Generalization of the Lorentz-Dirac equation to include spin

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For the classical point electron with *Zitterbewegung* (hence spin) we derive, after regularization, the radiation reaction force and covariant equations for the dynamical variables (x_μ, π_μ, v_μ, and $S_{\mu\nu}$), which reduce to the Lorentz-Dirac equation in the spinless limit.

Fifty years ago, Dirac[1] gave the final covariant form of the equation of the classical relativistic electron including the radiation reaction force culminating the long history of the classical radiation theory that began with Lorentz.[2] This equation we now call the Lorentz-Dirac (LD) equation. It is a nonlinear equation for the position $x(\tau)$ of a *spinless* electron, and reads in units $c=1$

$$m\ddot{x}_\mu = eF_{\mu\nu}\dot{x}^\nu + \tfrac{2}{3}e^2[\dddot{x}_\mu + (\ddot{x})^2\dot{x}_\mu] , \tag{1}$$

where m is a renormalized mass, dots are with respect to the proper time τ, and $F_{\mu\nu}$ is the external electromagnetic field. It forms the basis of all radiation problems in classical electrodynamics. Proper boundary conditions must be used, e.g., the vanishing of acceleration at $t=-\infty$, to eliminate the runaway solutions and "preacceleration." Otherwise, it is a fully consistent equation that works very well. The purpose of this work is to generalize Eq. (1) to a spinning electron in a way that comes very close to the quantum Dirac equation, and includes all the radiative effects.

The description of the electron spin in classical electrodynamics also has a very long history. Generally, the spinning electron is treated as a relativistic dipole[3–8] by adding to the Hamiltonian a term of the form $S_{\mu\nu}F^{\mu\nu}$, which is analogous to the passage from the Schrödinger (or Klein-Gordon) equation to the Pauli equation. However, the spin in the first-order Dirac equation is quite different and intricate. Here we shall use a more recent first-order classical spin theory which models the Dirac electron remarkably well and where the coupling to the electromagnetic field is via A_μ, as in the Dirac equation, and not via $F_{\mu\nu}$. In particular, the point charge performs classically a *Zitterbewegung* as its natural motion, and there is already the notion of the antiparticles on the classical level.[9] When this theory is quantized (by canonical quantization,[9] by Schrödinger quantization,[10] or by path-integral quantization[11]), we obtain precisely the Dirac equation. In the present work we determine the radiation reaction term to this equation from the self-field of the electron.

The action integral W is best formulated in terms of the two sets of canonically conjugate variables, the pair (x_μ, p_ν) and another pair of spin variables $(\bar{z}, z)$, where $z(\tau)$ is a classical c-number four-component spinor variable, with $\bar{z}=z^+\gamma^0$. It is given, now including the self-field, by

$$W = \int d\tau[i\lambda\bar{z}\dot{z} + p_\mu(\dot{x}^\mu - \bar{z}\gamma^\mu z) + e(A_\mu^{\text{ext}} + A_\mu^{\text{self}})\bar{z}\gamma^\mu z] . \tag{2}$$

The only fundamental constants here are λ, a constant of the dimension of h, and e, the coupling constant to A_μ. There is no mass constraint; the mass will come in as an integration constant, the value of the constant of motion $\mathcal{H}$.

The equations of motion resulting from (2) are (in units $\lambda=c=1$)

$$\begin{aligned} \dot{x}^\mu &= \bar{z}\gamma^\mu z , \\ \dot{p}^\mu &= eA^{\mu,\nu}\bar{z}\gamma_\nu z , \\ \dot{z} &= -i\gamma^\mu(p_\mu - eA_\mu)z , \\ \dot{\bar{z}} &= i\bar{z}\gamma^\mu(p_\mu - eA_\mu) , \end{aligned} \tag{3}$$

where $A_\mu = A_\mu^{\text{ext}} + A_\mu^{\text{self}}$ everywhere.

The self-field is determined by the current of the particle and is given by

$$A_\mu^{\text{self}}(x) = e\int d\tau' D(x - x(\tau'))\bar{z}(\tau')\gamma_\mu z(\tau') . \tag{4}$$

There are at least two main methods of regularization of the self-field. One, originally given by Dirac,[1] uses the finite part of the Green's function $\frac{1}{2}(D^{\text{ret}} + D^{\text{adv}})$; the other uses simply the retarded Green's function and analytic continuation.[12] There are also two methods at arriving at the Eq. (1). One via the energy-momentum tensor,[1] the other via the equations of motion.[12] We have applied both methods and obtained the same result, of course. Here we describe what we think to be the simplest approach using retarded potentials and the analytic continuation of the equations of motion.

The Hamiltonian form of the theory can also be formulated, in terms of the new set of real dynamical variables x_μ, p_μ, v_μ, and $S_{\mu\nu}$ (position, momentum, velocity, and spin variables), instead of the complex variables $\bar{z}$ and z, defined as follows:

$$\begin{aligned} v^\mu &= \bar{z}\gamma^\mu z , \\ \pi^\mu &= p^\mu - eA^\mu , \\ S^{\mu\nu} &= -\frac{i}{4}\bar{z}[\gamma^\mu, \gamma^\nu]z . \end{aligned} \tag{5}$$

From now on we shall work with the real dynamical variables $x, \dot{x}, \ddot{x}, \ldots$. Note that velocity and momenta are independent dynamical variables, which is also a characteristic of the Dirac equation.

The equations of motion take the simpler form

$$\begin{aligned} \dot{x}^{\mu} &= v^{\mu} , \\ \dot{v}^{\mu} &= -4S^{\mu\nu}\pi_{v} , \\ \dot{\pi}^{\mu} &= eF^{\mu\nu}v_{\nu} , \\ \dot{S}^{\mu\nu} &= v^{\mu}\pi^{\nu} - v^{\nu}\pi^{\mu} . \end{aligned} \tag{6}$$

We also note that, in contrast to spinless particles, $v_{\mu}v^{\mu} \neq 1$, due to *Zitterbewegung*, the helical motion of the charge around the center of mass. In fact (v^2-1) measures the deviation from the spinless particle and is a measure for spin, hence related to the variables $S^{\mu\nu}$. The self-field enters only in the equation for π^{μ}.

The method of deriving the radiation reaction consists in writing the retarded field at $X \equiv x(\tau+u)$ due to the charge located at $x = x(\tau)$ and then go to the limit $u \to 0$. The retarded field is given by[7]

$$\begin{aligned} F^{\mu\nu}_{\rm ret}&(X = x(\tau+u); x = x(\tau)) \\ &= \frac{e}{4\pi\epsilon_0 R^3}\{[(X-x)^{\mu}\ddot{x}^{\nu} - (X-x)^{\nu}\ddot{x}^{\mu}]R \\ &\qquad - (Q - \dot{x}^2)[(X-x)^{\mu}\dot{x}^{\nu} - (X-x)^{\nu}\dot{x}^{\mu}]\} , \end{aligned} \tag{7}$$

where

$$R \equiv (X-x)^{\sigma}\dot{x}_{\sigma}, \quad Q \equiv (X-x)^{\sigma}\ddot{x}_{\sigma} .$$

We now expand

$$\begin{aligned} x_{\mu}(\tau+u) &= x_{\mu} + u\dot{x}_{\mu} + \tfrac{1}{2}u^2\ddot{x}_{\mu} + \tfrac{1}{6}u^3\dddot{x}_{\mu} + \cdots , \\ \dot{x}_{\mu}(\tau+u) &= \dot{x}_{\mu} + u\ddot{x}_{\mu} + \tfrac{1}{2}u^2\dddot{x}_{\mu} + \cdots , \\ \ddot{x}_{\mu}(\tau+u) &= \ddot{x}_{\mu} + u\dddot{x}_{\mu} + \cdots , \\ R &= u\dot{x}^2 + \tfrac{1}{2}u^2\dot{x}\cdot\ddot{x} + \tfrac{1}{6}u^3\dot{x}\cdot\dddot{x} + \cdots , \\ Q &= u\dot{x}\cdot\ddot{x} + \tfrac{1}{2}u^2\ddot{x}^2 + \tfrac{1}{6}u^3\ddot{x}\cdot\dddot{x} + \cdots . \end{aligned} \tag{8}$$

Inserting these expressions into (7) and then into (6), expanding also $v_{\nu}(\tau+u)$ on the right-hand side, we have

$$\begin{aligned} \dot{\pi}^{\mu}(\tau+u) &= eF^{\mu\nu}_{\rm ext}(x(\tau+u))v_{\nu}(\tau+u) + \frac{e^2}{4\pi\epsilon_0 u\dot{x}^4}[\tfrac{1}{2}(\dot{x}^{\mu}\ddot{x}^{\nu} - \dot{x}^{\nu}\ddot{x}^{\mu})\dot{x}_{\nu}] \\ &\quad + \frac{e^2}{8\pi\epsilon_0\dot{x}^4}\left[(\dot{x}^{\mu}\ddot{x}^2 - \ddot{x}^{\mu}\dot{x}\cdot\ddot{x}) + \frac{\dot{x}\cdot\ddot{x}}{2\dot{x}^2}(\dot{x}^{\mu}\dot{x}\cdot\ddot{x} - \dot{x}^2\ddot{x}^{\mu}) + \tfrac{1}{3}(\dddot{x}^{\mu}\dot{x}^2 - \dot{x}^{\mu}\dot{x}\cdot\dddot{x}) + O(u)\right] . \end{aligned} \tag{9}$$

Here the third term is independent of u; the second term, going like $1/u$, is the mass renormalization term which we shall transform to time $(\tau+u)$, bring it to the left-hand side, and take the limit. The result is

$$\begin{aligned} \dot{\pi}^{\mu}_{\rm ren} &= eF^{\mu\nu}_{\rm ext}\dot{x}_{\nu} \\ &\quad + \frac{e^2}{4\pi\epsilon_0\dot{x}^4}\left[\frac{2}{3}(\dot{x}^2\dddot{x}^{\mu} - \dot{x}\cdot\dddot{x}\,\dot{x}^{\mu}) \right. \\ &\qquad \left. + \frac{9}{4}\frac{\dot{x}\cdot\ddot{x}}{\dot{x}^2}(\dot{x}^{\mu}\dot{x}\cdot\ddot{x} - \dot{x}^2\ddot{x}^{\mu})\right] . \end{aligned} \tag{10}$$

Introducing the projection tensor

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{v_{\mu}v_{\nu}}{v^2} , \tag{11}$$

which projects every four vector A^{μ} into a vector perpendicular to v^{μ}, our final equation (10) can also be written as $(\alpha \equiv e^2/4\pi\epsilon_0)$

$$\dot{\pi}^{\mu}_{\rm ren} = eF^{\mu\nu}_{\rm ext}v_{\nu} + \alpha\tilde{g}^{\mu\nu}\left[\frac{2}{3}\frac{\ddot{v}_{\nu}}{v^2} - \frac{9}{4}\frac{(v\cdot\dot{v})\dot{v}_{\nu}}{v^4}\right] . \tag{12}$$

In the limit $v^2 \to 1$, hence $v\cdot\dot{v} \to 0$, $\dot{\pi}^{\mu} \to m\ddot{x}^{\mu}$, we recover the Lorentz-Dirac equation (1) for spinless particles. We emphasize that because of spin, $v^2 \neq 1$, velocity and acceleration are not orthogonal, and π^{μ} and v^{μ} are linearly independent. All these are due to *Zitterbewengung*, hence due to spin. But we have the new orthogonality

$$v_{\mu}\dot{\pi}^{\mu} = 0 \tag{13}$$

showing that the new kinetic acceleration $\dot{\pi}^{\mu}$ is orthogonal to velocity. We shall now discuss the mass renormalization procedure in more detail, and give other forms of our final equation.

The component of π^{μ} normal to v^{μ} is, according to (11), $\pi^{\mu}_{\perp} = \tilde{g}^{\mu\nu}\pi_{\nu} = \pi^{\mu} - v^{\mu}(\mathcal{H}/v^2)$, so that $\pi^{\mu}_{\parallel} = -v_{\mu}(\mathcal{H}/v^2)$, where $\mathcal{H} = v^{\alpha}\pi_{\alpha} = v^{\alpha}\pi_{\alpha\parallel}$ is the "Hamiltonian" of the dynamical system (6). We can also write, using $S^{\mu\nu}$ in (6),

$$\pi^{\mu} = \frac{v^{\mu}\mathcal{H}}{v^2} + \frac{v_{\alpha}\dot{S}^{\alpha\mu}}{v^2} . \tag{14}$$

The renormalization term in (9) brought to the left-hand side, can be written as

$$-\frac{1}{2u}\frac{\alpha}{v^4}[v^{\mu}(v\cdot\dot{v}) - v^2\dot{v}^{\mu}] = \frac{1}{2u}\frac{\alpha}{(v^2)^{1/2}}\frac{d}{d\tau}\left[\frac{v^{\mu}}{(v^2)^{1/2}}\right] . \tag{15}$$

We shall now show that this term renormalizes the invariant Hamiltonian $\mathcal{H}$ only, whose value is the mass m. For this purpose we decompose $\dot{\pi}^{\mu}$, using (14), as follows:

$$\frac{d}{d\tau}(\pi^{\mu})=\frac{d}{d\tau}\left[\frac{1}{(v^2)^{1/2}}(v^2)^{1/2}\pi^{\mu}\right]$$
$$=\frac{1}{(v^2)^{1/2}}\frac{d}{d\tau}\left[\frac{v^{\mu}\mathcal{H}}{(v^2)^{1/2}}\right]$$
$$+\frac{1}{(v^2)^{1/2}}\frac{d}{d\tau}\left[\frac{v_{\alpha}\dot{S}^{\alpha\mu}}{(v^2)^{1/2}}\right]-\frac{(v\cdot\dot{v})}{v^2}\pi^{\mu}\ . \tag{16}$$

The only constant appears in the first term of (16), which is precisely of the form (15), hence we can define

$$\mathcal{H}_{\rm ren}\equiv\mathcal{H}+\frac{\alpha}{2u}\ , \tag{17}$$

which then defines $\pi^{\mu}_{\rm ren}$ and we obtain Eq. (12).

Finally, we rewrite Eq. (12) in order to compare it with the Bhabba-Corben equation.[3] Using (14) in (12) we have

$$\frac{d}{d\tau}\pi^{\mu}=\frac{d}{d\tau}\left[\mathcal{H}\frac{v_{\mu}}{v^2}+\frac{v_{\alpha}\dot{S}^{\alpha\mu}}{v^2}\right]$$
$$=eF^{\mu\nu}_{\rm ext}v_{\nu}+\alpha\tilde{g}^{\mu\nu}\left[\frac{2}{3}\frac{\ddot{v}_{\nu}}{v^2}+\frac{9}{4}\frac{(v\cdot\dot{v})\dot{v}_{\nu}}{v^4}\right]. \tag{18}$$

In the Bhabba-Corben equations, v^2 is taken to be unity; hence $v\cdot\dot{v}=0$ and the term with coefficient $\frac{9}{4}$ is missing, and $v_{\mu}S^{\mu\nu}=0$. Both of these relations do not hold here; the BC equation is an approximation to ours. The new $v\cdot\dot{v}$ and the spin equations can also be written as

$$v_{\mu}\dot{v}^{\mu}=-4\frac{v_{\mu}}{(v^2)^{1/2}}S^{\mu\nu}\frac{v^{\alpha}}{(v^2)^{1/2}}\dot{S}_{\alpha\nu}\ ,$$
$$\dot{S}^{\mu\nu}=v^{\mu}\frac{v_{\alpha}}{v^2}\dot{S}^{\alpha\nu}+\frac{v^{\nu}v_{\alpha}}{v^2}\dot{S}^{\mu\alpha}\ . \tag{19}$$

Finally, we emphasize briefly the significance of the new formalism for spin and of the result obtained in this paper.

It may appear strange to use complex spinors z and Dirac matrices in a classical theory. The reason for this is to have a geometric *symplectic theory* of spin on the one hand, and to have the correct configuration and phase space for spin, which is different from that of a spinning top, as mentioned above. The physical observables are, of course, all real. One can see this either by considering the real and imaginary parts of z as dynamical variables (which satisfy oscillator equations), or by passing to the new set of dynamical variables v and S satisfying Eqs. (6). This system shows that velocity and momenta are independent dynamical variables. This crucial point, which also characterizes the quantum Dirac particle, is lost if one models spin as a top or as a dipole moment.

The simplest and most elegant covariant symplectic action principle, Eq (2), is formulated in terms of an invariant time τ. Although for a spinless point particle, τ can be identified with the proper time of the charge, this turns out to be not the case for a spinning particle. But τ can be related to the proper time of an appropriately defined "center of mass," rather a center of mass can be so introduced that τ is the proper time of this fictious center of mass with coordinates X_{μ} such that $dX^{\mu}/d\tau=p^{\mu}/m$; hence X^{μ} moves like a relativistic particle of mass m. The charge performs a helical motion around it (*Zitterbewegung*); hence its velocity $v^2\neq1$. As to the dimensions, we have put $c=1$ and $\lambda=1$; hence we have the same units as quantum theory with $c=\hbar=1$; γ^{μ} are dimensionless numbers, so are z and $\bar{z}$; $\bar{z}\gamma^{\mu}z$ has the same dimension as $\dot{x}=dx^{\mu}/d\tau$.

Our main result is Eq. (12) [or (18)]. We believe that it is the first relativistic symplectic formulation of both coordinates and spin and the first significant generalization of the Lorentz-Dirac equation since 1938. In the second term of (12) we have the LD term $\frac{2}{3}(\ddot{v}^{\nu}/v^2)\tilde{g}^{\mu\nu}$ but also the *new term* $-\frac{9}{4}[(v\cdot\dot{v})\dot{v}^{\nu}/v^4]$. Another difference from the LD equation is that on the left-hand side we have $\dot{\pi}^{\mu}$ instead of $m\ddot{x}^{\mu}$. The Bhabba-Corben equation[6] is not derived from an action principle, but from considerations of energy conservation of a magnetic dipole moment, and the new term $-\frac{9}{4}[(v\cdot\dot{v})\dot{v}^{\nu}/v^4]$ is missing. They have assumed a mass point with charge g_1 and dipole moment g and put $v^2=1$, $S_{\mu\nu}S^{\mu\nu}=0$, and $S_{\mu\nu}v^{\nu}=0$ from the beginning.

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