIAN MODEL

In preceding sections we have studied the two-photon CEL's starting from the exact atom-field interaction Hamiltonian V_j given by Eq. (2.5); especially, the off-

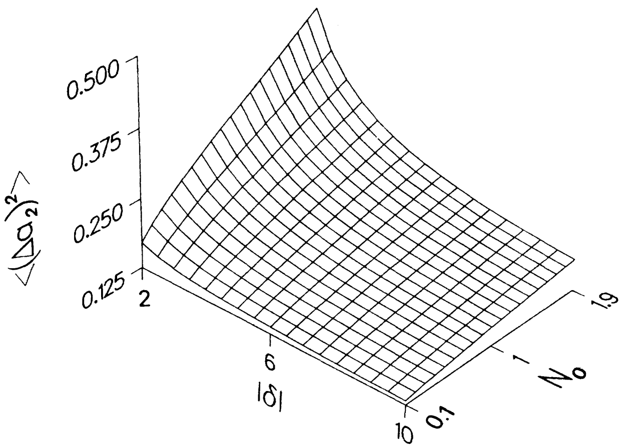

FIG. 7. Same as Fig. 6 except $\rho_{aa}=3N_0/4(1+2N_0)$ [see Eq. (5.31)].

resonant two-photon CEL has been examined in detail in Sec. V. On the other hand, the (off-resonant) two-photon CEL has also been analyzed in Ref. 19 starting from an effective atom-field interaction Hamiltonian for a two-photon transition. The effective interaction Hamiltonian is generally regarded to be valid in the off-resonant case with large one-photon detunings.

In this section we compare the predictions for the off-resonant two-photon CEL obtained in this paper with those obtained in Ref. 19, and discuss the validity domain of the effective interaction Hamiltonian. The comparison is made in the case of the actual two-photon resonance $\omega_{ac}=\omega_a-\omega_c=2\nu$ and the equal coupling constants $g_1=g_2\equiv g$. Also g has been taken to be real.

This comparison will be very interesting for people working on the two-photon processes, in which the effective interaction Hamiltonian is used generally. Recently, Ref. 31 has made such a comparison for the photon-number equations in a (cascade) two-mode two-photon laser. In addition, by using the exact Hamiltonian, Zhu and Li[13] found two peaks in the photon-number distribution of an ordinary (single-mode) two-photon laser.

The effective atom-field interaction Hamiltonian $\hbar V_j'$ used in Ref. 19 is

$$V_j'=g'|a^j\rangle\langle c^j|a^2+g'(a^\dagger)^2|c^j\rangle\langle a^j| \tag{6.1}$$

in the Schrödinger picture, which should be compared with V_j in Eq. (2.5). (We change the notations used in Ref. 19 to distinguish them from the notations used in this paper.) Here g' is the effective atom-field coupling constant for the two-photon transition between levels a and c, and has been identified in Ref. 19 to be

$$g'=g^2/\Delta \tag{6.2}$$

in the limit $|\delta|\gg 1$ from a comparison with the exact, but linear theory of Ref. 5. The field master equation for the off-resonant two-photon CEL obtained from the exact Hamiltonian V_j is Eq. (2.15) (with $\rho_{bb}=\bar{\rho}_{ab}=\bar{\rho}_{bc}=0$), and that obtained from the effective Hamiltonian V_j' is Eq. (2.17′) of Ref. 19. (We use the primed equation numbers to indicate equations in Ref. 19.) The loss parts of the two master equations are the same. The gain parts of Eq. (2.17′) consist of terms containing ρ_{nm}, $\rho_{n+2,m+2}$, $\rho_{n-2,m-2}$, $\rho_{n,m\pm 2}$, and $\rho_{n\pm 2,m}$ only, which represents two-photon processes. On the other hand, the gain part of the Eq. (2.15) (with $\rho_{bb}=\bar{\rho}_{ab}=\bar{\rho}_{cc}=0$) has additional terms containing $\rho_{n-1,m-1}$, $\rho_{n+1,m+1}$, $\rho_{n-1,m+1}$, and $\rho_{n+1,m+1}$, which represent one-photon processes. We rewrite α, β, and S in Eq. (2.19′) as

$$\alpha'=\frac{2r_a g'^2}{\Gamma^2},\quad \beta'=\frac{8r_a g'^4}{\Gamma^4},\quad S'=\frac{r_a g'}{\Gamma}\ . \tag{6.3}$$

A careful examination of (2.15), however, shows that one cannot say when the one-photon-process terms can be neglected compared with the two-photon-process terms. Thus it is not convenient to discuss the validity of the effective Hamiltonian V_j' directly from the master equations (2.15) and (2.17′). After all, the two master equations were not used directly to calculate the first and second moments of the laser field.

To see the accuracy of the effective Hamiltonian in predicting the first and second moments of the field, it is better to use the Fokker-Planck equations (3.7) and (5.4′) for the Q function, which are uniquely represented by their drift and diffusion coefficients.

A. First moments

The phase ϕ_0, mean photon number $\mathcal{E}_{10}^2$, and the mode pulling $\nu-\Omega$ of the laser field are exclusively determined by the drift coefficients of the Fokker-Planck equations (3.7) and (5.4′), which are Eqs. (5.2) and (5.3) from the exact Hamiltonian V_j [i.e., for Eq. (3.7)] and Eqs. (5.14′) and (5.15′) from the effective Hamiltonian V_j' [i.e., for Eq. (5.4′)].

1. *Laser phase*

This work predicts that the laser phase will be locked to either ϕ_0 (i.e., $\mathcal{E}_{10}>0$) or $\phi_0+\pi$ (i.e., $\mathcal{E}_{10}<0$) in the steady state, where ϕ_0 is given by Eq. (5.1); Ref. 19 showed that the steady-state laser phase is ϕ_0^1 or $\phi_0^1+\pi$ [see Eq. (4.22′)], where

$$\phi_0^1=\tfrac{1}{2}\theta-\tfrac{1}{4}\pi\ ,\qquad \theta=\arg(g'\bar{\rho}_{ac})\ . \tag{6.4}$$

Using Eq. (6.2) (or assuming that g' has the same sign as Δ), it is easy to see that

$$\phi_0=\phi_0^1\ . \tag{6.5}$$

Thus the exact and effective Hamiltonians give the same steady-state laser phase.

2. *Drift coefficients*

Comparing the drift coefficients d_1 and d_2 in the amplitude and phase quadratures given by Eqs. (5.2) with those given by Eqs. (5.14′), one sees that $\frac{1}{2}(G-\gamma)$ and A_ϕ correspond to A_1 and A_2, respectively. With the help of Eqs. (2.17), (6.2), and (6.3) we find that $\frac{1}{2}(G-\gamma)$ reduces to A_1 when $|\mathcal{E}|^2=\mathcal{E}_1^2+\mathcal{E}_2^2$ satisfies

$$\delta^2 \gg |\mathcal{E}|^2\beta/\alpha \gg 1 \ , \tag{6.6}$$

with the following correspondence:

$$\alpha|\delta|^{-1}=2S' \ , \tag{6.7a}$$

$$\beta/(2\alpha|\delta|)=\sqrt{\beta'/\alpha'} \ . \tag{6.7b}$$

On the other hand, under no circumstance [including the limit of (6.6)] will A_ϕ reduce to

$$A_2=\nu-\Omega-4|S'\rho_{ac}|\mathcal{E}_1\mathcal{E}_2(\mathcal{E}_1^2+\mathcal{E}_2^2)^{-1}$$

in Ref. 19. Thus the two drift coefficients d_1's (d_2's) in the amplitude (phase) quadrature do not agree with each other in general.

3. Laser intensity

Since $\mathcal{E}_{20}=0$, the two drift coefficients d_1's in the amplitude quadrature *do* agree with each other in the steady state when inequalities (5.33) are satisfied. Notice that, due to Eq. (6.7b), the normalized mean photon number N_0' defined by Eq. (5.35) is exactly the same as the normalized mean photon number N_0' defined in Eq. (4.27′). Consequently, the laser gain $G(N_0',\phi_0)$ and the mean photon number n_0 obtained from the effective Hamiltonian V_j' are accurate under the conditions (5.33), as is evident from Eqs. (5.34), (6.7a), and (4.23′). Moreover, the amplitude locking strength $A_{11}=(\partial d_1/\partial\mathcal{E}_1)_0$ derived from the exact Hamiltonian reduces to that from the effective one under the conditions (5.33), as can be seen from Eqs. (5.36) and (5.19a′). In addition, $A_{12}=(\partial d_1/\partial\mathcal{E}_2)_0$ vanishes for both Hamiltonians [see Eqs. (5.13) and (5.17′)], which is independent of the condition (5.33).

4. Mode pulling and phase locking

One thing that the effective Hamiltonian V_j' failed to predict is the mode pulling $\nu-\Omega$ at the exact two-photon resonance $\omega_{ac}=2\nu$, as pointed out in Ref. 19. While the effective Hamiltonian method gives no mode pulling $\Omega=\frac{1}{2}\omega_{ac}=\nu$, the exact Hamiltonian method shows that there exists in fact a mode pulling $\nu-\Omega=\frac{1}{2}\omega_{ac}-\Omega=(\omega_{ab}-\omega_{bc})\tilde{\gamma}/2\Gamma$ [see Eq. (5.11)]. The reason for this is that A_ϕ does not reduce to A_1. This also leads to $A_{21}\neq 0$ [see Eqs. (5.12b) and (5.14b)], in contrast to $A_{21}=0$ from the effective Hamiltonian method [see Eq. (5.17′)]. The "phase locking strengths" $A_{22}=(\partial d_2/\partial\mathcal{E}_2)_0$ from the two Hamiltonians, however, coincide with each other when

$$\delta^2 \gg N_0, 1 \tag{6.8}$$

i.e., $\Delta^2 \gg (2g\mathcal{E}_{10})^2, \Gamma^2$. Also under the conditions (6.8) the importance of $A_{21}\neq 0$ becomes small, since $|A_{21}/A_{22}| \ll 1$ in such a case.

To finish Sec. VI A, we conclude that the accuracy of the effective Hamiltonian V_j' in predicting the steady-state operation of the off-resonant two-photon CEL depends on which quantities we are concerned with, ranging from the always correct laser phase to never correct mode pulling. Overall, in order for the effective Hamiltonian V_j' to be valid in calculating the first moments of the laser field, the necessary conditions are (6.8) and the sufficient conditions are (5.33), with the exception of the mode pulling.

B. Second moments

The quadrature variances of the field are determined by the steady-state diffusion coefficients D_{jk}^0 and the locking strengths A_{jk}, as indicated by Eqs. (5.21), (5.22) and (5.21′). We compare the phase- and amplitude-quadrature variances separately in the following.

1. Phase-quadrature variance

We first note that the effective Hamiltonian gives $\langle(\Delta a_2)^2\rangle$ which is never affected by D_{11}^0 and D_{12}^0 [see Eq. (5.21′)]. This does not agree with Eq. (5.22) in general. Under the conditions (6.8), however, $|A_{21}/A_{22}| \ll 1$ and thus the last term in Eqs. (5.22) can be neglected compared with the first two terms, as is true in Secs. V D and V E. Namely, $\langle(\Delta a_2)^2\rangle$ obtained in Ref. 19 may be accurate under the conditions (6.8). (i) Comparing Eq. (5.21′) with Eq. (5.38) in Sec. V E, we find that there is a difference between the noise levels calculated from the two Hamiltonians,

$$\langle(\Delta a_2)^2\rangle_{\text{exact}}-\langle(\Delta a_2)^2\rangle_{\text{effective}}=\frac{N_0'}{8|\bar{\rho}_{ac}|}\geq\frac{N_0'}{4} \ , \tag{6.9}$$

where $\rho_{aa}+\rho_{cc}=1$ has been used. This difference is solely due to the difference in two steady-state phase diffusion coefficients D_{22}^0's, and is proportional to N_0'. Equation (6.9) means that the effective Hamiltonian V_j' underestimates the phase noise level at least in the region of (5.33). This can also be seen from Fig. 4, which plots the minimum variances $\langle(\Delta a_2)^2\rangle_{\min}$, Eqs. (5.39) and (5.22a′) obtained from the two Hamiltonians. (Two ρ_{aa} values are different.) On the other hand, Fig. 4 also shows that the basic features about $\langle(\Delta a_2)^2\rangle_{\min}$ obtained from the two Hamiltonians are the same. (ii) To compare Eq. (5.21′) with Eq. (5.27) in Sec. V D, we first need to expand Eq. (5.21′) in the limit $N_0' \ll 1$, yielding

$$\langle(\Delta a_2)^2\rangle_{\text{effective}}=\tfrac{1}{8}+\frac{\frac{1}{2}N_0'(3\rho_{aa}+\rho_{cc})}{8|\bar{\rho}_{ac}|} \ . \tag{6.10}$$

Then the comparison between Eqs. (5.27) and (6.10) gives the difference

$$\langle(\Delta a_2)^2\rangle_{\text{exact}}-\langle(\Delta a_2)^2\rangle_{\text{effective}}=\frac{N_0'+\rho_{aa}|\delta|^{-1}}{8|\bar{\rho}_{ac}|} \tag{6.11}$$

in the limit $|\delta| \gg 1 \gg N_0$, where $\rho_{aa}+\rho_{cc}=1$ has been used. Once again this difference arises from the difference in the steady-state diffusion coefficients D_{22}^0's.

Equation (6.11) indicates that the effective Hamiltonian underestimates the fluctuations in the a_2 quadrature in the limit $|\delta| \gg 1 \gg N_0$ too. Nevertheless, the difference given by Eq. (6.11) is much smaller than $\frac{1}{4}$.

2. *Amplitude-quadrature variance*

The variance in the amplitude quadrature, $\langle(\Delta a_1)^2\rangle$, has been omitted in Ref. 19. For the purpose of the comparison we give it here,

$$\langle(\Delta a_1)^2\rangle_{\text{effective}} = \frac{D_{11}(\mathcal{E}_{10},\mathcal{E}_{20})}{|\partial d_1(\mathcal{E}_{10},\mathcal{E}_{20})/\partial\mathcal{E}_2|} - \frac{1}{4} = \frac{2(1+3N_0'^4)|\bar{\rho}_{ac}| + N_0'(1+N_0'^2)(3\rho_{aa}+\rho_{cc})}{8N_0'[4N_0'|\bar{\rho}_{ac}| + (1-N_0'^2)(\rho_{cc}-\rho_{aa})]}, \tag{6.12}$$

which is also much larger than $\frac{1}{4}$ if $N_0' \ll 1$. Comparing Eq. (6.12) with Eq. (5.21), one sees that both $\langle(\Delta a_1)^2\rangle$ are independent of D_{22}^0. Under the conditions (5.33) we find from Eqs. (6.12) and (5.37) that the difference between the two a_1's variances is

$$\langle(\Delta a_1)^2\rangle_{\text{exact}} - \langle(\Delta a_1)^2\rangle_{\text{effective}} = \frac{N_0'(1-3N_0'^2)|\bar{\rho}_{ac}| + 2(\rho_{cc}-\rho_{aa})}{4[4N_0'|\bar{\rho}_{ac}| + (1-N_0'^2)(\rho_{cc}-\rho_{aa})]}, \tag{6.13}$$

which can be positive, negative, or zero. As in the previous case for $\langle(\Delta a_2)^2\rangle$, this difference is due to the difference in the two steady-state amplitude diffusion coefficients D_{11}^0's. When $N_0' \ll 1$ and $\rho_{cc}-\rho_{aa}$ is not too small, the value of Eq. (6.13) is approximately $\frac{1}{2}$, which is a relatively small correction to $\langle(\Delta a_1)^2\rangle_{\text{effective}}$.

To conclude Sec. VI B, we observe that the quadrature variances calculated from the effective Hamiltonian V_j' are accurate only at the far-below-threshold region $N_0' \ll 1$, and the effective Hamiltonian always underestimates the phase-quadrature variance. In predicting the field fluctuations in a two-photon laser, there is no guarantee for the effective Hamiltonian to be valid in general, even under the conditions (5.33) which ensure the first moments of the field.

VII. CONCLUSION

We have developed a quantum theory of the two-photon CEL's by using the exact atom-field interaction Hamiltonian. This theory is to all order in the atom-field coupling constant g $(=g_1=g_2)$ and treats the resonant and off-resonant two-photon CEL's in a unified approach. By assuming an actual two-photon resonance $2\nu=\omega_{ac}$ but allowing one-photon detuning δ $(=\Delta/\Gamma)$, we derive the master equation for the reduced field-density operator and then transform it into a Fokker-Planck equation for the antinormal-ordering Q function. The master equation and the Fokker-Planck equation are valid for arbitrary initial atomic conditions, with or without initial atomic coherences, and for an arbitrary cavity-loss rate. The Fokker-Planck equation takes relatively simple form and makes an analytic study of the two-photon CEL's possible.

For the resonant two-photon CEL in which $\delta=0$ but $\bar{\rho}_{ab}\neq 0$ and $\bar{\rho}_{bc}\neq 0$, there is no threshold since the atomic coherences $\bar{\rho}_{ab}$ and $\bar{\rho}_{bc}$ act as a driving force. When $\bar{\rho}_{ab}$ and $\bar{\rho}_{bc}$ have the same phase $\theta_{ab}=\theta_{bc}$, the phase of the laser field will be locked to $\phi_0=\theta_{ab}-\frac{1}{2}\pi$ and phase squeezing can occur. More than 90% squeezing can be obtained for a proper choice of the initial atomic variables, which is in the no-population-inversion region with the laser intensity far below saturation. By linearizing the Fokker-Planck equation a steady-state Q function is found.

For the off-resonant two-photon CEL in which $\delta\neq 0$ and $\bar{\rho}_{ac}\neq 0$ but $\rho_{bb}=\bar{\rho}_{ab}=\bar{\rho}_{bc}=0$, there exists a threshold $G(0,0)=\gamma$ (i.e., the linear gain equals the cavity loss). Below threshold $G(0,0)<\gamma$ but $G_{\max}(\mathcal{E}_1,0)>\gamma$ a stable laser field can be obtained by triggering the field. Above threshold $G(0,0)>\gamma$ the laser field will build up from a vacuum and become stable. There are two phase-locking points $(\pm\mathcal{E}_{10},0)$ and the laser field will be locked to one of them depending on the initial field fluctuations (or triggering). Larger phase squeezing occurs in the larger $|\delta|$, far-below-saturation intensity $(N_0 \ll 1)$, and no-population-inversion region. A maximum of 50% squeezing can be approached near the threshold when $|\delta| \gg 1$. The properties of the off-resonant two-photon CEL are investigated more closely near the threshold and in the region $\delta^2 \gg N_0 \gg 1$ [i.e., $\Delta^2 \gg (2g\mathcal{E}_{10})^2 \gg \Gamma^2$], respectively, and approximate steady-state Q functions are obtained, which consists of two identical Gaussian peaks centered at the two locking points $(\pm\mathcal{E}_{10},0)$. In the case $\delta^2 \gg N_0 \gg 1$, we find the minimum variance in the phase quadrature as a function of the initial atomic variables for a given $N_0'=N_0/2|\delta|$, which increases monotonically with increasing N_0'. In general, the phase squeezing is possible only when N_0 is less than a certain value for a given $|\delta|$.

These results for the off-resonant two-photon CEL have been compared in detail with those obtained in Ref. 19. For the laser operation (except for the mode pulling), the effective Hamiltonian V_j' is found to be valid under the conditions $\delta^2 \gg N_0 \gg 1$. For the quadrature variances, one needs an additional condition $N_0' \ll 1$ (i.e., far below saturation) to ensure the validity of the effective Hamiltonian.

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